G_1 CLASS ELEMENTS IN A BANACH ALGEBRA

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Abstract. Let A be a complex unital Banach algebra with unit 1. An element $a \in A$ is said to be of G_1-*class* if

$$\|(z-a)^{-1}\| = \frac{1}{\mathrm{d}(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a).$$

Here $d(z, \sigma(a))$ denotes the distance between z and the spectrum $\sigma(a)$ of a. Some examples of such elements are given and also some properties are proved. It is shown that a G_1-class element is a scalar multiple of the unit 1 if and only if its spectrum is a singleton set consisting of that scalar. It is proved that if T is a G_1 class operator on a Banach space X, then every isolated point of $\sigma(T)$ is an eigenvalue of T. If, in addition, $\sigma(T)$ is finite, then X is a direct sum of eigenspaces of T.

1. Introduction

Let T be a normal operator on a complex Hilbert space H and λ a complex number not lying in the spectrum $\sigma(T)$ of T. Then it is known that the distance between λ and $\sigma(T)$ is given by $\frac{1}{\|(\lambda I - T)^{-1}\|}$. It is also known that there are many other operators that are not normal but still satisfy this property. Putnam called such operators as operators satisfying G_1 condition and investigated properties of such operators in [7], [8]. In particular, he proved that if T is a G_1 class operator, then every isolated point of $\sigma(T)$ is an eigenvalue of T and every G_1 class operator on a finite dimensional Hilbert space is normal.

In this note we extend this concept of G_1 class operators to operators on a Banach space and more generally to elements of a complex Banach algebra and investigate the properties of such elements. The next section contains some preliminary definitions and results that are used throughout. In Section 3, we give definition of a G_1 class element in a complex unital Banach algebra, give some examples and prove a few elementary properties of such elements. In particular, it is proved that every element of a uniform algebra is of G_1 class and conversely if every element of a complex unital Banach algebra A is of G_1 class, then A is commutative, semisimple and hence isomorphic and homeomorphic to a uniform

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algebra. The last section deals with the spectral properties of G_1 class elements and contains the main results of this note. In particular, it is proved that if T is a G_1 class operator on a Banach space X, then every isolated point of $\sigma(T)$ is an eigenvalue of T. Further, if, in addition, $\sigma(T)$ is finite, then X is a direct sum of eigenspaces of T. In this sense T is "diagonalizable" and hence this result can be considered to be an analogue of the Spectral Theorem for such operators.

An overall aim of such a study can be to obtain an analogue of the Spectral Theorem for G_1 class operators. Though at present we are far away from this goal, the present results can be considered a small step in that direction. Next natural step should be to try to prove a similar result for compact operators of G_1 class. Another way of looking at this study is an attempt to answer the following question: "To what extent does the spectrum of an element determine the element?" This question has a long and interesting history. It has appeared under different names at different times such as "Spectral characterizations", "hearing the shape of a drum",[2] "$T = I$ problem" [12] etc. The results in this note say that the spectrum of a G_1 class element gives a fairly good information about that element.

We shall use the following notations throughout this article. Let
$B(w, r) := \{z \in \mathbb{C} : |z - w| < r\}$, the open disc with the centre at w and radius r,
$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$, the closed disc with the centre at w and radius r,
$A + D(0, r) = \bigcup\limits_{a \in A} D(a; r)$ for $A \subseteq \mathbb{C}$ and $d(z, K) = \inf\{|z - k| : k \in K\}$, the distance between a complex number z and a closed set $K \subseteq \mathbb{C}$.
Let $\delta\Omega$ denote the boundary of a set $\Omega \subseteq \mathbb{C}$.
$\mathbb{C}^{n \times n}$ denotes the space of square matrices of order n and $B(X)$ denotes the set of bounded linear operators on a Banach space X.

2. Preliminaries

Since our main objects of study are certain elements in a Banach algebra, we shall review some definitions related to a Banach algebra. Many of these definitions can be found in the book [1]. Some material in this section is also available in the review article [6].

Definition 2.1. Spectrum: Let A be a complex unital Banach algebra with unit 1. For $\lambda \in \mathbb{C}$, $\lambda.1$ is identified with λ. Let $\text{Inv}(A) = \{x \in A : x \text{ is invertible in } A\}$ and $\text{Sing}(A) = \{x \in A : x \text{ is not invertible in } A\}$. The *spectrum* of an element $a \in A$ is defined as:

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \text{Sing}(A)\}$$

The *spectral radius* of an element a is defined as:

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

Its value is also given by the Spectral Radius Formula,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_n \|a^n\|^{\frac{1}{n}}$$

The complement of the spectrum of an element a is called the *resolvent set of* a and is denoted by $\rho(a)$.

Thus when $A = C(X)$, the algebra of all continuous complex valued functions on a compact Hausdorff space X and $f \in A$,then the spectrum $\sigma(f)$ of f coincides with the range of f.

Similarly when $A = \mathbb{C}^{n\times n}$, the algebra of all square matrices of order n with complex entries and $M \in A$, the spectrum $\sigma(M)$ of M is the set of all eigenvalues of M.

Definition 2.2. Numerical Range Let A be a Banach algebra and $a \in A$. The *numerical range* of a is defined by

$$V(a) := \{f(a) : f \in A', f(1) = 1 = \|f\|\},$$

where A' denotes the dual space of A, the space of all continuous linear functionals on A..

The *numerical radius* $\nu(a)$ is defined as

$$\nu(a) := \sup\{|\lambda| : \lambda \in V(a)\}$$

Let A be a Banach algebra and $a \in A$. Then a is said to be *Hermitian* if $V(a) \subseteq \mathbb{R}$.

If A is a is a C^* algebra(also known as B^* algebra), then an element $a \in A$ is Hermitian if and only if it is self-adjoint. [1]

Definition 2.3. Spatial Numerical Range

Let X be a Banach space and $T \in B(X)$. Let $X^{'}$ denote the dual space of X. The *spatial numerical range* of T is defined by

$$W(T) = \{f(Tx) : f \in X^{'}, \|f\| = f(x) = 1 = \|x\|\}.$$

For an operator T on a Banach space X, the spatial numerical range $W(T)$ and the numerical range $V(T)$, where T is regarded as an element of the Banach algebra $B(X)$, are related by the following:

$$\overline{\mathrm{Co}}\, W(T) = V(T)$$

where $\overline{\mathrm{Co}}\, E$ denotes the closure of the convex hull of $E \subseteq \mathbb{C}$.

The following theorem gives the relation between the spectrum and numerical range.

Theorem 2.4. *Let A be a complex unital Banach algebra with unit 1 and $a \in A$. Then the numerical range $V(a)$ is a closed convex set containing $\sigma(a)$. Thus $\overline{Co}(\sigma(a)) \subseteq V(a)$. Hence
$r(a) \leq \nu(a) \leq \|a\| \leq e\nu(a)$.*

A proof of this can be found in [1].

Corollary 2.5. *Let A be a complex unital Banach algebra with unit 1 and $a \in A$. If a is Hermitian, then $\sigma(a) \subseteq \mathbb{R}$.*

We now discuss another important and popular set related to the spectrum, namely pseudospectrum. We begin with its definition.

Definition 2.6. Pseudospectrum Let A be a complex Banach algebra, $a \in A$ and $\epsilon > 0$. The ϵ-pseudospectrum $\Lambda_\epsilon(a)$ of a is defined by

$$\Lambda_\epsilon(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \epsilon^{-1}\}$$

with the convention that $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

This definition and many results in this section can be found in [5]. The book [10] is a standard reference on Pseudospectrum. It contains a good amount of information about the idea of pseudospectrum, (especially in the context of matrices and operators), historical remarks and applications to various fields. Another useful source is the website [11].

The following theorems establish the relationships between the spectrum, the ϵ-pseudospectrum and the numerical range of an element of a Banach algebra.

Theorem 2.7. *Let A be a Banach algebra, $a \in A$ and $\epsilon > 0$. Then*

$$d(\lambda, V(a)) \leq \frac{1}{\|(\lambda - a)^{-1}\|} \leq d(\lambda, \sigma(a)) \quad \forall \lambda \in \mathbb{C} \setminus \sigma(a). \tag{1}$$

Thus

$$\sigma(a) + D(0; \epsilon) \subseteq \Lambda_\epsilon(a) \subseteq V(a) + D(0; \epsilon). \tag{2}$$

A proof of this Theorem can be found in [5].

The following theorem gives the basic information about the analytical functional calculus for elements of a Banach algebra.

Theorem 2.8. *Let A be a Banach algebra and $a \in A$. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\sigma(a)$ and Γ be a contour that surrounds $\sigma(a)$ in Ω. Let $H(\Omega)$ denote the set of all analytic functions in Ω and let $P(\Omega)$ denote the set of all polynomials in z with $z \in \Omega$. We recall the definition of $\tilde{f}(a)$ in the analytical functional calculus as*

$$\tilde{f}(a) = \frac{1}{2\pi i} \int_\Gamma (z - a)^{-1} f(z) dz \tag{3}$$

Then the map $f \to \tilde{f}(a)$ is a homomorphism from $H(\Omega)$ into A that extends the natural homomorphism $p \to p(a)$ of $P(\Omega)$ into A and

$$\sigma(\tilde{f}(a)) = \{f(z) : z \in \sigma(a)\}$$

A proof of this Theorem can be found in [1].

3. G_1 class elements

In this section, we give definition, some examples and elementary properties of G_1 class elements. It is possible to view this definition as motivated by considering the question of equality in some of the inclusions given in Theorem 2.7.

Definition 3.1. Let A be a Banach algebra and $a \in A$. We define a to be of G_1-*class* if

$$\|(z-a)^{-1}\| = \frac{1}{\mathrm{d}(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a). \tag{4}$$

Remark 3.2. The idea of G_1-class was introduced by Putnam who defined it for operators on Hilbert spaces. (See [7],[8].) It is known that the G_1-class properly contains the class of seminormal operators (that is, the operators satisfying $TT^* \le T^*T$ or $T^*T \le TT^*$) and this class properly contains the class of normal operators. Using the Gelfand- Naimark theorem [1], we can make similar statements about elements in a C^* algebra.
G_1-class operators on a finite dimensional Hilbert space are normal[7].

In particular, normal elements are hyponormal. In general, the equation (4) may hold, for every $z \in \mathbb{C} \setminus \sigma(a)$, for an element a of a C^*-algebra even though a is not normal.
For example, we may consider the right shift operator R on $\ell^2(\mathbb{N})$. It is not normal but $\Lambda_\epsilon(R) = \sigma(R) + D(0;\epsilon) = D(0; 1+\epsilon)\, \forall \epsilon > 0$. The operator R is, however, a hyponormal operator.

We now deal with a natural question: What are G_1 class elements in an arbitrary Banach algebra?

The following lemma is elementary and gives a characterization of a G_1 class element in terms of its pseudospectrum.

Lemma 3.3. *Let A be a Banach algebra and $a \in A$. Then*

$$\Lambda_\epsilon(a) = \sigma(a) + D(0;\epsilon) \quad \forall \epsilon > 0 \tag{5}$$

iff a is of G_1-class.

A proof of this Lemma can be found in [5].

As one may expect, most natural candidates to be G_1 class elements are scalars, that is, scalar multiples of the identity 1.

Theorem 3.4. *Let A be a complex Banach algebra with unit 1 and $a \in A$.*
(i) If $a = \mu$ for some complex number μ, then a is of G_1 class and $\sigma(a) = \{\mu\}$.
(ii) If a is of G_1 class, then $\alpha a + \beta$ is also of G_1 class for every complex numbers α, β.
(iii) If a is of G_1 class and $\sigma(a) = \{\mu\}$, then $a = \mu$.

A proof of this is straight forward. It also follows easily from 3.3 and Corollary 3.17 of [5]. We include it here for the sake of completeness.

Proof. (i) Let $a = \mu$ for some complex number μ. Then clearly $\sigma(a) = \{\mu\}$. Hence for all $z \in \mathbb{C} \setminus \sigma(a)$, we have $z \neq \mu$. Thus $\|(z-a)^{-1}\| = \frac{1}{|z-\mu|} = \frac{1}{\mathrm{d}(z,\sigma(a))}$. This shows that a is of G_1 class.
(ii) Next suppose that a is of G_1 class and $b = \alpha a + \beta$ for some complex numbers α, β. We want to prove that b is of G_1 class. If $\alpha = 0$, then it follows from (i). So assume that $\alpha \neq 0$. Let $w \notin \sigma(b) = \{\alpha z + \beta : z \in \sigma(a)\}$. Then $z := \frac{w-\beta}{\alpha} \notin \sigma(a)$ and since a is of G_1 class, $\|(z-a)^{-1}\| = \frac{1}{\mathrm{d}(z,\sigma(a))}$. Now $\|(w-b)^{-1}\| = \|(\alpha z+\beta-(\alpha a+\beta))^{-1}\| = \frac{1}{|\alpha|}\|(z-a)^{-1}\| = \frac{1}{|\alpha| d(z,\sigma(a)} = \frac{1}{d(\alpha z,\sigma(\alpha a))} = \frac{1}{d(w,\sigma(b))}$. This shows that b is of G_1 class.
(iii) Suppose a is of G_1 class and $\sigma(a) = \{\mu\}$. Let $b = a - \mu$. Then by (ii), b is of G_1 class and $\sigma(b) = \{0\}$. Let $\epsilon > 0$ and C denote the circle with the centre at 0 and radius ϵ traced anticlockwise. Then for every $z \in C$, $\|(z-b)^{-1}\| = \frac{1}{d(z,\sigma(b)} = \frac{1}{|z-0|} = \frac{1}{\epsilon}$. Also

$$b = \frac{1}{2\pi i}\int_C z(z-b)^{-1}dz$$

Hence $\|b\| \leq \frac{1}{2\pi} 2\pi\epsilon\epsilon\frac{1}{\epsilon} = \epsilon$. Since this holds for every $\epsilon > 0$, we have $b = 0$, that is $a = \mu$. □

Remark 3.5. The above Theorem has a relevance in the context of a very well known classical problem in operator theory known as *"$T = I$? problem"*. This problem asks the following question: *Let T be an operator on a Banach space. Suppose $\sigma(T) = \{1\}$. Under what additional conditions can we conclude $T = I$?* A survey article [12] contains details of many classical results about this problem.

From the above Theorem it follows that if T is of G_1 class and $\sigma(T) = \{1\}$, then we can conclude that $T = I$. In other words "T is of G_1 class" works as an additional condition in the "$T = I$ problem".

Next we show that every Hermitian idempotent element is of G_1 class. A version of this result was included in the thesis [4].

Theorem 3.6. *Let A be a complex unital Banach algebra with unit 1 and $a \in A$. If a is a Hermitian idempotent element, then a is of G_1 class. Also, if a is of G_1 class and $\sigma(a) \subseteq \{0,1\}$, then a is a Hermitian idempotent.*

Proof. Supose a is a Hermitian idempotent element. If $a = 0$ or $a = 1$, then a is of G_1 class by (i) of Theorem 3.4. Next, let $a \neq 0, 1$. Then $\sigma(a) = \{0, 1\}$ and by Theorem 1.10.17 of [1], $\|a\| = r(a) = 1$. Now Corollary 3.18 of [5] implies that $\Lambda_\epsilon(a) = D(0, \epsilon) \cup D(1, \epsilon)$ for every $\epsilon > 0$. Hence a is of G_1 class by Lemma 3.3.

Next suppose a is of G_1 class and $\sigma(a) \subseteq \{0, 1\}$. If $\sigma(a) = \{0\}$, then $a = 0$ by (ii) of Theorem 3.4. Similarly, if $\sigma(a) = \{1\}$, then $a = 1$. So assume that $\sigma(a) = \{0, 1\}$. Then by Lemma 3.3, $\Lambda_\epsilon(a) = D(0, \epsilon) \cup D(1, \epsilon)$ for every $\epsilon > 0$. Hence by 3.18 of [5], a is a Hermitian idempotent element. □

The abundance or scarcity of G_1 class elements in a given Banach algebra depends on the nature of that Banach algebra. There exist extreme cases, that is, there are Banach algebras in which every element is of G_1 class. On the other hand, there are also Banach algebras in which the scalars are the only elements of G_1 class. We shall see examples of both types below. Before that, we need to review a relation between the spectrum and numerical range of an element of G_1 class. Recall that the numerical range of an element of a Banach algebra is a compact convex subset of $\mathbb{C}$ containing its spectrum, and hence it also contains the closure of the convex hull of the spectrum. The next proposition shows that the equality holds in case of elements of G_1 class.

Proposition 3.7. *Let A be a complex unital Banach algebra and $a \in A$. Suppose a is of G_1-class. Then $V(a) = \overline{Co}\,(\sigma(a))$, the closure of the convex hull of the spectrum of a and $\|a\| \leq e\, r(a)$.*

A proof of this can be found in [5].

Corollary 3.8. *Let A be a complex unital Banach algebra. Suppose $a \in A$ is of G_1-class and $\sigma(a) \subseteq \mathbb{R}$. Then a is Hermitian.*

It is shown in the next theorem that every element in a uniform algebra is of G_1 class. Also a partial converse of this statement is proved. We may recall that a *uniform algebra* is a unital Banach algera satisfying $\|a\|^2 = \|a^2\|$ for every $a \in A$. Every complex uniform algebra is commutative by a theorem of Hirschfeld and Zelazko [1]. Then it follows by Gelfand theory [1] that such an algebra is isomertically isomorphic to a *function algebra*, that is, a uniformly closed subalgebra of $C(X)$ that contains the constant function 1 and separates the points of X, where X is the maximal ideal space of A.

Theorem 3.9. *(See also Theorem 3.15 of* [5]*) Let A be a complex unital Banach algebra with unit* 1.
(i) If A is a uniform algebra, then every element in A is of G_1 class.

(ii) If every element of A is of G_1 class, then A is commutative, semisimple and hence isomorphic and homeomorphic to a uniform algebra.

Proof. (i) The Spectral Radius Formula implies that $\|a\| = r(a)$ for every $a \in A$. Now let $a \in A$ and $\lambda \notin \sigma(a)$. Then

$$\begin{aligned}
\|(\lambda - a)^{-1}\| &= r((\lambda - a)^{-1}) \\
&= \sup\{|z| : z \in \sigma((\lambda - a)^{-1})\} \\
&= \sup\{\frac{1}{|\lambda - \mu|} : \mu \in \sigma(a)\} \\
&= \frac{1}{\inf\{|\lambda - \mu| : \mu \in \sigma(a)\}} \\
&= \frac{1}{d(\lambda, \sigma(a)}
\end{aligned}$$

. This shows that a is of G_1 class.

(ii) By Proposition 3.7, $\|a\| \leq er(a)$ for all $a \in A$. Hence A is commutative by a theorem of Hirschfeld and Zelazko [1]. Also, the condition $\|a\| \leq er(a)$ for all $a \in A$ implies that A is semisimple and hence the spectral radius $r(.)$ is a norm on A. Clearly, $r(a^2) = (r(a))^2$ for every $a \in A$. Hence A is a uniform algebra under this norm. Also the inequality $r(a) \leq \|a\| \leq er(a)$ for all $a \in A$ implies that the identity map is a homeomorphism between these two algebras. $\square$

Next we consider an example of a Banach algebra in which scalars are the only elements of G_1 class.

Example 3.10. (See also Example 2.16 and Remark 2.20 of [3])

Let $A = \{a \in \mathbb{C}^{2\times 2} : a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}\}$ with the norm given by $\|a\| = |\alpha| + \|\beta|$. Suppose $a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in A$ is of G_1 class. Then since $\sigma(a) = \{\alpha\}$, it follows by Theorem 3.4(iii) that $a = \alpha$. (This means $\beta = 0$.)

4. Spectral properties of G_1 class elements

In this section, we show that G_1 class elements have some properties that are very similar to the properties of normal operators on a complex Hilbert space. For example, if H is a complex Hilbert space, T is a normal operator on H and λ is an isolated point of $\sigma(T)$, then λ is an eigenvalue of T. We show that a similar property holds for a bounded operator of G_1 class on a Banach space. For that we need the following theorem about isolated points of the spectrum of a G_1 class element in a Banach algebra.

Theorem 4.1. *Let A be a complex unital Banach algebra with unit* 1. *Suppose a is of G_1-class and λ is an isolated point of $\sigma(a)$. Then there exists an idempotent element $e \in A$ such that $ae = \lambda e$ and $\|e\| = 1$.*

Proof. If $\sigma(a) = \{\lambda\}$, then by 3.4(iii), $a = \lambda$ and we can take $e = 1$.

Next assume that $\sigma(a) \setminus \{\lambda\}$ is nonempty. Let D_1 and D_2 be disjoint open neighbourhoods of λ and $\sigma(a) \setminus \{\lambda\}$ respectively. Define

$$f(z) = \begin{cases} 1 & \text{if} \quad z \in D_1 \\ 0 & \text{if} \quad z \in D_2 \end{cases}$$

Then f is analytic in $D_1 \cup D_2$. Let $e = \tilde{f}(a)$. Then since $f^2 = f$, we have $e^2 = e$, that is, e is an idempotent element and $\|e\| \geq 1$. To prove other assertions, choose $\epsilon > 0$ in such a way that for every $z \in \Gamma_1 := \{w \in \mathbb{C} : |w - \lambda| = \epsilon\}$, λ is the nearest point of $\sigma(a)$ and $\Gamma_1 \subseteq D_1$. Then for every such z, $d(z, \sigma(a)) = |z - \lambda| = \epsilon$, hence $\|(z-a)^{-1}\| = \frac{1}{\epsilon}$. Now let Γ_2 be any closed curve lying in D_2 and enclosing $\sigma(a) \setminus \{\lambda\}$ and let $\Gamma = \Gamma_1 \cup \Gamma_2$. Then

$$e = \tilde{f}(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z-a)^{-1} dz = \frac{1}{2\pi i} \int_{\Gamma_1} (z-a)^{-1} dz$$

Hence

$$\|e\| \leq \frac{1}{2\pi} \frac{1}{\epsilon} 2\pi\epsilon = 1$$

This shows that $\|e\| = 1$.

Now define $g(z) = (z - \lambda) f(z)$. Then $|g(z)| \leq \epsilon$ for all $z \in \Gamma_1$. Note that

$$ae - \lambda e = \tilde{g}(a) = \frac{1}{2\pi i} \int_{\Gamma} g(z)(z-a)^{-1} dz = \frac{1}{2\pi i} \int_{\Gamma_1} g(z)(z-a)^{-1} dz$$

Hence

$$\|ae - \lambda e\| \leq \frac{1}{2\pi}\, \epsilon\, \frac{1}{\epsilon}\, 2\pi\epsilon = \epsilon$$

Since this holds for every $\epsilon > 0$, we have $ae - \lambda e = 0$.

□

Corollary 4.2. *Let X be a complex Banach space, $T \in B(X)$ be of G_1 class and λ be an isolated point of $\sigma(T)$. Then λ is an eigenvalue of T.*

Proof. By Theorem 4.1, there exists an idempotent element $P \in B(X)$ such that $\|P\| = 1$ and $TP = \lambda P$. Clearly P is a nonzero projection operator on X. Let $x \neq 0$ be an element of the range $R(P)$ of P. Then $P(x) = x$. Hence $T(x) = TP(x) = \lambda P(x) = \lambda x$. Thus λ is an eigenvalue of T. □

Some ideas in the proof of the next theorem can be compared with the proof of Theorem C in [9] that deals with similar results about hyponormal operators on a Hilbert space.

Theorem 4.3. *Let A be a complex unital Banach algebra with unit 1. Suppose a is of G_1-class and $\sigma(a) = \{\lambda_1, \ldots, \lambda_m\}$ is finite. Then there exist idempotent elements $e_1, \ldots, e_m$ such that*

(1) $\|e_j\| = 1$, $ae_j = \lambda_j e_j$ *for* $j = 1, \ldots, m$, $e_j e_k = 0$ *for* $j \neq k$,

$$e_1 + \ldots + e_m = 1$$

and

$$a = \lambda_1 e_1 + \ldots + \lambda_m e_m.$$

(2) If p is any polynomial, then

$$p(a) = p(\lambda_1)e_1 + \ldots + p(\lambda_m)e_m.$$

(3) In particular,

$$(a - \lambda_1) \ldots (a - \lambda_m) = 0.$$

(4) If λ is a complex number such that $\lambda \neq \lambda_j$ for $j = 1, \ldots, m$, then

$$(\lambda - a)^{-1} = \frac{1}{\lambda - \lambda_1} e_1 + \ldots + \frac{1}{\lambda - \lambda_m} e_m.$$

(5) If a function f is analytic in a neighbourhood of $\sigma(a)$, then

$$\tilde{f}(a) = f(\lambda_1)e_1 + \ldots + f(\lambda_m)e_m$$

Proof. If $m = 1$, then by Theorem 3.4(iii), $a = \lambda_1$. Hence we can take $e_1 = 1$ and all the conclusions follow trivially. Next we assume $m > 1$. Let $D_1, \ldots, D_m$ be mutually disjoint neighbourhoods of $\lambda_1, \ldots, \lambda_m$ respectively and let $D = \cup_{j=1}^m D_j$. Now for each $j = 1, \ldots, m$, define a function f_j on D by

$$f_j(z) = \begin{cases} 1 & \text{if} \quad z \in D_j \\ 0 & \text{if} \quad z \notin D_j \end{cases}$$

Let $e_j = \tilde{f}_j(a)$. Then it follows as in Theorem 4.1 that each e_j is an idempotent, $\|e_j\| = 1$ and $ae_j = \lambda_j e_j$. Since for $j \neq k$, $f_j f_k = 0$, we have $e_j e_k = 0$.
Further $f_1 + \ldots + f_m = 1$ implies $e_1 + \ldots + e_m = 1$.
Next

$$\begin{aligned} a &= a1 \\ &= a(e_1 + \ldots + e_m) \\ &= ae_1 + \ldots + ae_m \\ &= \lambda_1 e_1 + \ldots + \lambda_m e_m. \end{aligned}$$

This proves (1).
Now since $e_j^2 = e_j$ for each j and $e_j e_k = 0$ for $j \neq k$, we have

$$a^2 = \lambda_1^2 e_1 + \ldots + \lambda_m^2 e_m$$

and in general for any power k,

$$a^k = \lambda_1^k e_1 + \ldots + \lambda_m^k e_m.$$

It follows easily from this that for any polynomial p, we have

$$p(a) = p(\lambda_1)e_1 + \ldots + p(\lambda_m)e_m.$$

Thus (2) is proved.

Now consider the polynomial p given by $p(z) = (z-\lambda_1)\ldots(z-\lambda_m)$. Then $p(\lambda_j) = 0$ for each j. Hence $p(a) = 0$, that is, $(a-\lambda_1)\ldots(a-\lambda_m) = 0$. This completes the proof of (3).

Now suppose λ is a complex number such that $\lambda \neq \lambda_j$ for $j = 1, \ldots, m$. Let

$$b = \frac{1}{\lambda - \lambda_1}e_1 + \ldots + \frac{1}{\lambda - \lambda_m}e_m.$$

Then in view of (1), we have

$$\begin{aligned}(\lambda - a)b &= [(\lambda-\lambda_1)e_1 + \ldots + (\lambda - \lambda_m)e_m][\frac{1}{\lambda-\lambda_1}e_1 + \ldots + \frac{1}{\lambda-\lambda_m}e_m] \\ &= 1\end{aligned}$$

Similarly, we can prove $b(\lambda - a) = 1$ implying (4).

Next suppose a function f is analytic in a neighbourhood Ω of $\sigma(a)$ and Γ is a closed curve lying in Ω and surrounding $\sigma(a)$. Then

$$\begin{aligned}\tilde{f}(a) &= \frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}dz \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)[\frac{1}{z-\lambda_1}e_1 + \ldots + \frac{1}{z-\lambda_m}e_m]dz \\ &= (\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\lambda_1}dz)e_1 + \ldots + (\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\lambda_m}dz)e_m \\ &= f(\lambda_1)e_1 + \ldots + f(\lambda_m)e_m\end{aligned}$$

□

Remark 4.4. Note that the conclusions (2) and (4) of the above Theorem are special cases of (5).

Now we apply the above Theorem to a bounded operator on a Banach space.

Theorem 4.5. *Let X be a complex Banach space. Suppose $T \in B(X)$ is of G_1 class and $\sigma(T) = \{\lambda_1, \ldots, \lambda_m\}$ is finite. Then*

(1) Each λ_j is an eigenvalue of T. In fact, there exist projections P_j such that for each j, the range of P_j is the eigenspace corresponding to the eigenvalue λ_j and X is the direct sum of these eigenspaces. In other words, T is

"diagonalizable". Also $\|P_j\| = 1$ *and* $TP_j = \lambda_j P_j$ *for each* j, $P_j P_k = 0$ *for* $j \neq k$,

$$P_1 + \ldots + P_m = I$$

and

$$T = \lambda_1 P_1 + \ldots + \lambda_m P_m.$$

(2)

$$(T - \lambda_1 I) \ldots (T - \lambda_m I) = 0.$$

(3) If a function f *is analytic in a neighbourhood of* $\sigma(T)$*, then*

$$\tilde{f}(T) = f(\lambda_1)P_1 + \ldots + f(\lambda_m)P_m$$

Proof. It follows from Corollary 4.2 that each λ_j is an eigenvalue of T. The existence and properties of projections P_j follow from Theorem 4.3. Let $X_j = R(P_j)$, the range of P_j. The property $TP_j = \lambda_j P_j$ implies that X_j is the eigenspace of T corresponding to the eigenvalue λ_j for each j. Also $P_j P_k = 0$ for $j \neq k$ implies that $X_j \cap X_k = \{0\}$ for $j \neq k$. It follows from

$$P_1 + \ldots + P_m = I$$

that X is the sum of X_j. This shows that X is the direct sum of these eigenspaces. □

Remark 4.6. Let X and T be as in the above Theorem. Since the conclusion (1) says that X has a basis consisting of eigenvectors of T and T is a linear combination of projections, it can be called Spectral Theorem for such operators. Similarly, the conclusion (2) is an analogue of the Caley-Hamilton Theorem. If, in particular, X is a Hilbert space, then every projection of norm 1 is orthogonal and hence Hermitian(self-adjoint). Thus each P_j is self-adjoint and hence T is normal. This result is also proved in [8].

Suppose X is finite dimensional. Then the above Theorem says that every G_1 class operator on X is diagonalizable.

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