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New bounds on the Ramsey number $r(I_m, L_n)$

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Abstract

We investigate the Ramsey number $r(I_m, L_n)$ which is the smallest natural number k such that every oriented graph on k vertices contains either an independent set of size m or a transitive tournament on n vertices. Continuing research by Larson and Mitchell and earlier work by Bermond we establish two new upper bounds for $r(I_m, L_3)$ which are paramount in proving $r(I_4, L_3) = 15 < 23 = r(I_5, L_3)$ and $r(I_m, L_3) = \Theta(m^2/\log m)$, respectively. We furthermore elaborate on implications of the latter on upper bounds for $r(I_m, L_n)$.

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1. Introduction

In this paper the minimal number ℓ for which every oriented graph[3] of order ℓ either contains an independent set of cardinality m or a transitive induced subtournament of order n is studied. This minimal number ℓ is denoted by $r(I_m, L_n)$.

The case $m = 2$ received a decent amount of attention, it is known that $r(I_2, L_3) = 4$, c.f. [9], that $r(I_2, L_4) = 8$, c.f. [7] and that $r(I_2, L_5) = 14$ and $r(I_2, L_6) = 28$, c.f. [18]. The general asymptotic behaviour of $r(I_2, L_n)$ was studied as well, Stearns in [21] showed that $r(I_2, L_n) \leqslant 2^{n-1}$, this was later improved to $r(I_2, L_n) \leqslant 7 \cdot 2^{n-4}$ for $n > 4$ by Reid and Parker in [18] and to $r(I_2, L_n) \leqslant 55 \cdot 2^{n-7}$ for $n > 6$ by Sánchez-Flores in [19]. Erdős and Moser established $r(I_2, L_n) \geqslant 2^{(n-1)/2}$ in [7]. This case was furthermore studied in the papers [14, 13] and [20].

By contrast, cases in which $m > 2$ were only studied in considerably fewer papers. In [5], Bermond proved $r(I_3, L_3) = 9$ mainly by providing an example establishing the lower bound. The numbers $r(I_m, L_n)$ for $m > 2$ were last revisited two decades ago by Larson and Mitchell [12]. There they proved $r(I_m, L_3) \leqslant m^2$ using a degree argument and showed $r(I_4, L_3) > 13$. This left open three possibilities for the number $r(I_4, L_3)$, the arguably easiest case among the hitherto open ones.

There seems to be a noticable gap between the knowledge about undirected Ramsey numbers $r(I_m, K_n)$ and that on oriented Ramsey numbers $r(I_m, L_n)$ and hereby we are attempting a step in closing it. The numbers $r(I_m, K_3)$ are known for $1 < m < 10$, they are $3, 6, 9, 14, 18, 23, 28, 36$. The last of these values was established in 1982 by Grinstead and Roberts in [10]. More information on small Ramsey numbers can be found in Radziszowski's survey [17]. Moreover since Kim in [11] established a lower bound of appropriate order of magnitude, we know that $r(I_m, K_3) = \Theta(m^2/\log m)$.

Even though we are later going to establish an asymptotically better

[3] We use the adjective "oriented" over "directed" as the graphs under discussion contain at most one edge between any two vertices. Likewise, the graphs are all loopless.

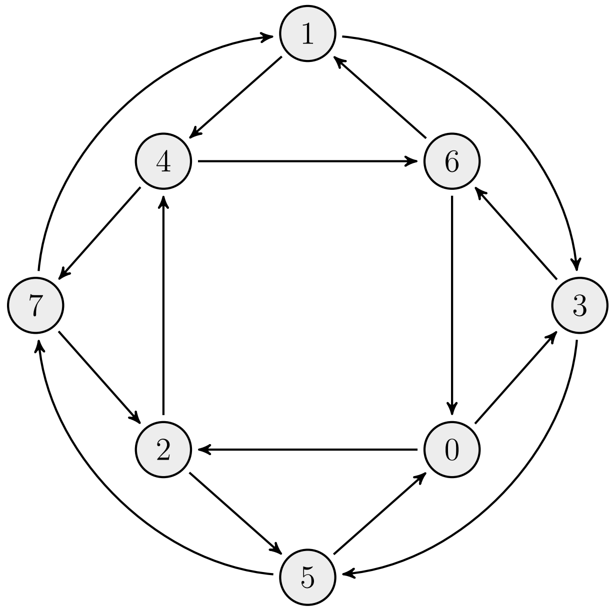

Figure 1: Bermond's $\{I_3, L_3\}$-free graph on 8 vertices.

third bound, in Section 3 we provide an upper bound of $m^2 - m + 3$ for $r(I_m, L_3)$ which is better than both the aforementioned asymptotically better bound and the Larson-Mitchell-bound for $m \leqslant 2^{508}$. More importantly, it allows for the determination of $r(I_m, L_3)$ for $m \in \{4, 5\}$ by giving the correct values. Subsequently, in Section 4, we construct oriented graphs witnessing $r(I_4, L_3) > 14$ and $r(I_5, L_3) > 22$. Thereby we prove the following.

Theorem 1.1. *$r(I_4, L_3) = 15$ and $r(I_5, L_3) = 23$.*

Since any orientation of an $\{I_m, K_3\}$-free graph is $\{I_m, L_3\}$-free, $r(I_m, L_3) \geq r(I_m, K_3)$. Moreover, since every orientation of a graph which contains a K_4 will contain an L_3, $r(I_m, L_3) \leq r(I_m, K_4)$. In Section 5, we use a result of Alon [3] to show that $r(I_m, L_3)$ behaves more like $r(I_m, K_3)$. That is we show the following.

Theorem 1.2. *$r(I_m, L_3) = \Theta(m^2/\log m)$.*

Then we follow an argument of Ajtai, Komlós, and Szemerédi to give, for each $n \geqslant 3$, asymptotic upper bounds on $r(I_m, L_n)$ of the same order as the best known upper bounds for $r(I_m, K_n)$.

More concretely, we extend the result above to the following:

Theorem 1.3. *For each natural number m and each natural number $n \geqslant 3$, there exists a universal constant C_n such that $r(I_m, L_n) \leqslant C_n m^{n-1}/(\log m)^{n-2}$.*

Finally—in an appendix—we provide a formula which gives the best known upper bounds for small values of m and n.

The numbers $r(I_m, L_n)$ are of interest also due to their connection to ordinal Ramsey theory, c.f. [24, Chapter 7] and [1, Chapter 2]. In particular, [9, Theorem 25] amounts to the following:

Theorem 1.4 (Erdős and Rado [9])**.** *$r(\omega m, n) = \omega r(I_m, L_n)$ for all natural numbers m and n.*

In [8] Erdős and Rado showed that for any infinite initial ordinal κ and any natural numbers m and n there is a natural number ℓ such that $r(\kappa m, n) \leqslant \kappa\ell$. They conjectured that ℓ never depends on κ. In [4] Baumgartner settled this conjecture affirmatively.

Theorem 1.5 (Baumgartner [4])**.** *$r(\kappa m, n) = \kappa r(I_m, L_n)$ for all infinite initial ordinals κ.*

2. Preliminaries

Let v be a vertex of an oriented graph $D = (V, A)$. We denote the *in-neighbourhood* of v by $N^-(v)$ and the *out-neighbourhood* of v by $N^+(v)$. Formally, we have $N^-(v) = \{w \in V : (w, v) \in A\}$ and $N^+(v) = \{w \in V : (v, w) \in A\}$. We denote the vertices non-adjacent to v by $I(v)$, formally we have $I(v) = V \setminus (\{v\} \cup N^-(v) \cup N^+(v))$. We denote $|N^-(v)|$ by $d^-(v)$ and $|N^+(v)|$ by $d^+(v)$. We call $d^-(v)$ the *in-degree* of v and $d^+(v)$ its *out-degree*. Whenever we refer to the *degree* of v simpliciter, we mean the sum $d^-(v) + d^+(v)$ of its in- and out-degrees. An oriented graph is n-regular, whenever $d^-(v) = d^+(v) = n$ for all its vertices v.

Lemma 2.1. *Let m and n both be natural numbers, let $D = (V, A)$ be an $\{I_m, L_n\}$-free oriented graph and let $v \in V$. Then the following holds:*

1. *The induced subgraphs on $N^-(v)$ and $N^+(v)$ are $\{I_m, L_{n-1}\}$-free.*
2. *The induced subgraph on $I(v)$ is $\{I_{m-1}, L_n\}$-free.*

Proof. To show the first assertion suppose towards a contradiction that $N^-(v)$ contains a set of vertices T such that the induced subgraph on T is the transitive tournament of size $n-1$. Then $\{v\} \cup T$ is the transitive tournament of size n. This contradicts that D is L_n-free. For the induced subgraph on $N^+(v)$ one may argue analogously.

To show the second assertion suppose towards a contradiction that $I(v)$ contains an independent set I of size $m-1$. Then $\{v\} \cup I$ is an independent set of size m. This contradicts that D is I_m-free. □

This has the following consequences for the case $n = 3$.

Corollary 2.2. *Let m be a natural number, let $D = (V, A)$ be an $\{I_m, L_3\}$-free oriented graph and let $v \in V$. Then $N^-(v)$ and $N^+(v)$ are independent sets. Particularly, $d^-(v), d^+(v) \leqslant m - 1$.*

We now provide a recursive upper bound for $r(I_m, L_n)$.

Lemma 2.3. *We have $r(I_{m+1}, L_{n+1}) \leqslant 2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 1$ for all natural numbers m and n. Furthermore, if an $\{I_{m+1}, I_{n+1}\}$-free oriented graph $D = (V, A)$ has order $2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 2$, then all $v \in V$ satisfy*

1. *$d^-(v) = d^+(v) = r(I_{m+1}, L_n) - 1$ and*
2. *$|I(v)| = r(I_m, L_{n+1}) - 1$.*

Proof. Let D be an $\{I_{m+1}, L_{n+1}\}$-free oriented graph. Let $v \in D$. By Lemma 2.1, $N^-(v)$ and $N^+(v)$ have at most size $r(I_{m+1}, L_n) - 1$ each, and $I(v)$ has at most size $r(I_m, L_{n+1}) - 1$. Hence,

$$\begin{aligned} |V| &\leqslant |\{v\}| + |N^-(v)| + |N^+(v)| + |I(v)| \\ &\leqslant 1 + 2(r(I_{m+1}, L_n) - 1) + r(I_m, L_{n+1}) - 1 \\ &= 2r(I_{m+1}, L_n) + r(I_m, L_{n+1}) - 2. \end{aligned}$$

This implies the assertion. □

The following lemma goes back to Larson and Mitchell, c.f. [12]. It follows from Corollary 2.2 in connection with Lemma 2.3. We will later improve on it with Proposition 3.4.

Lemma 2.4 (Larson and Mitchell). *$r(I_m, L_3) \leqslant m^2$ for all natural numbers $m \geqslant 2$.*

A proof of the following lemma can be found in [21] so we do state, yet not prove it.

Lemma 2.5. *$r(I_2, L_n) \leqslant 2^{n-1}$ for all natural numbers $n \geqslant 2$.* □

3. Improving the Larson-Mitchell Upper Bound

In this section we improve Lemma 2.4 and show that $r(I_m, L_3) \leqslant m^2 - m + 3$ for all $m \geqslant 3$. This upper bound turns out to be tight for $m \in \{3,4,5\}$.

If $D = (V, A)$ is an oriented graph and $B, C \subset V$, let $E(B, C)$ denote the set of edges between vertices in B and vertices in C, irrespective of their direction. Formally we have $E(B, C) = A \cap ((B \times C) \cup (C \times B))$.

For the following lemma and its proof, note that whenever we refer to a triangle without specifying that it be transitive or cyclic, it may be either. In particular, when we say that a subset S of vertices in D contains a triangle, we mean the oriented subgraph of D induced by S contains either a transitive or a cyclic triangle.

Lemma 3.1. *Up to isomorphism there is exactly one $\{I_3, L_3\}$-free oriented graph D on eight vertices. It has the following properties:*

1. *D is 2-regular,*
2. *every triple of vertices of D contains at least one edge,*
3. *the non-neighbourhood of any vertex of D induces a triangle,*
4. *any set of 5 vertices in D either contains a triangle or the induced underlying unoriented graph is isomorphic to C_5,*
5. *any set of 6 vertices in D contains a triangle.*

Proof. As $r(I_2, L_3) = 4$ and $r(I_3, L_2) = 3$, the bound in Lemma 2.3 is tight. Hence, the oriented graph is 2-regular. The second and third assertion follow from D being I_3-free.

Let M be a set of five vertices of D. We assume that M does not contain a triangle. We can ignore the orientation of the edges. Let $x \in M$. By part 3, $|I(x) \cap M| \leqslant 2$. If $|I(x) \cap M| \leqslant 1$, then $|M \cap (N^+(x) \cup N^-(x))| \geqslant 3$, so part 2 implies the assertion. Hence, $|I(x) \cap M| = 2$ for all $x \in M$. Hence, the induced underlying subgraph on M is isomorphic to a cycle of length 5. This implies the fourth assertion. The fifth assertion follows similarly.

Now we show the uniqueness of D. W.l.o.g. the vertex set of D is $\{0,1,2,3,4,5,6,7\}$, where $N^+(0) = \{2,3\}$, $N^-(0) = \{5,6\}$, and $I(0) = \{1,4,7\}$. As $I(0)$ is $\{I_2, L_3\}$-free, w.l.o.g. we have the edges

$$(1,4), (4,7), \text{ and } (7,1)$$

in D. By definition, we have

$$(0,2), (0,3), (5,0), \text{ and } (6,0)$$

in D. As every vertex in D has degree 4,

$$|E(N^+(0) \cup N^-(0), I(0))| = 4|I(0)| - 2|E(I(0), I(0))| = 12 - 6 = 6.$$

Hence,

$$2|E(N^+(0), N^-(0))| = 4 \cdot 3 - |E(N^+(0) \cup N^-(0), I(0))| = 6.$$

As D is L_3-free, the three edges in $E(N^+(0), N^-(0))$ go from $N^+(0)$ to $N^-(0)$, so w.l.o.g. we can assume that the edges

$$(3, 6), (2, 5), \text{ and } (3, 5)$$

are in D. As 3 has in-degree 2, there is one edge from $I(0)$ to 3, w.l.o.g. that is $(1, 3)$. As $I(3)$ is $\{I_2, L_3\}$-free, the edges $(7, 2)$ and $(2, 4)$ are in D. Similarly, $I(2)$ is $\{I_2, L_3\}$-free, so $(6, 1)$ is an edge of D. As the out- and in-degrees of all vertices are 2, the edges $(5, 7)$ and $(4, 6)$ are in D. Now we have given all 16 oriented edges of D without loss of generality. □

The unique $\{I_3, L_3\}$-free oriented graph on eight vertices may be defined on $\mathbb{Z}_8$ by setting both $x \mapsto x + 1$ and $x \mapsto x - 2$, see Figure 1.

Lemma 3.2. *An $\{I_4, L_3\}$-free oriented graph on fourteen vertices contains at least 38 edges.*

Proof. We show the statement by contradiction. Let $D = (V, A)$ be a 14-vertex $\{I_4, L_3\}$-free oriented graph with $|A| < 42$. By Corollary 2.2, every vertex has in-degree and out-degree at most 3. Since $|A| < 42$, the sum of in-degrees is less than 42 and hence there exists a vertex v with in-degree at most 2. By Lemma 2.1, $d^-(v) + d^+(v) \geqslant 14 - r(I_3, L_3) = 5$. Since $d^+(v) \leqslant 3$, we have $d^-(v) = 2$ and $d^+(v) = 3$.

Since $N^+(v)$ is already an independent set of size 3, and D is I_4-free, each of the 8 vertices in $I(v)$ is adjacent to at least one vertex of $N^+(v)$. Hence $|E(N^+(v), I(v))| \geqslant 8$.

Similarly, we get $|E(N^-(v), I(v))| \geqslant 5$: Assume to the contrary that $|E(N^-(v), I(v))| < 5$. Let F denote the vertices in $I(v)$ which are not in an edge of $E(N^-(v), I(v))$. As $|I(v)| = 8$, we have $|F| \geqslant 4$. As $r(I_2, L_3) = 4$, we find an independent set F' of size 2 in F and thus $F' \cup N^-(v)$ is an independent set of size 4. This contradicts that D is I_4-free.

By Lemma 3.1, $|E(I(v), I(v))| = 16$. Let y be the number of vertices $w \in I(v)$ with $d^-(w) + d^+(w) = 6$. Then

$$\begin{aligned} 45 &\leqslant |E(N^+(v), I(v))| + |E(N^-(v), I(v))| + 2 \cdot |E(I(v), I(v))| \\ &= 6y + 5(8 - y). \end{aligned}$$

Hence, $y \geqslant 5$. This, together with the handshaking lemma, ensures that there are at most 8 vertices u with $d^-(u) + d^+(u) = 5$. As all the vertices w in D satisfy $d^-(w) + d^+(w) \in \{5, 6\}$, we have $|A| \geqslant (8 \cdot 5 + 6 \cdot 6)/2 = 38$. □

One can improve the previous argument to show that there are at least 41 edges, but it is slightly more tedious and not needed in the following.

Lemma 3.3. *Let m be a natural number and suppose that $D = (V, A)$ is an $\{I_m, L_3\}$-free oriented graph. Let $v \in V$ with $d^-(v) = m - 1$ and $w \in V$ with $d^+(w) = m - 1$. Then*

$$|E(N^-(v), I(v))| \geqslant 2(|I(v)| - m + 1)$$

and

$$|E(N^+(w), I(w))| \geqslant 2(|I(w)| - m + 1).$$

Proof. By Corollary 2.2, $N^-(v)$ is an independent set of size $m - 1$. Notice that each $x \in I(v)$ is adjacent to at least one vertex of $N^-(v)$ as otherwise $N^-(v) \cup \{x\}$ is an independent set of size m. We call $x \in I(v)$ a *private neighbour* (with respect to $N^-(v)$) if x has exactly one neighbour in $N^-(v)$. We claim that a vertex $u \in N^-(v)$ is adjacent to at most two private neighbours.

Suppose that u is adjacent to three private neighbours, call them x, y and z. If x, y and z are all adjacent, then the induced subgraph on $\{u, x, y, z\}$ is an $\{I_2, L_3\}$-free graph. This contradicts $r(I_2, L_3) = 4$. If without loss of generality x and y are not adjacent, then $\{x, y\} \cup N^-(v) \setminus \{u\}$ is an independent set of size m. This contradicts thats D is I_m-free. This shows our claim.

Hence, each $u \in N^-(v)$ is adjacent to at most two private neighbours in $I(v)$. Let P denote the set of private neighbours in $I(v)$ (with respect to $N^-(v)$). Since every vertex in $I(v) \setminus P$ has at least two neighbours in $N^-(v)$ and $|P| \leqslant 2|N^-(v)| = 2d^-(v)$, we have:

$$\begin{aligned} |E(N^-(v), I(v))| \geqslant& |P| + 2(|I(v)| - |P|) = 2|I(v)| - |P| \\ \geqslant& 2(|I(v)| - d^-(v)) = 2(|I(v)| - m + 1). \end{aligned}$$

An analogous argument shows

$$|E(N^+(w), I(w))| \geqslant 2(|I(w)| - m + 1).$$

The assertion follows. □

Proposition 3.4. *If m is a natural number, where $m \geqslant 2$, then (1) every $\{I_m, L_3\}$-free oriented graph on $m^2 - m + 2$ vertices has at least $(m^2 - m + 2)(2m - 3)/2$ edges and that (2) $r(I_m, L_3) \leqslant m^2 - m + 3$.*

Proof. The proposition will be established by induction on m. As there are no $\{I_2, L_3\}$-free oriented graphs on four vertices, the statement of the proposition is vacuously true in the case $m = 2$. By Lemmas 2.4 and 3.1(1) it holds in case $m = 3$ as well. Henceforth we assume $m \geqslant 3$ and the truth of the proposition for m. We show it holds for $m + 1$ as well. To this end, let $D = (V, A)$ an $\{I_{m+1}, L_3\}$-free graph with $|V| \geqslant (m + 1)^2 - (m + 1) + 2 = m^2 + m + 2$.

Note that this induction is slightly twisted. First we use the induction hypothesis on (2) for m to show (1) for $m + 1$. Then we use the induction hypothesis on (1) and (2) for m to show (2) for $m + 1$.

In order to show that there are no fewer edges in D than claimed, consider that as D is $\{I_{m+1}, L_3\}$-free, by Lemma 2.3 the non-neighbourhood $I(v)$ of any vertex $v \in V$ induces an $\{I_m, L_3\}$-free oriented subgraph $D_v = \big(I(v), A \restriction (I(v) \times I(v))\big)$ of D. By our induction hypothesis we have $|I(v)| \leqslant m^2 - m + 2$ and hence $d^-(v) + d^+(v) \geqslant (m^2 + m + 2) - 1 - (m^2 - m + 2) = 2(m + 1) - 3$ for all $v \in V$. Hence the number of edges in D is at least

$$\frac{1}{2} \sum_{v \in V} \big(d^-(v) + d^+(v)\big) \geqslant \frac{1}{2} \sum_{v \in V} \big(2(m + 1) - 3\big) = \frac{|V|\big(2(m + 1) - 3\big)}{2}.$$

For $|V| = m^2 + m + 2$, this shows (1) for $m + 1$.

It remains to show that $|V| > m^2 + m + 2$ cannot occur. Assume towards a contradiction that $|V| = m^2 + m + 3$. As $r(I_{m+1}, L_2) = m + 1$ and, by induction hypothesis, $r(I_m, L_3) \leqslant m^2 - m + 3$, we have equality in Lemma 2.3. By Lemma 2.3, $d^-(v) = d^+(v) = m$ and $|I(v)| = m^2 - m + 2$ for all $v \in V$.

Let us fix v. As $d^-(v) = d^+(v) = m$, we can apply Lemma 3.3 and obtain

$$|E(N^-(v) \cup N^+(v), I(v))| \geqslant 4(|I(v)| - m) = 4(m^2 - 2m + 2). \qquad (1)$$

As $d^-(w) = d^+(w) = m$ for $w \in I(v)$, we have that

$$\begin{aligned} |E(N^-(v) \cup N^+(v), I(v))| + 2|E(I(v), I(v))| \\ = 2m \cdot |I(v)| = 2m(m^2 - m + 2). \end{aligned} \tag{2}$$

Now we will employ our knowledge about the degrees and the induction hypothesis for the number of edges in an $\{I_m, L_3\}$-free oriented graph on $m^2 - m + 2$ vertices. We distinguish three cases (a), (b), and (c). Let $\ell := |E(N^-(v) \cup N^+(v), I(v))|$. Case (a): If $m = 3$, then, by Lemma 3.1, $|E(I(v), I(v))| \geqslant 16$. Then, by (2), $\ell \leqslant 16$. But, by (1), $\ell \geqslant 20$. This is a contradiction. Case (b): If $m = 4$, then by Lemma 3.2, $|E(I(v), I(v))| \geqslant 38$. Then, by (2), $\ell \leqslant 36$. But, by (1), $\ell \geqslant 40$. Again, this is a contradiction. Case (c): If $m > 4$, then we have $|E(I(v), I(v))| \geqslant (2m-3)(m^2 - m + 2)/2$ by the induction hypothesis. By (2),

$$\ell = |E(N^-(v) \cup N^+(v), I(v))| \leqslant 3m^2 - 3m + 6.$$

As $m > 4$, this contradicts (1). □

4. Constructive Lower Bounds

Observation 4.1. $r(I_4, L_3) = 15$.

Proof. By Proposition 3.4, $r(I_4, L_3) \leqslant 15$. The oriented $\{I_4, L_3\}$-free graph in Figure 2 may be defined on $\mathbb{Z}_{14}$ by setting both $x \mapsto x + 1$ and $x \mapsto x - 2$ for all $x \in \mathbb{Z}_{14}$ and moreover $x \mapsto x + 4$ if x is even and $x \mapsto x - 6$ if x is odd. □

We want to remark that there is no oriented $\{I_4, L_3\}$-free Cayley graph on 14 vertices.

Observation 4.2. $r(I_5, L_3) = 23$.

Proof. By Proposition 3.4, $r(I_5, L_3) \leqslant 23$. The oriented $\{I_5, L_3\}$-free graph in Figure 3 may be defined on $\mathbb{Z}_{22}$ by setting both $x \mapsto x + 1$, $x \mapsto x + 4$, $x \mapsto x - 5$ and $x \mapsto x + 10$ for all $x \in \mathbb{Z}_{22}$. □

Both observations together imply Theorem 1.1.

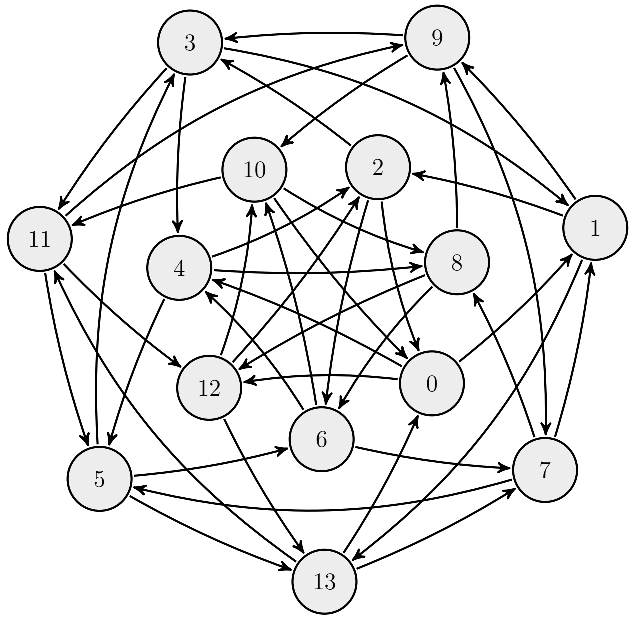

Figure 2: An oriented graph showing $r(I_4, L_3) > 14$.

5. Probabilistic Upper Bounds

In this section, we use a result of Alon to show that $r(I_m, L_3)$ is in $O(m^2/\log m)$. This bound is better than the one in Proposition 3.4 for large enough m. Moreover, this is tight upto multiplicative constants since $r(I_m, L_3) \geqslant r(I_m, K_3)$. Then we follow an upper bound argument of Ajtai, Komlós and Szemerédi for $r(I_m, K_n)$ to obtain upper bounds of commensurate order for $r(I_m, L_n)$.

Note that ld stands for *logarithm dualis*, the logarithm to base 2.

Proposition 5.1 ([3, Prop. 2.1]). *Let $G = (V, E)$ be a graph on v vertices with maximum degree $d \geqslant 1$, in which the neighbourhood of any vertex is r-colourable. Then*

$$\alpha(G) \geqslant \frac{v \operatorname{ld} d}{160 d \operatorname{ld}(r+1)}.$$

Corollary 5.2.

$$r(I_m, L_3) \leqslant \frac{508 m^2}{\operatorname{ld} m} \text{ for all natural numbers } m \geqslant 2.$$

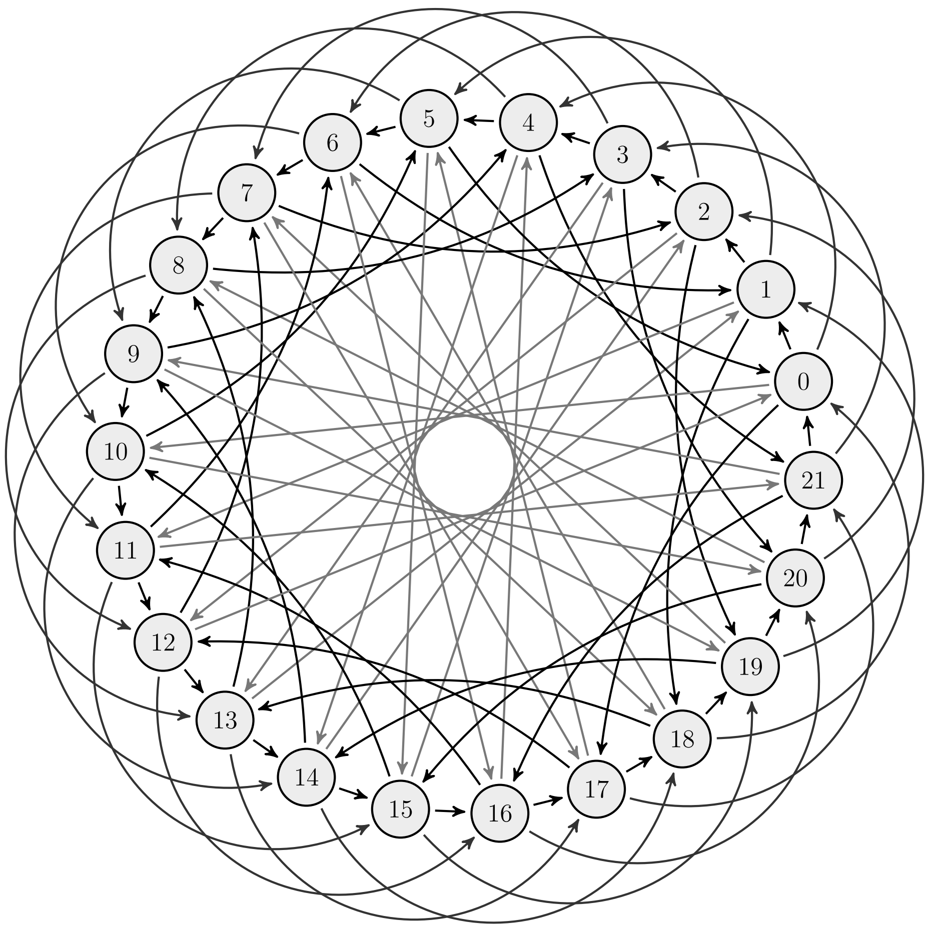

Figure 3: An oriented graph showing $r(I_5, L_3) > 22$.

Proof. Assume towards a contradiction that there are a natural number m and an oriented graph D on $v := 508m^2/\operatorname{ld} m$ vertices with no transitive triangle and no independent set of size m. Let G be the undirected graph obtained from D by forgetting the directions of the edges. Let d denote the maximal degree of a vertex in G. We distinguish two overlapping cases:

Firstly we assume $d \leqslant 253$. Then, by Turán's bound, we have

$$m > \frac{v}{d+1} = \frac{508m^2}{(d+1)\operatorname{ld} m} \geqslant \frac{508m^2}{(253+1)\operatorname{ld} m} = \frac{508m^2}{254\operatorname{ld} m} = \frac{2m^2}{\operatorname{ld} m}.$$

This clearly implies $\operatorname{ld} m > 2m$, so $m > 2^{2m}$, a contradiction.

Secondly we assume $d \geqslant 3$. Since $d < 2m$ by Corollary 2.2 and $\operatorname{ld}(x)/x$ is decreasing for $x > e$,

$$\frac{\operatorname{ld} d}{d} \geqslant \frac{\operatorname{ld}(2m)}{2m},$$

Note that the neighbourhood of any vertex x is 2-colourable since it consists of the in- and out-neighbourhoods of x, both of which are independent sets so by Proposition 5.1, we may conclude that

$$\begin{aligned} m > \alpha(G) \geqslant & \frac{508m^2}{\operatorname{ld} m} \cdot \frac{\operatorname{ld}(d)}{160 \cdot d \operatorname{ld} 3} \geqslant \frac{508m^2}{\operatorname{ld} m} \cdot \frac{\operatorname{ld}(2m)}{160 \cdot 2m \operatorname{ld} 3} \\ = & \frac{508m(\operatorname{ld} 2 + \operatorname{ld} m)}{320 \operatorname{ld} 3 \operatorname{ld} m} = \frac{127m}{80 \operatorname{ld} 3}\left(1 + \frac{1}{\operatorname{ld} m}\right). \end{aligned}$$

It follows that $127 < 80 \operatorname{ld} 3 < 126.8$ which is a contradiction. ☐

Due to Kim [11], $r(I_m, K_3) \geqslant \Theta(m^2/\log m)$. Since $r(I_m, L_3) \geqslant r(I_m, K_3)$, this shows Theorem 1.2.

We follow an argument by Ajtai, Komlós, and Szemerédi for $\{I_m, K_n\}$-free graphs [2] to obtain another upper bound for $\{I_m, L_n\}$-free graphs.

Following the proof of [2, Lemma 4], which is a standard application of Chebyshev's and Markov's inequalities, we obtain the following lemma.

Lemma 5.3. *Let D be an oriented graph with v vertices, e edges, h transitive triangles, and average degree d. Let $0 < p < 1$. Then there exists an induced subgraph D' of D with v' vertices, e' edges, h' transitive triangles, and average degree d' satisfying*

$$v' \geqslant vp/2, \qquad e' \leqslant 3ep^2 \qquad h' \leqslant 3hp^3, \qquad d' \leqslant 6dp.$$

We also need the average version of Alon's bound. The constant in the bound can be easily verified from the proof there.

Theorem 5.4 ([3, Theorem 1.1])**.** *Let $G = (V, E)$ be a graph on v vertices with average degree $d \geqslant 1$, in which the neighbourhood of any vertex is r-colourable. Then*

$$\alpha(G) \geqslant \frac{v \operatorname{ld}(2d)}{640d \operatorname{ld}(r+1)}.$$

Lemma 5.5. *Let $\varepsilon \leqslant 1$ be positive. If D is an oriented graph with v vertices, average degree $d \geq 1$, and $h \leqslant vd^{2-\varepsilon}$ transitive triangles, then*

$$\alpha(D) \geqslant \frac{\varepsilon v \operatorname{ld} d}{2^{15} d}.$$

Proof.

$$\text{Let } p = \frac{\sqrt{15}-3}{6d^{1-\varepsilon/2}}.$$

By Lemma 5.3, we obtain an oriented graph D' with $v' \geqslant vp/2$ vertices, average degree $d' \leqslant 6dp = (\sqrt{15}-3)d^{\varepsilon/2}$, and $h' \leqslant 3hp^3$ transitive triangles. Since $d^{2-\varepsilon}p^2 = (\sqrt{15}-3)^2/6^2 = (4-\sqrt{15})/6$, we get

$$h' \leqslant 3hp^3 \leqslant 3vd^{2-\varepsilon}p^3 \leqslant \left(4-\sqrt{15}\right)\frac{vp}{2} \leqslant (4-\sqrt{15})v'.$$

Deleting one vertex from each of the transitive triangles in D' gives us an L_3-free oriented graph D'' on $v'' \geqslant (\sqrt{15}-3)v' \geqslant (\sqrt{15}-3)vp/2$ vertices. So the neighbourhood of any vertex in D'' is 2-colourable. If d'' denotes the average degree of D'', then

$$d'' \leqslant \frac{d'}{\sqrt{15}-3} \leqslant \frac{6dp}{\sqrt{15}-3} = d^{\varepsilon/2}.$$

We distinguish two cases:

First we assume that $d'' \leqslant 1959$. Then by Caro-Wei, c.f. [6, 23], we get

$$\begin{aligned}\alpha(D'') \geqslant & \frac{v''}{d''+1} \geqslant \frac{v''}{1960} \geqslant \frac{(\sqrt{15}-3)vp}{3920} = \frac{(\sqrt{15}-3)^2 v}{23520d^{1-\varepsilon/2}} = \frac{6(4-\sqrt{15})v}{23520d^{1-\varepsilon/2}} \\ = & \frac{(4-\sqrt{15})v}{3920d^{1-\varepsilon/2}} \geqslant \frac{v}{30864d^{1-\varepsilon/2}} = \frac{v\sqrt{d^{\varepsilon}}}{2^4 \cdot 1929d} \geqslant \frac{v \operatorname{ld} d^{\varepsilon}}{2^{15}d} = \frac{\varepsilon v \operatorname{ld} d}{2^{15}d}.\end{aligned}$$

The last inequality follows from $\mathrm{ld}(x)/\sqrt{x}$ having a global maximum of value smaller than $2/(e \ln 2)$, which is less than $2^{11}/1929$.

Now we assume that $d'' \geqslant e$. Then, as the function $\mathrm{ld}(x)/x$ is decreasing above e and by Theorem 5.4,

$$\begin{aligned}\alpha(D) \geqslant \alpha(D'') &\geqslant \frac{1}{640 \,\mathrm{ld}\, 3} v'' \frac{\mathrm{ld}\, d''}{d''} \geqslant \frac{1}{640 \,\mathrm{ld}\, 3} \frac{vp(\sqrt{15}-3)}{2} \frac{\mathrm{ld}(d^{\varepsilon/2})}{d^{\varepsilon/2}} \\ &\geqslant \frac{1}{640 \,\mathrm{ld}\, 3} \frac{v(\sqrt{15}-3)}{2} \frac{(\sqrt{15}-3)}{6d} \mathrm{ld}(d^{\varepsilon/2}) \\ &\geqslant \frac{v(4-\sqrt{15})\varepsilon \,\mathrm{ld}\, d}{2^9 5 d \,\mathrm{ld}\, 3} \geqslant \frac{v\varepsilon \,\mathrm{ld}\, d}{2^9 5 d(4+\sqrt{15}) \,\mathrm{ld}\, 3} \geqslant \frac{\varepsilon v \,\mathrm{ld}\, d}{2^{15} d}.\end{aligned}$$

□

Theorem 5.6. *For all natural numbers $m, n \geqslant 2$,*

$$r(I_m, L_n) \leqslant 2^{17n} \cdot \frac{m^{n-1}}{(\mathrm{ld}\, m)^{n-2}}.$$

Proof. We prove the bound by induction on n. We already know that $r(I_m, L_2) \leqslant m$ and, by Corollary 5.2, $r(I_m, L_3) \leqslant 2^9 \cdot \frac{m^2}{\mathrm{ld}\, m}$. Fix $n \geqslant 4$ and assume that the claim is true for $n-1$ and $n-2$.

Suppose that D is an L_n-free oriented graph on

$$v \geqslant 2^{17n} \cdot \frac{m^{n-1}}{(\mathrm{ld}\, m)^{n-2}}$$

vertices. We will argue that $\alpha(D) \geqslant m$. Let $\varepsilon = \frac{7}{8n-8}$. Furthermore, let d and $\bar{d}$ denote the maximum and average degrees of the vertices in D, respectively.

Case 1.. $\bar{d} \leqslant 7$. Then by Turán's bound,

$$\alpha(D) \geqslant \frac{2^{17n} m^{n-1}}{(7+1)(\mathrm{ld}\, m)^{n-2}} \geqslant \frac{2^{17} m^{n-1}}{8(\mathrm{ld}\, m)^{n-2}} \geqslant \frac{2^{14} m^{n-1}}{(\mathrm{ld}\, m)^{n-2}} \geqslant 2^{14} m \geqslant m.$$

Case 2.. $\bar{d} \geqslant 3$ and the number of transitive triangles in D is at least $v \cdot d^{2-\varepsilon}$. The graph D contains at most $vd/2$ edges. By double counting there exists an oriented edge $e = (a, b)$ in D such that (a, b) lies in at least

$$\frac{vd^{2-\varepsilon}}{vd/2}$$

transitive triangles of the form $\{(a,b),(b,v),(a,v)\}$. Let V_e denote the set of vertices v such that $\{(a,b),(b,v),(a,v)\}$ is a transitive triangle of D. Then $|V_e| \geqslant 2d^{1-\varepsilon}$.

If there is an oriented subgraph H isomorphic to L_{n-2} in the subgraph D' induced on V_e, then the induced subgraph on $\{a,b\} \cup V(H)$ is isomorphic to L_n. Hence, D' is $\{I_m, L_{n-2}\}$-free. Hence,

$$2d^{1-\varepsilon} \leqslant |V_e| < r(I_m, L_{n-2}).$$

Hence,

$$\begin{aligned} d &< \left(\frac{r(I_m, L_{n-2})}{2}\right)^{\frac{1}{1-\varepsilon}} < \left(2^{17n-35} \cdot \frac{m^{n-3}}{(\operatorname{ld} m)^{n-4}}\right)^{1+\frac{7}{8n-15}} \\ &= 2^{17n-35+\frac{119n-245}{8n-15}} \cdot \frac{m^{n-3+\frac{7n-21}{8n-15}}}{(\operatorname{ld} m)^{n-4+\frac{7n-28}{8n-15}}} < 2^{17n-35+15} \cdot \frac{m^{n-3+\frac{7}{8}}}{(\operatorname{ld} m)^{n-4}} \\ &= 2^{17n-20} \cdot \frac{m^{n-\frac{17}{8}}}{(\operatorname{ld} m)^{n-4}}. \end{aligned}$$

By Turán's bound,

$$\alpha(D) \geqslant \frac{v}{d+1} \geqslant \frac{v}{2d} \geqslant \frac{2^{19} m^{\frac{9}{8}}}{(\operatorname{ld} m)^2} \geqslant 2^{12} m \geqslant m,$$

as $\sqrt[8]{m}(\operatorname{ld} m)^{-2}$ has a minimum of $(e \log 2/16)^2 > 2^{-7}$ at e^{16}.

Case 3.. $\bar{d} \geqslant 3$ and there are fewer than $v \cdot d^{2-\varepsilon}$ transitive triangles in D. By Lemma 2.1(1), we have

$$d < 2r(I_m, L_{n-1}) \leqslant 2^{17n-16} \cdot \frac{m^{n-2}}{(\operatorname{ld} m)^{n-3}}.$$

As $\mathrm{ld}(x)/x$ is decreasing for $x \geqslant 3$, by Lemma 5.5,

$$\begin{aligned}
\alpha(D) \geqslant & \frac{\varepsilon v \cdot \mathrm{ld}\,\bar{d}}{2^{15}\bar{d}} \geqslant \frac{\varepsilon v\,\mathrm{ld}\,d}{2^{15}d} \geqslant \frac{2^{17n-15}\varepsilon m^{n-1}\,\mathrm{ld}\,d}{d(\mathrm{ld}\,m)^{n-2}} \\
> & \frac{2\varepsilon m(17n-16+(n-2)\,\mathrm{ld}\,m-(n-3)\,\mathrm{ld}\,\mathrm{ld}\,m)}{\mathrm{ld}\,m} \\
= & 7m \cdot \frac{17n-16+(n-2)\,\mathrm{ld}\,m-(n-3)\,\mathrm{ld}\,\mathrm{ld}\,m}{4(n-1)\,\mathrm{ld}\,m} \\
= & \frac{7m}{4}\left(\frac{17}{\mathrm{ld}\,m}+\frac{1}{(n-1)\,\mathrm{ld}\,m}+\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)+\frac{1}{n-1}\right) \\
\geqslant & \frac{7m}{4}\left(\frac{17}{\mathrm{ld}\,m}+\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)+\frac{1}{n-1}\right) \geqslant m.
\end{aligned}$$

To see that the last inequality is true, we distinguish two subcases. First we assume that $(\mathrm{ld}\,m)^4 \geqslant m$. Then $\mathrm{ld}\,m \leqslant 16$, so:

$$\begin{aligned}
& \frac{7m}{4}\left(\frac{17}{\mathrm{ld}\,m}+\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)+\frac{1}{n-1}\right) \\
\geqslant & \frac{7m}{4}\left(\frac{17}{\mathrm{ld}\,m}+\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)\right) \\
\geqslant & \frac{7m}{4}\cdot\frac{17}{\mathrm{ld}\,m} \geqslant \frac{7m}{4}\cdot\frac{17}{16} \geqslant \frac{119m}{64} \geqslant \frac{9m}{5} \geqslant m.
\end{aligned}$$

Now we assume that $(\mathrm{ld}\,m)^4 \leqslant m$. Also recall that $n \geqslant 4$. Then

$$\begin{aligned}
& \frac{7m}{4}\left(\frac{17}{\mathrm{ld}\,m}+\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)+\frac{1}{n-1}\right) \\
\geqslant & \frac{7m}{4}\left(\left(1-\frac{2}{n-1}\right)\left(1-\frac{\mathrm{ld}\,\mathrm{ld}\,m}{\mathrm{ld}\,m}\right)+\frac{1}{n-1}\right) \\
\geqslant & \frac{7m}{4}\left(\left(1-\frac{2}{n-1}\right)\left(1-\frac{1}{4}\right)+\frac{1}{n-1}\right) \\
\geqslant & \frac{7m}{4}\left(\left(1-\frac{2}{n-1}\right)\frac{3}{4}+\frac{1}{n-1}\right) \geqslant \frac{7m}{4}\left(\frac{3}{4}-\frac{1}{2n-2}\right) \\
\geqslant & \frac{7m}{4}\left(\frac{3}{4}-\frac{1}{6}\right) \geqslant \frac{7m}{4}\cdot\frac{7}{12} \geqslant \frac{49m}{48} \geqslant m.
\end{aligned}$$

□

This implies Theorem 1.3.

6. Coda

There are more open problems in finite combinatorics stemming from set theory. Determining $r(I_3, L_4)$ would continue our work and seems feasible given the size of the candidates for examples of $\{I_3, L_4\}$-free graphs.

Finally, for the Ramsey numbers $r(\omega^m, n)$ formulae have been found for all natural numbers $m \neq 4$ and all natural numbers n by Nosal in [16, 15]. The determination of the numbers $r(\omega^4, n)$ by a formula, however, has still to be accomplished.

A Formula for Small m and n

We provide the following—admittedly slightly baroque—formula. It gives asymptotically suboptimal upper bounds for $r(I_m, L_n)$ but provides the state of the art for small m and n. Let

$$v(m,n) := \sum_{i=0}^{n-2} \binom{i+m-1}{i+1} 2^i - \binom{m+n-6}{m-4} 2^{n-3} + 1.$$

The following proposition can be proved from Lemma 2.3 by induction on m and n

Proposition 6.1. *We have $r(I_m, L_n) \leqslant v(m,n)$ for all natural numbers m and n with $m \geqslant 2$ and $n \geqslant 3$.*

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