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NON-INVERTIBILITY OF CERTAIN ALMOST MATHIEU OPERATORS

R. BALASUBRAMANIAN, S. H. KULKARNI, AND R. RADHA

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ABSTRACT. It is shown that the almost Mathieu operators of the type $Te_n = e_{n-1} + \lambda sin(2nr)e_n + e_{n+1}$ where λ is real and r is a rational multiple of π and $\{e_n : n = 1, 2, 3, ...\}$, an orthonormal basis for a Hilbert space, is not invertible.

Let H be a Hilbert space with an orthonormal basis $\{e_n : n = 1, 2, 3, ...\}$. An important class of tridiagonal operators used in mathematical physics are almost Mathieu operators which are defined by

$$Te_n \;=\; e_{n-1} + \lambda cos(2n\pi\alpha + \theta)e_n + e_{n+1},$$

$\alpha,\ \lambda,\ \theta$ are real. Certain questions regarding the Lebesgue measure of the spectra of such operators seem to have received a good deal of attention in the literaure. (See [1], [2], [4].) However, the question of invertibility of such operators seems to be unexplored. In this note we prove that the almost Mathieu operators of the type

$$Te_n \;= e_{n-1} + \lambda sin(2nr)e_n + e_{n+1},$$

λ real, r a rational multiple of π are not invertible. Since every separable Hilbert space is isometrically isomorphic to ℓ^2, the main theorem is proved for operators on ℓ^2.

Theorem 0.1. *Let V be an infinite tridiagonal matrix whose diagonal elements are $d_1, d_2, ...\ d_m, 0, -d_m, ..., -d_1, 0$ repeated in the same order and off diagonal entries are* 1. *Then V defines a bounded linear operator on ℓ^2 and V is not invertible.*

Proof. That V defines a bounded linear operator on ℓ^2 is straightforward. To show that V is not invertible, we prove that V is not onto. In particular we aim to show that e_1 is not in the range of V. Let $x = (\alpha_1, \alpha_2, ...) \in \ell^2$ such that $Vx = (1, 0, 0, ...)$. Then $\alpha_1 d_1 + \alpha_2 = 1$ and

$$\alpha_{n-1} + \alpha_n\lambda_n + \alpha_{n+1} = 0, \qquad n = 1, 2, 3, ...,$$

where λ_n are the diagonal elements of the matrix, *viz.* $d_1, d_2, ..., d_m, 0, -d_m, ...,$ $-d_1, 0$. We first consider a block of $2m+3$ equations for $n = m+1$ to $3m+3$. For $n = 2m+2$, $\lambda_n = 0$, we have $\alpha_{2m+3} = -\alpha_{2m+1}$. Next we consider the two

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equations adjacent to the above for $n = 2m+1$ (with $\lambda_n = -d_1$) and $n = 2m+3$ (with $\lambda_n = d_1$)

$$\alpha_{2m} - d_1\alpha_{2m+1} + \alpha_{2m+2} = 0,$$

$$\alpha_{2m+2} + d_1\alpha_{2m+3} + \alpha_{2m+4} = 0,$$

This yields (using $\alpha_{2m+1} = -\alpha_{2m+3}$) $\alpha_{2m+4} = \alpha_{2m}$. Proceeding in this way we can prove by induction $\alpha_{2m+2+k} = (-1)^k\alpha_{2m+2-k}$ for $k = 0, 1, 2, ..., m+1$. In particular $\alpha_{3m+3} = (-1)^{m+1}\alpha_{m+1}$ and $\alpha_{3m+2} = (-1)^m\alpha_{m+2}$. Now, the next block of $2m+3$ equations for $n = 3m+3$ to $5m+5$ is exactly same as the previous block. Hence as above, $\alpha_{5m+5} = (-1)^{m+1}\alpha_{3m+3} = \alpha_{m+1}$. Thus $\alpha_n = \pm\alpha_{m+1}$ for $n = m+1, 3m+3, 5m+5, ...$ Since $x \in \ell^2$, we have $\alpha_{m+1} = 0$. Similarly $\alpha_{m+2} = 0$. However, then $x = 0$ and so $Vx = e_1$ is impossible. □

Now we consider a separable Hilbert sapce H with an orthonormal basis $\{e_n : n = 1, 2, 3, ...\}$ and the almost Mathieu operator of the type

$$Te_n = e_{n-1} + \lambda sin(2nr)e_n + e_{n+1}$$

where λ is real, r is rational multiple of π say $\frac{p\pi}{q}$. Then using the properties of the sine function, we see that T is a matrix of the type defined in Theorem 0.1 with a suitable choice of m. Thus we conclude the following result.

Corollary 0.2. *Let $Te_n = e_{n-1} + \lambda sin(2nr)e_n + e_{n+1}$, λ real and r a rational multiple of π. Then T is not invertible.*

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The Institute of Mathematical Sciences, C.I.T. Campus, Madras-600 113, India
E-mail address: balu@imsc.ernet.in

Department of Mathematics, Indian Institute of Technology, Madras-600 036, India
E-mail address: shk@acer.iitm.ernet.in

Department of Mathematics, Anna University, Madras-600 025, India
E-mail address: radharam@annauniv.edu
E-mail address: radharam@imsc.ernet.in