PROPER HOLOMORPHIC MAPPINGS OF BALANCED DOMAINS IN \mathbb{C}^n

JAIKRISHNAN JANARDHANAN

ABSTRACT. We extend a well-known result, about the unit ball, by H. Alexander to a class of balanced domains in \mathbb{C}^n , n > 1. Specifically: we prove that any proper holomorphic self-map of a certain type of balanced, finite-type domain in \mathbb{C}^n , n > 1, is an automorphism. The main novelty of our proof is the use of a recent result of Opshtein on the behaviour of the iterates of holomorphic self-maps of a certain class of domains. We use Opshtein's theorem, together with the tools made available by finiteness of type, to deduce that the aforementioned map is unbranched. The monodromy theorem then delivers the result.

1. Introduction

The central result of this paper is:

Theorem 1.1. Let $\Omega \subset \mathbb{C}^n$, n > 1, be a smoothly bounded balanced domain of (D'Angelo) finite type. Assume that Ω has a smooth defining function that is plurisubharmonic in Ω . Then every proper holomorphic self-map $F: \Omega \to \Omega$ is an automorphism.

We recall that a domain Ω is said to be balanced if whenever $z \in \Omega$ and $\lambda \in \mathbb{C}, |\lambda| \leq 1$, then we have $\lambda z \in \Omega$. The above theorem is motivated by the famous result of H. Alexander [Ale77] showing the non-existence of non-injective proper holomorphic self-maps of the unit ball in \mathbb{C}^n , n > 1— which the above theorem generalizes. Since its appearance, Alexander's theorem has been extended in many different ways. We refer the interested reader to Section 3 of the survey article by Forstnerič [For93]. One such result is that of Bedford and Bell [BB82], which extends Alexander's theorem to bounded pseudoconvex domains having real-analytic boundaries. Recent work relating to Alexander's theorem has, therefore, focused on domains whose boundaries need not be real-analytic. See, for instance, the work of Berteloot [Ber98] and Coupet, Pan and Sukhov [CPS99, CPS01].

An alternative strategy for deriving a conclusion similar to that of Alexander's theorem, especially when a domain does *not* have real-analytic boundary, is to impose a condition on its automorphism group. With a hypothesis on the automorphism group, there are often interesting interactions between the symmetries of the domain and the boundary geometry. These interactions allow one to simplify the structure of the branch locus of a branched proper holomorphic self-map.

However, there is a new ingredient to our proof, which we briefly introduce here. In a relatively recent paper, Opshtein proved the following result:

Result 1.2 ([Ops06], Théorème A). Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a smoothly bounded domain having a smooth defining function that is plurisubharmonic in D. Let $f: D \to D$ be a

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proper holomorphic self-map that is recurrent. Then the limit manifold of f is necessarily of dimension higher than 1.

Refer to Section 3 for the definitions of the terms that occur in the above result. In [Ops06], Opshtein suggests that his results could serve as a new set of tools for establishing Alexander-type theorems. We found Opshtein's viewpoint very useful in the context of Theorem 1.1: Result 1.2 plays a key role in our proof of Theorem 1.1 (which we shall presently explain).

Another influence behind this work is the following result:

Result 1.3 (Coupet, Pan and Sukhov [CPS99]). Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded balanced pseudoconvex domain of (D'Angelo) finite type. Then every proper holomorphic self-map $F: \Omega \to \Omega$ is an automorphism.

Remark 1.4. In a later paper [CPS01], Coupet, Pan and Sukhov have extended their result to smoothly bounded pseudoconvex quasi-circular domains of finite type in \mathbb{C}^2 .

Let Ω and F be as in Theorem 1.1. Our proof of Theorem 1.1 and the proof of Result 1.3 share some key ideas. As Ω is simply connected, to prove that F is an automorphism it suffices — in view of the monodromy theorem — to establish that F is unbranched. Thus, a natural approach to proving Theorem 1.1 is to assume that F is branched, and use this assumption together with the properties of Ω to reach a contradiction. To this end, we prove the following result that establishes an important property of the branch locus of F.

Proposition 1.5. Let $\Omega \subset \mathbb{C}^n$, n > 1, be a smoothly bounded balanced pseudoconvex domain of (D'Angelo) finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping, and assume that the branch locus $V_F := \{z \in \Omega : \mathsf{Jac}_{\mathbb{C}}(F)(z) = 0\} \neq \emptyset$. Let X be an irreducible component of V_F . Then for each $z \in X$, the set $(\mathbb{C} \cdot z) \cap \Omega$ is contained in X.

We ought to point out that Proposition 1.5 is a hypothetical statement. If F as in Theorem 1.1 were branched, then it would have the above structure. The thrust of our proof is that F can never be branched.

Coupet, Pan and Sukhov have proved a version of Proposition 1.5 for domains in \mathbb{C}^2 in which the domain need not be balanced, but is only required to admit a transverse \mathbb{T} -action. The point that is worth highlighting here is that by restricting ourselves to balanced domains, we are able to give an almost entirely elementary proof of Proposition 1.5, and that these methods have one significant payoff: we do not have to assume in the above results that the \mathbb{T} -action is transverse. We bypass the need for transversality by using some results from dimension theory. These results are insensitive to dimension, which allows us to state and prove Theorem 1.1 in \mathbb{C}^n for all n > 1.

A key object used in the proofs of Proposition 1.5 and Theorem 1.1 is the function $\tau: \partial\Omega \to \mathbb{Z}_+ \cup \{0\}$ introduced by Bedford and Bell (see [BB82, Bel84]). The number $\tau(p)$ is the order of vanishing in the tangential directions of the Levi determinant of a smoothly bounded pseudoconvex domain at the point $p \in \partial\Omega$. The function τ has been used to study the branching behaviour of proper holomorphic mappings in many earlier results. It turns out that, owing to the hypothesis that Ω is of (D'Angelo) finite type, τ is bounded on $\partial\Omega$. We refer the reader to Section 2 for a more precise discussion.

Using Proposition 1.5, one can prove that $F^{-1}\{0\} = \{0\}$. It is at this point that our methods and the methods of Coupet-Pan-Sukhov diverge. We make use of Opshtein's

theorem to deal with the complexities posed by dimension. The key steps of Theorem 1.1 may be summarized as follows:

- We begin by assuming that F is branched.
- If z_0 is a point in $F^{-1}\{0\}$ that is not 0, it follows from Proposition 1.5 that $(\mathbb{C} \cdot z_0) \cap \overline{\Omega}$ is contained in a sequence of distinct irreducible components of the branch loci of the iterates F^k , $k = 1, 2, 3, \ldots$ We note that, by Lemma 2.8, F extends holomorphically to a neighbourhood of $\overline{\Omega}$, whence these irreducible components extend through $\partial \Omega$.
- We pick a point $q \in (\mathbb{C} \cdot z_0) \cap \partial \Omega$. Each iterate of F must be branched at q. Using this fact, and that q is located on the closures of *distinct* irreducible components of the branch loci of F^k , one can show using a result of Bell [Bel84] that τ must be unbounded on $\partial \Omega$. This is impossible, whence $F^{-1}\{0\} = \{0\}$.
- In particular, F fixes 0, whence there exists a limit manifold, call it M, associated to the iterates of F. The circular symmetry of Ω makes it possible to deduce that M is the intersection of a linear subspace of \mathbb{C}^n with Ω , and that one may assume, without loss of generality, that $F|_M$ is given by

$$(z_1, \ldots, z_m, 0, \ldots, 0) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m, 0, \ldots, 0).$$

- If m > 1 then we can find a point $p \in \overline{M} \cap \partial\Omega$ that also lies in the (prolongation of) the branch locus of F. We examine the orbit of p under the action of the group generated by $F|_M$. The behaviour of the function τ along this orbit contradicts the upper semi-continuity of τ . Hence, $m \leq 1$, which, however, contradicts Opshtein's theorem: Result 1.2 above.
- This proves that our assumption that F is branched must be false.

As is evident from the above outline, we will need some definitions and facts from the theory of (iterative) dynamics of holomorphic self-maps. These will be presented in Section 3. Section 4 is devoted to stating and proving certain propositions that are essential to our proofs. The proofs of Proposition 1.5 and Theorem 1.1 will be presented in Section 5.

Before, we proceed further, we clarify that, in this paper, whenever we use use the word "smooth", it will refer to \mathcal{C}^{∞} -smoothness unless specified otherwise. By a smoothly bounded domain, we shall mean a bounded domain whose boundary is \mathcal{C}^{∞} -smooth.

2. Boundary Geometry

In this section, we shall summarize the properties of the function τ alluded to above. For the sake of completeness, we first give the definition of D'Angelo finite type.

Definition 2.1. Let $D \subset \mathbb{C}$ be a domain, and let $f: D \to \mathbb{C}$ be a smooth function. We define the *multiplicity* of f at $p \in D$ to be the least positive integer k such that the homogeneous polynomial of degree k in the Taylor series of (f - f(p)) around p is not identically zero, defining it to be $+\infty$ if no such k exists. The multiplicity of a \mathbb{C}^n -valued function is defined to be the minimum of the multiplicities of its components. We denote the multiplicity of a function f at a point p by $v_p(f)$.

Definition 2.2 (D'Angelo). Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, and let $p \in M$. Let r be a defining function for M in some neighbourhood of the point p. We say that p is a point of *finite type* (also known as finite 1-type) if there is a constant C > 0 such that

$$\frac{v_0(r \circ \phi)}{v_0(\phi)} \le C,$$

whenever $\phi: \mathbb{D} \to \mathbb{C}^n$ is a non-constant analytic disk such that $\phi(0) = p$.

It is simple to prove that the above definition is independent of the choice of the defining function r. Whenever we say below that a bounded domain is of finite type, we shall mean that each boundary point is of finite type in the sense of the above definition.

We now define a function τ . This function was introduced by Bedford and Bell [BB82], and has been used in many results to examine the branching behaviour of proper holomorphic mappings. It is used multiple times in our proof of Theorem 1.1

Definition 2.3. Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and r a smooth defining function for D. Define

$$\Lambda_r := \det \begin{bmatrix} 0 & r_{z_i} \\ r_{\overline{z}_j} & r_{z_i \overline{z}_j} \end{bmatrix}_{i,j=1}^n,$$

the determinant of the Levi-form of r (for a justification of this terminology, see [DK99, Section 2]). For $p \in \partial D$, we define $\tau(p)$ to be the smallest non-negative integer m such that there is a tangential differential operator T of order m on ∂D such that $T\Lambda_r(p) \neq 0$.

Remark 2.4. As any other defining function r' can be written as $h \cdot r$, where h is a positive smooth function defined on some neighbourhood U of the boundary point p, we see that the number $\tau(p)$ is independent of the choice of r. Note that by the pseudoconvexity of D, $\Lambda_r(p) \geq 0 \ \forall p \in \partial D$, and $\Lambda_r(p) = 0$ if and only if p is a point of weak pseudoconvexity.

Observe that τ is an upper semi-continuous function on ∂D .

Suppose $f: D_1 \to D_2$ is a proper holomorphic mapping between bounded pseudoconvex domains with \mathcal{C}^{∞} -smooth boundaries that extends smoothly to a ∂D_1 -open neighbourhood of a point $p \in \partial D_1$. Let ρ be a defining function for D_2 such that $\rho \circ f$ is a local defining function for ∂D_1 near p (see [Bed84, Remark 2]). It follows that

$$\Lambda_{\rho \circ f}(z) = |\mathsf{Jac}_{\mathbb{C}}(f)(z)|^2 \Lambda_{\rho}(f(z)),$$

for $z \in \partial D_1$ and close to p, from which the next result is straightforward to prove.

Let us establish the following notation that we shall use in the remainder of this paper. Given any open set $D \subset \mathbb{C}^n$ and a holomorphic map $f: D \to \mathbb{C}^n$, V_f will be defined as:

$$V_f := \{ z \in D : \mathsf{Jac}_{\mathbb{C}}(f)(z) = 0 \}.$$

Result 2.5 (Bell [Bel84]). Given a proper holomorphic mapping $g: D_1 \to D_2$ between bounded pseudoconvex domains in \mathbb{C}^n , n > 1, with smooth boundaries, if g extends smoothly to ∂D_1 in a neighbourhood of $p \in \partial D_1$, then $\tau(p) \geq \tau(g(p))$, and when $\tau(p) \neq \infty$, the following are equivalent:

- (i) $\tau(p) = \tau(g(p));$
- (ii) g extends to a local diffeomorphism at p;
- (iii) $p \notin \overline{V}_a$.

The next two results give an idea why the τ function is relevant to our main theorem.

Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type and let $p \in \partial D$. It can be argued that $\tau(p)$ is finite. As the D'Angelo 1-type of p is finite, if we pick a non-zero vector $V \in T_p(\partial D) \cap iT_p(\partial D)$, then by definition there exists a $k \in \mathbb{Z}_+$ such that the homogeneous polynomial of degree k in the Taylor expansion of $\zeta \mapsto r(p + \zeta V)$ around $0 \in \mathbb{C}$ is *not* identically zero (r here is a defining function of D). However, as this k depends on the choice of $V \in T_p(\partial D) \cap iT_p(\partial D)$, it becomes quite technical to produce a *single* finite-order tangential differential operator T at p such that $T\Lambda_r(p) \neq 0$. We could not find an elementary proof of Result 2.6 (see below) in the literature, although it has been made use of a number of times; see, for instance, [Pan91]. A recent work of Nicoara [Nic12, Main Theorem 1.1] provides an effective upper bound for τ in terms of the D'Angelo 1-type. For our purposes, the following consequence of Nicoara's result suffices:

Result 2.6. Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain that is of finite type in the sense of D'Angelo. Then there is an $m \in \mathbb{Z}_+$ such that $\tau(p) \leq m \ \forall p \in \partial D$.

The next result is by Coupet, Pan and Sukhov [CPS01]. Since the statement of the result below is slightly different from that of [CPS01, Lemma 1], and its proof is sufficiently important for our purposes, we provide a proof of it. Here, and elsewhere in this paper, F^{ν} will denote the ν -th iterate of F.

Lemma 2.7 (a paraphrasing of Lemma 1 of [CPS01]). Let $\Omega \subset \mathbb{C}^n$, n > 1, be a bounded pseudoconvex domain with \mathbb{C}^{∞} -smooth boundary that is also of finite type. Let $F: \Omega \to \Omega$ be a proper holomorphic mapping (which extends smoothly to $\partial\Omega$ by [Cat87] together with [BC82] or [DF82]). Assume that $V_F \neq \emptyset$ and let L_1 be any irreducible component of V_F . Inductively select any irreducible component of $F^{-1}(L_{\nu})$, $\nu = 1, 2, 3, \ldots$, and denote it by $L_{\nu+1}$. Then:

- a) L_{ν} is an irreducible component of $V_{F^{\nu}}$.
- b) $L_i \neq L_j$ if $i \neq j$.

Proof. It is elementary to see that $V_{F^{\nu+1}} \supset F^{-1}(V_{F^{\nu}})$. From this (a) is immediate. Note that the restriction of F to each $L_{\nu+1}$ is proper, and consequently $F(L_{\nu+1}) = L_{\nu}$. We now show that the stated procedure for constructing $\{L_{\nu}\}$, irrespective of the choices made, ensures that $L_{\nu+1} \neq L_j \ \forall j \leq \nu, \nu = 1, 2, 3, \ldots$ Suppose not, and let m be the smallest positive integer such that $L_m = L_{m+p}$ for some positive integer p. If m > 1, then $F(L_m) = F(L_{m+p})$, and so $L_{m-1} = L_{m+p-1}$, contradicting the definition of m. So, m = 1 and $L_1 = L_{1+p}$. Since $F^p(L_{1+p}) = L_1$, we have $F^p(L_1) = L_1$. As $L_1 \subset V_{F^p}$, for any $q \in \overline{L}_1 \cap \partial \Omega$, we see, by Result 2.5, that

$$0 \le \cdots \tau(F^{kp}(q)) < \tau(F^{(k-1)p}(q)) < \cdots < \tau(q). \tag{1}$$

This, in view of Result 2.6, contradicts the finite-type hypothesis on $\partial\Omega$.

The extendability of F as stated in the previous lemma — and, indeed, a stronger conclusion — is guaranteed when Ω is as in Theorem 1.1. We shall need this stronger conclusion, presented in the next lemma, in our proof. This lemma is an easy consequence of [Bel82, Theorem 2]; see [BJ14, Lemma 4.2] for a proof.

Lemma 2.8. Let $f: D_1 \to D_2$ be a proper holomorphic map between bounded balanced domains. Assume that the intersection of every complex line passing through 0 with ∂D_1 is a circle. Then f extends holomorphically to a neighbourhood of \overline{D}_1 .

3. Dynamics of holomorphic mappings

In this section we summarize some material from the theory of (iterative) dynamics of holomorphic self-maps of a taut manifold. Given complex manifolds X and Y, Hol(X,Y) will denote the space of holomorphic mappings from X to Y, where the topology on Hol(X,Y) is the compact-open topology. We are interested in the set $\Gamma(f)$ which is defined

to be the set of all limit points of the iterates of a holomorphic mapping $f \in \text{Hol}(X, X)$, where X is a taut complex manifold. Of course, $\Gamma(f)$ might be empty. The following result describes the possible behaviours of the iterates.

Result 3.1 ([Aba89], Chapter 2.1). Let X be a taut manifold, and $f \in \operatorname{Hol}(X,X)$. Then either the sequence $\{f^k\}$ of iterates of f is compactly divergent, or there exists a complex submanifold M of X and a holomorphic retraction $\rho: X \to M$ (i.e., $\rho^2 = \rho$) such that every limit point $h \in \operatorname{Hol}(X,X)$ of $\{f^k\}$ is of the form $h = \gamma \circ \rho$, where γ is an automorphism of M. Moreover,

- (1) even ρ is a limit point of the sequence $\{f^k\}$,
- (2) $f|_M$ is an automorphism of M.

Definition 3.2. With the notation as in Result 3.1, we say that f is non-recurrent if the sequence $\{f^k\}$ of iterates of f is compactly divergent. Otherwise, we say that f is recurrent, and we call the map ρ the limit retraction, and the manifold M the limit manifold.

The behaviour of the iterates of a holomorphic self-map of a taut manifold X depends on whether f has a fixed point or not. The following theorem gives a quantitative description of the behaviour of the complex derivative f' at a fixed point of f; see [Aba89, Theorem 2.1.21] for a proof.

Result 3.3. Let X be a taut complex manifold, and let $f \in \text{Hol}(X,X)$ have some fixed point $z_0 \in X$. Then

- (1) the spectrum of $f'(z_0)$ is contained in $\overline{\mathbb{D}}$;
- (2) $T_{z_0}X$ admits a $f'(z_0)$ -invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $f'(z_0)|_{L_N}$ is contained in \mathbb{D} , the spectrum of $f'(z_0)|_{L_U}$ is contained in $\partial \mathbb{D}$ and $f'(z_0)|_{L_U}$ is diagonalizable;
- (3) L_U is the complex tangent space at z_0 of the limit manifold of f.

The space L_U is called the unitary space of f at z_0 .

The next result gives quite precise information about the set $\Gamma(f)$; see [Aba89, Corollary 2.4.4] for a proof.

Result 3.4. Let X be a taut manifold, and let $f \in \text{Hol}(X,X)$ be recurrent with limit retraction $\rho: X \to M$. Then $\Gamma(f)|_M$ is isomorphic to a compact abelian subgroup of Aut(M); in particular, it is the closed subgroup generated by $f|_M \in \text{Aut}(M)$.

We point out that Result 3.1 guarantees that $f|_M$ is an automorphism of M.

To conclude this section, we reiterate that one of the main results of [Ops06] plays a central role in the final step in our proof of Theorem 1.1. This is Result 1.2 above. All the terms occurring in that result have now been defined in this section.

4. Some essential propositions

We begin with some material on complex geodesics. In what follows, given a domain $D \subset \mathbb{C}^n$, we will denote the Kobayashi pseudo-distance by K_D , and the infinitesimal Kobayashi metric by κ_D . We will denote the Poincaré metric and distance on \mathbb{D} by ω and $\mathbf{p}_{\mathbb{D}}$, respectively.

The notion of complex geodesics was introduced by Vesentini ([Ves81]) and is a useful tool in the study of holomorphic mappings.

Definition 4.1. Let $D \subset \mathbb{C}^n$ be a domain, and let $\phi : \mathbb{D} \to D$ be a holomorphic map.

- (i) Let $a, b \in D$. The map ϕ is said to be a K_D -geodesic for (a, b) if there exist points $x, y \in \mathbb{D}$ such that $\phi(x) = a$, $\phi(y) = b$, and $\mathsf{p}_{\mathbb{D}}(x, y) = K_D(a, b)$.
- (ii) Let $a \in D$ and $V \in \mathbb{C}^n \setminus \{0\}$. The map ϕ is said to be a κ_D -geodesic for (a, V) if there exists a number $\alpha \in \mathbb{C}$ and a point $x \in \mathbb{D}$ such that $\phi(x) = a$, $V = \alpha \phi'(x)$, and $\kappa_D(a; V) = \omega(x; \alpha)$.

We need one more notion before we can state an important uniqueness result for certain geodesics.

Definition 4.2. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that a point $a \in \partial D$ is holomorphically extreme if there is no non-constant holomorphic mapping $\phi : \mathbb{D} \to \overline{D}$ such that $\phi(0) = a$.

Example 4.3. Every boundary point of a smoothly bounded pseudoconvex domain of finite type is holomorphically extreme. This follows easily from the fact that any such point admits a continuous plurisubharmonic barrier. The existence of a continuous plurisubharmonic barrier follows from [Sib87] together with [Cat84]. Thus every boundary point of Ω in Theorem 1.1 is holomorphically extreme.

The following uniqueness result for K_D -geodesics illustrates the importance of the above definition; see [JP93, Proposition 8.3.5] for a proof.

Result 4.4. Let $D \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain, and let $a \in D, a \neq 0$, be such that $a/M_D(a) \in \partial D$ is holomorphically extreme, where M_D is the Minkowski functional of D. Then the mapping

$$\phi_a: \mathbb{D} \ni \lambda \mapsto \lambda a/M_D(a)$$

is the unique (modulo $Aut(\mathbb{D})$) K_D -geodesic (κ_D -geodesic) for (0,a) (resp., $(0,a/M_D(a))$).

We now prove a proposition that is a simple consequence of the above uniqueness result. This proposition has already been obtained by Vesentini in the more general context of reflexive Banach spaces. We use his techniques to give a simple proof in our special case.

Proposition 4.5. Let $D \subset \mathbb{C}^n$ be a bounded, balanced pseudoconvex domain, all of whose boundary points are holomorphically extreme. Let $\rho: D \to D$ be a holomorphic retraction such that $\rho(0) = 0$. Then $M := \rho(D) = D \cap V$, where

$$V := \{ v \in \mathbb{C}^n : \rho'(0)v = v \}.$$

Proof. If $\rho \equiv 0$, then there is nothing to prove. So assume that ρ is non-constant. From Result 3.1, it follows that M is a connected complex submanifold of D whose complex tangent space at 0 is V. Let $v \in V \cap D$, $v \neq 0$. From Result 4.4 and our assumption on ∂D , it follows that the mapping

$$\phi: \mathbb{D} \ni \lambda \longmapsto \lambda v/M_D(v) \in D$$

is the unique (modulo Aut(\mathbb{D})) κ_D -geodesic for $(0, v/M_D(v))$. Note that

$$\kappa_D(\rho \circ \phi(0); (\rho \circ \phi)'(0)) = \kappa_D(0; v/M_D(v)) = \kappa_D(\phi(0); v/M_D(v)),$$

whence $\rho \circ \phi$ is also a κ_D -geodesic for $(0, v/M_D(v))$. By uniqueness, it follows that $\rho \circ \phi = \phi \circ \psi$, where $\psi \in \operatorname{Aut}(\mathbb{D})$. Substituting $\psi^{-1}(M_D(v)) (\in \mathbb{D})$ into this equation, we see that v lies in the image of ρ . Hence $V \cap D \subset M$. Now, V and M have the same dimension. Since M is connected, it follows from the principle of analytic continuation that $M = D \cap V$. \square

5. Proofs of Proposition 1.5 and Theorem 1.1

If we assume that the map $F: \Omega \to \Omega$, as stated in Theorem 1.1 is branched, then our assumptions on the geometry of Ω stated in the first sentence of Theorem 1.1 give us a structural result for the branch locus V_F of F. We begin with the proof of this result — i.e., Proposition 1.5. We mention here that some of the arguments used in the proof were inspired by the arguments used in [CP07].

A comment about notation: in what follows, $\dim_H(S)$ will denote the Hausdorff dimension of the set $S \subset \mathbb{C}^n$.

The proof of Proposition 1.5. Let X_1, \ldots, X_m be the distinct irreducible components of the variety V_F . By Lemma 2.8, F extends holomorphically to a neighbourhood N of $\overline{\Omega}$. For the moment, let \widetilde{F} denote this extension.

Lemma 5.1. Let X be an arbitrary irreducible component of the variety V_F (viewed as a subvariety of Ω). Let $E := \overline{X} \setminus X$. There exists a non-empty open (relative to E) subset $\omega_X \subset E$ such that, for each $p \in \omega_X$, there exists a connected neighbourhood $N_p \ni p$ satisfying

$$\dim_H(E \cap N_p) \ge 2n - 3.$$

Proof. Let \widetilde{F} be as introduced just prior to the lemma. Note that X lies in an irreducible component \widetilde{X} of $V_{\widetilde{F}}$. Let S denote the subvariety of singular points of \widetilde{X} . Applying the maximum modulus principle on the irreducible variety X to the functions

$$\left. \frac{\partial \mathsf{Jac}_{\mathbb{C}}(F)}{\partial z_{j}} \right|_{X}, \ \ j=1,2,\dots,n$$

(which vanish simultaneously in X precisely on $S \cap X$), we see that $E \cap S \subsetneq E$. Let $\omega_X := E \setminus S$. Since non-singularity is an open condition on \widetilde{X} , ω_X is open relative to E. Pick a point $p \in \omega_X$. By definition, $\exists r_p > 0$ such that $\overline{B}(p, r_p) \cap S = \emptyset$ and $\widetilde{X} \cap B(p, r_p)$ is a complex submanifold of $B(p, r_p)$.

Claim: For each $r \in (0, r_p), \ (\widetilde{X} \setminus \overline{\Omega}) \cap B(p, r) \neq \emptyset$.

Assume this is false. Then, $\exists r \in (0, r_p)$ such that $\widetilde{X} \cap B(p, r) \subset \overline{\Omega}$. This implies that there exists a non-constant holomorphic map $\psi : \mathbb{D} \to \mathbb{C}^n$ such that $\psi(\mathbb{D}) \subset \widetilde{X} \cap B(p, r) \cap \overline{\Omega}$ and $\psi(\mathbb{D}) \cap \partial\Omega \neq \emptyset$. But this is impossible as every point of $\partial\Omega$ is holomorphically extreme (see Example 4.3). Hence the claim.

Now, let $r_p^* \in (0, r_p)$ be so so small that $B(p, r_p^*) \setminus \partial \Omega$ has exactly two connected components (possible as $\partial \Omega$ is an imbedded smooth submanifold),

$$B(p,r_p^*) \setminus \partial \Omega = C^+ \sqcup C^-.$$

By our above claim, $C^{\pm} \cap \widetilde{X} \neq \emptyset$. Thus $\partial \Omega \cap \widetilde{X} \cap B(p, r_p^*)$ disconnects the manifold $\widetilde{X} \cap B(p, r_n^*)$.

In what follows, \dim_I will denote the *inductive dimension*. The precise definition is rather involved and we refer the reader to [HW41, Chapters II and III]. The fact that we need is Corollary 1 to [HW41, Theorem IV.4]: since $\partial\Omega \cap \widetilde{X} \cap B(p, r_n^*)$ disconnects $\widetilde{X} \cap B(p, r_n^*)$

$$\dim_{I}(\partial\Omega \cap \widetilde{X} \cap B(p, r_{p}^{*})) \ge \dim_{\mathbb{R}}(\widetilde{X} \cap B(p, r_{p}^{*})) - 1 = 2n - 3.$$
(2)

It is well-known that the Hausdorff dimension dominates the inductive dimension; see for instance [HW41, Chapter VII, §4]. By (2), therefore, writing $N_p := B(p, r_p^*)$,

$$\dim_H(N_p \cap E) \ge 2n - 3.$$

Since the above is true for any $p \in \omega_X$, we are done.

Since the extension \widetilde{F} of F is uniquely determined by the latter, for the remainder of the proof we shall not use different symbols for $F: \Omega \to \Omega$ and \widetilde{F} .

Let E be as in Lemma 5.1. Let us fix a point $z_0 \in E$ for the moment. From Result 2.5, all the points of E are necessarily weakly pseudoconvex. Note that $\tau(e^{i\theta}z_0) = \tau(z_0)$. From the fact that τ is upper semi-continuous, it follows that the set $\{w \in \partial\Omega : \tau(w) < \tau(F(z_0)) + 1\}$ is open in $\partial\Omega$, and consequently so is its inverse image under F, $\{z \in \partial\Omega : \tau(F(z_0)) < \tau(F(z_0)) + 1\}$. The latter set obviously contains z_0 . This implies that for θ close to 0, we must have $\tau(F(e^{i\theta}z_0)) \leq \tau(F(z_0)) < \tau(z_0) = \tau(e^{i\theta}z_0)$ which, by Result 2.5, implies that for θ close to 0, we have $e^{i\theta}z_0 \in \overline{V_F}$. Restricting $\operatorname{Jac}_{\mathbb{C}}(F)$ to the set $\mathbb{D} \cdot z_0$, and observing that the boundary-values of this restriction vanish on an arc of $\partial\mathbb{D}$, we see that $\operatorname{Jac}_{\mathbb{C}}(F)$ must vanish on the set $\mathbb{D} \cdot z_0$. As $z_0 \in E$ was arbitrary, we get that for each $z \in E$, $\mathbb{D} \cdot z \subset X_i$, for some i. Let us define

$$E_i := \{ w \in E : \mathbb{D} \cdot w \subset X_i \}, \ i = 1, \dots, m.$$

Since, by Lemma 5.1, E is of Hausdorff dimension at least 2n-3, there is an $i_0, 1 \le i_0 \le m$, such that $\dim_H(E_{i_0}) \ge 2n-3$. Let us call this set E'. Then:

$$\bigcup_{z\in E'} \mathbb{D} \cdot z \subset X_{i_0}.$$

As X and X_{i_0} are irreducible varieties the intersection of whose closures is of Hausdorff dimension at least 2n-3, it must be that $X=X_{i_0}$. This follows from the stratification of $X \cap X_{i_0}$ [Chi89, Chapter 1, § 5.5], the properties of the Hausdorff measure and [Chi89, Corollary 1 of Chapter 1, § 5.3].

We have proved that

$$\bigcup_{z \in E'} \mathbb{D} \cdot z \subset X. \tag{3}$$

Now fix $\lambda \in \mathbb{D}$, and consider the holomorphic function $h_{\lambda}(z) := \mathsf{Jac}_{\mathbb{C}}(F)(\lambda z)$ defined on X. From what we have shown, (3) in particular, h_{λ} vanishes on a subset of Hausdorff dimension at least 2n-3 of the irreducible variety X. Hence h_{λ} must vanish identically on X, and this is true for each $\lambda \in \mathbb{D}$. Thus, we have shown that, given $z \in X$, $(\mathbb{D} \cdot z) \cap \Omega \subset V_F$. Since V_F comprises finitely many irreducible components, by a similar argument as in the previous paragraph (with the role of E now taken by E), we actually have $(\mathbb{D} \cdot z) \cap \Omega \subset E$. By analytic continuation, it follows that if E0, then E1 then E2 then E3 then E3 and E4 then E5 then E5 then E5 then E6 then E7 then E8 then E9 analytic continuation, it follows that if E8 then E9 then

We now have all the tools needed to prove our main theorem. Owing to the fact that the domain Ω admits a smooth defining function that is plurisubharmonic in Ω , it is a fortiori pseudoconvex. We shall use this fact without explicit mention in our proof.

The proof of Theorem 1.1. Observe that Ω is contractible. Thus, to prove that F is an automorphism it suffices, in view of the monodromy theorem, to establish that F is unbranched. So we will assume that F is branched and reach a contradiction. We will not,

hereafter, remark upon the well-definedness of quantities such as $Jac_{\mathbb{C}}(F)(p)$ for $p \in \partial\Omega$. In view of Lemma 2.8, these are indeed well-defined.

Step 1. Proving that $F^{-1}\{0\} = \{0\}$. If not, there is point $0 \neq z_0 \in \Omega$ such that $F(z_0) = 0$. From Lemma 2.7, we have distinct irreducible subvarieties $L_i \subset V_{F^i}$ such that $F|_{L_{i+1}}: L_{i+1} \to L_i$ is a proper holomorphic mapping. By Proposition 1.5, each L_i contains 0, whence we can select each L_i as described in Lemma 2.7 and ensure that $z_0 \in L_i \ \forall i \geq 2$. Again from Proposition 1.5, it follows that $\Lambda := (\mathbb{C} \cdot z_0) \cap \Omega \subset L_i \ \forall i \geq 2$. This means that the sets $F^k(\Lambda) \subset L_2, \forall k \in \mathbb{Z}_+$. Let $q \in (\mathbb{C} \cdot z_0) \cap \partial \Omega = \overline{\Lambda} \setminus \Lambda$. It is elementary that

$$V_{F^n} = \bigcup_{k=0}^{n-1} (F^k)^{-1}(V_F), \quad n \in \mathbb{Z}_+,$$

with the understanding that $F^0 = \mathrm{id}_{\Omega}$. Thus — we refer to the recipe for the L_i 's in Lemma 2.7 — $L_2 \subset V_{F^{2k}} \forall k \in \mathbb{Z}_+$. Note that $q \in \overline{L_2} \setminus L_2$. At this stage, we are precisely in the situation prior to (1) in the proof of Lemma 2.7, except that q belongs to the (prolongation of) the branch locus of F^2 . Therefore, it follows as in the proof of Lemma 2.7 (taking p=2 in the relevant argument), that

$$0 \le \dots < \tau(F^{2k}(q)) < \dots < \tau(F^4(q)) < \tau(F^2(q)) < \tau(q),$$

which contradicts the conclusion of Result 2.6. Our claim follows.

We should point out (although we shall not make use of it below) that from a theorem of Bell [Bel82, Theorem 1], it follows that F is a polynomial mapping.

Step 2. Analyzing the limit manifold of F. As 0 is a fixed point of F, it follows that F is recurrent. Let $\rho: \Omega \to M$ be the limit retraction. As Ω is pseudoconvex and of finite type, by Example 4.3 every point in $\partial\Omega$ is holomorphically extreme. Consequently, from Proposition 4.5 and Result 3.3 it follows that $M = L_U \cap \Omega$, where L_U is the unitary space of F at 0. Recall that $F'(0)|_{L_U}$ is diagonalizable; see Result 3.3. So without loss of generality, (replacing Ω by a suitable linear image and conjugating F by a suitable linear operator, if needed) we may assume that $L_U = \mathbb{C}^m \times \{0_{\mathbb{C}^{n-m}}\}$, and that $F'(0)|_{L_U}$ is given by

$$(z_1, z_2, \dots, z_m, 0, \dots, 0) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_m} z_m, 0, \dots, 0).$$
 (4)

By Cartan's uniqueness theorem, it also follows that $F|_M \equiv F'(0)|_M$.

Step 3. Proving that dim $M \leq 1$. Suppose dim M > 1. From the previous steps, we have that $0 \in V_F \cap M$. Therefore the set $\{z \in M : \mathsf{Jac}_{\mathbb{C}}(F)(z) = 0\}$ is a non-empty analytic subvariety of M. As dim M > 1, it follows that there exists a point $p \in \overline{M} \cap \partial \Omega$ such that $\mathsf{Jac}_{\mathbb{C}}(F)(p) = 0$.

From Result 3.4, and the fact that the maps $F^k|_M$, $k \in \mathbb{Z}$, are all of the special form (4), it follows that there is a strictly increasing sequence $\{n_k\} \in \mathbb{N}$ such that $F^{n_k}|_{\overline{M}} \to (F|_{\overline{M}})^{-1}$ uniformly on \overline{M} . Let $p_k := (F|_{\overline{M}})^{-n_k}(p)$. By Result 2.5, it follows that $\tau(p_{k+1}) \geq \tau(p_k) \geq \tau(p)$. But, $p_k \to F(p)$, and as $p \in \overline{V}_F$, $\tau(F(p)) < \tau(p)$, which means that

$$\limsup_{k \to \infty} \tau(p_k) \ge \tau(p) > \tau(F(p)),$$

which contradicts the fact that τ is upper semi-continuous. This proves that $\dim(M) \leq 1$.

The conclusion of Step 3 is in conflict with the conclusion of Opshtein's theorem, i.e., Result 1.2. Therefore, F is unbranched. As Ω is simply connected, it follows from the monodromy theorem (just continue the germ of a local inverse of F at 0 along paths in Ω from 0 to $z \in \Omega$: the germ obtained at z will be a local inverse by the permanence of relations) that F is an automorphism.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560012, INDIA $E\text{-}mail\ address:}$ jaikrishnan@math.iisc.ernet.in