Quenching through Dirac and semi-Dirac points in optical Lattices: Kibble-Zurek scaling for anisotropic Quantum-Critical systems

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We propose that Kibble-Zurek scaling can be studied in optical lattices by creating geometries that support, Dirac, Semi-Dirac and Quadratic Band Crossings. On a Honeycomb lattice with fermions, as a staggered on-site potential is varied through zero, the system crosses the gapless Dirac points, and we show that the density of defects created scales as $1/\tau$, where τ is the inverse rate of change of the potential, in agreement with the Kibble-Zurek relation. We generalize the result for a passage through a semi-Dirac point in d dimensions, in which spectrum is linear in m parallel directions and quadratic in rest of the perpendicular $(d-m)$ directions. We find that the defect density is given by $1/\tau^{m\nu_{||}z_{||}+(d-m)\nu_{\perp}z_{\perp}}$ where $\nu_{||}, z_{||}$ and $\nu_{\perp}, z_{\perp}$ are the dynamical exponents and the correlation length exponents along the parallel and perpendicular directions, respectively. The scaling relations are also generalized to the case of non-linear quenching.

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The Kibble-Zurek (KZ) scaling [1, 2, 3, 4, 5, 6] of defect density in the final state of a quantum many-body system following a slow passage across a quantum critical point, has been an exciting area of recent research. The KZ argument predicts that the scaling of the defect density is universal and is given as $n \sim 1/\tau^{\nu d/(nuz+1)}$ where τ is the inverse rate of change of a parameter, d is the spatial dimension and ν and z are the correlation length and dynamical exponents, respectively, associated with the quantum critical point [7, 8] across which the system is swept. Following the initial predictions, a plethora of theoretical studies have been carried out [10, 11, 12, 13, 14, 15, 16, 17, 18] to explore the defect generation and the entropy production using different quenching schemes across critical points [10], quantum multicritical points [18], gapless phases [13], along gapless lines [16], etc. The possibility of experimental observations in a spin-1 Bose condensate [19] and also on ions trapped in optical lattices [20, 21] has provided a tremendous boost to the related theoretical studies.

Here, we propose that Kibble-Zurek scaling can be studied in optical lattices with fermionic atoms by creating geometries that support Dirac, semi-Dirac and Quadratic Band Crossings. For example, a Honeycomb lattice consists of two interpenetrating triangular lattices. If the two sublattices are controlled separately, one can create a staggered on-site potential, which creates a gap in the spectrum [22, 23]. We will call the Honeycomb lattice with a staggered potential a gapped Graphene Hamiltonian in analogy with Graphene [22]. The system can be loaded with atoms when one of the sublattices has a much lower energy than the other. As the staggered potential is varied through zero, the system will cross through gapless Dirac point, and the creation of defect density can be studied as a function of the rate of change of potential. Similarly, two interpenetrating square-lattices can lead to Quadratic Band Crossing [24] and either a deformed Honeycomb system [25, 26], or a 3-band system can lead to semi-Dirac points[27], where the spectrum is linear along one axis and quadratic along another. Anisotropic quantum critical points associated with spectra that are linear in some directions and quadratic in others arise in systems as diverse as semiconductor hetero-structures [28] and He^3 [29]. We will calculate a generalized Kibble-Zurek scaling for such anisotropic Quantum critical systems.

Many previous theoretical studies are on one-dimensional quantum spin systems some of which can be exactly solved via Jordan-Wigner transformation [30]. On the other hand, the quenching dynamics of non-integrable spin chains have been studied using adiabatic perturbation theory [4] or exact diagonalization and time-dependent density matrix renormalization techniques [16, 17]. Both the methods have been very successful in yielding exciting results for the scaling of defect density following a slow quantum quench in pure as well as random systems [17]. There are also a few results for the quenching dynamics of higher dimensional systems, e.g., in ref. [13] the exact solvability of a two-dimensional Kitaev model [31] was utilized.

In this paper, we propose a kind of time-dependent perturbation that would be very difficult to achieve in solid state systems but should be possible in optical lattices. Because the underlying geometry in optical lattices is controlled by lasers, one should be able to change them in such a way that the system passes through special band crossings. The salient feature of our proposal on Honeycomb and other lattices is that the time-dependent perturbation preserves crystal momentum. Thus, as long as the interactions between particles are weak, the system factorizes into independent momentum sectors. For each

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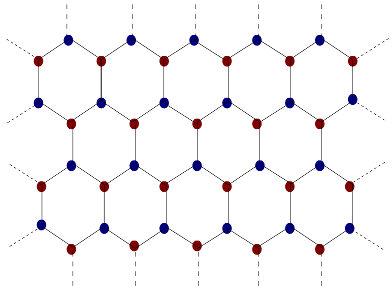

FIG. 1: A schematic picture of a honeycomb lattice consisting of two interpenetrating triangular lattices shown in different colors. In cold-atoms, this system can be controlled by two sets of lasers.

value of the momentum, there is a probability that the system can undergo transition from the lower to the upper eigenstate. This can be analyzed using the Landau-Zener transition formula [32] to provide an exact result for the defect creation in the quenching process. Integration over all momentum leads to the desired KZ scaling relations. Furthermore, quenching through the semi-Dirac point yields a more general form of the KZ scaling for spatially anisotropic quantum critical points. In principle, Dirac-like band crossings are of tremendous interest in themselves, in contexts such as topological insulators.[33, 34] But, whether one can find a perturbation that drives the system through such a crossing will have to be examined in individual cases.

We focus primarily on a tight-binding model of fermions moving on a Honeycomb lattice. This lattice has two inequivalent sites per unit cell (see Fig. 1) and hence the system is described in the momentum space in terms of de-coupled 2×2 matrices [22]. Assuming the system to be half-filled, the honeycomb lattice of the tight-binding Hamiltonian is critical at two inequivalent points of the Brillouin zone and there is Dirac cone of excitations with linear dispersion around this point. If a staggered on-site potential (called the mass term) which differentiates between two inequivalent sites is added to the tight binding Hamiltonian, a gap is created in the spectrum [23]. We assume that the magnitude of the mass term (denoted by μ) is quenched linearly from $-\infty$ to $-\infty$ at a linear rate $\mu = t/\tau$ so that the system crosses the gapless point $\mu = 0$ at a time $t = 0$. When $\mu = -\infty$, only one sublattice of the HC lattice will be occupied which amounts to saying that for every Fourier mode k, one of the basis states is occupied. If the mass term is slowly changed, the state would follow adiabatically until the Dirac point at $t = 0$ where the spectrum is gapless and thereafter system no longer follows the ground state adiabatically. We here show that the density of defect thus generated is in complete agreement with the KZ prediction. We then generalize to the case of quenching through a (d, m) semi-Dirac point [27] at which the spectrum is linear in m-directions and quadratic in rest of the $(d - m)$-directions. Exploiting again the 2×2 form of the resulting Hamiltonian and the Landau-Zener transition formula we arrive at a generalized KZ scaling form $1/\tau^{m\nu_{||}z_{||}+(d-m)\nu_{\perp}z_{\perp}}$ where $\nu_{||}, z_{||}$ and $\nu_{\perp}, z_{\perp}$ are the dynamical exponent and the correlation length exponents along the parallel directions and perpendicular directions, respectively. The same scaling relation is also expected to be valid for quenching through a (d, m) quantum Lifhshitz point [35].

Let us start from the general Hamiltonian of fermions moving on a honeycomb lattice

$$H = \sum_{k=0}^{\pi} \psi_k^{\dagger} H_k \psi_k \qquad (1)$$

where H_k is the reduced 2×2 matrix $(H_k)_{11} = (H_k)_{22} = 0$ and $(H_k)_{12} = (H_k)_{12}^* = -t[e^{ik_x a} + 2e^{-ik_x a/2}\cos(k_y\sqrt{3}a/2)$, where t is the hopping term and a is the lattice spacing. In the presence of a mass term that arises due to a staggered on-site potential μ [23], the reduced Hamiltonian takes the form

$$\tilde{H}_k = \begin{bmatrix} \mu & \Delta \\ \Delta^* & -\mu \end{bmatrix} \qquad (2)$$

where $\Delta(k) = (H_k)_{12}$ as defined before. The off-diagonal term can be simplified by expanding around the Dirac point $(0, \frac{4\pi}{3\sqrt{3}a})$ as $\Delta(k) = -t\left[e^{iq_x a} + 2e^{-iq_x a/2}\cos((\frac{4\pi}{3\sqrt{3}a} + q_y)\frac{\sqrt{3}a}{2})\right]$ where $k_x = q_x$ and $k_y = \frac{4\pi}{3\sqrt{3}a} + q_y$. In the limit of $q_x, q_y \to 0$, we arrive at a simpler form of Eq. 2

$$\tilde{H}_k = \begin{bmatrix} \mu & v_f(q_y - iq_x) \\ v_f(q_y + iq_x) & -\mu \end{bmatrix} \qquad (3)$$

where $v_f = 3ta/2$.

During the quenching $\mu = t/\tau$, the general state vector at an instant t can be written as $|\psi_k(t)\rangle = C_{1k}(t)|1_k\rangle + C_{2k}(t)|2_k\rangle$ where the basis vectors are $|1_k\rangle$ and $|2_k\rangle$. With the initial condition $|C_{1k}(t \to -\infty)|^2 = 1$, the non-adiabatic transition probability due to the passage through the Dirac point is given as $p_k = |C_{1k}(t \to +\infty)|^2$. Using the Hamiltonian (2), the time evolution of the system is described in terms of the Schrödinger equations

$$\begin{aligned} i\frac{\partial}{\partial t}C_{1k}(t) &= \frac{t}{\tau}C_{1k}(t) + \Delta(k)C_{2k}(t) \\ i\frac{\partial}{\partial t}C_{2k}(t) &= -\frac{t}{\tau}C_{2,k}(t) + \Delta^*(k)C_{1k}(t). \end{aligned} \qquad (4)$$

The above equations describes two-time dependent levels $\pm\sqrt{(t/\tau)^2 + \Delta^2}$ approaching each other and there is an avoided level crossing at $t = 0$. The non-adiabatic transition probability can be calculated using the Landau-Zener transition formula [32] for each Fourier mode k given by $p_k = \exp(-\pi|\Delta|^2\tau)$. We therefore obtain the density of defects integrating over the modes $n = \int p_k d^2k$. The non-adiabatic transition becomes prominent only in the vicinity of the Dirac point where the

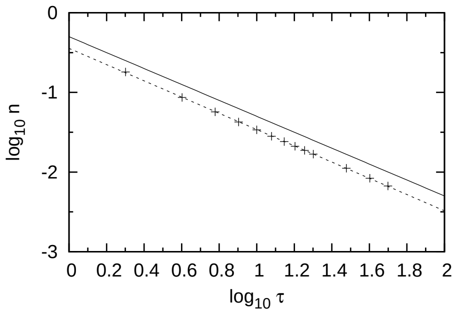

FIG. 2: The density of defect generated due to quenching through a two-dimensional Dirac point, as in Graphene, obtained through the numerical integration of the Schrödinger equations 4. The data clearly shows a $1/\tau$-decay of the defect density which is indicated with a straight line of slope -1.

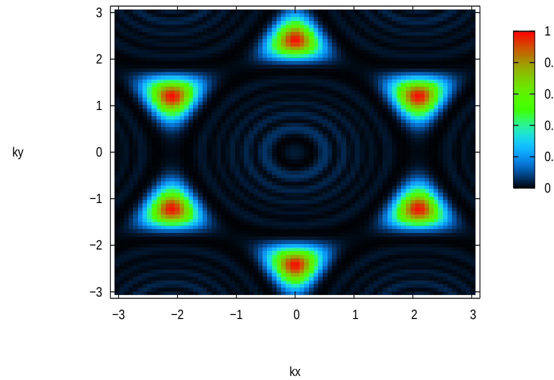

FIG. 3: Momentum distribution of particles created in the upper band is shown in an extended Brillouin zone. The centers of the six bright regions form the corners of the first Brillouin Zone and house the two inequivalent Dirac points.

gap closes, we therefore use the form of Δ valid close to the Dirac point given in Eq. (6) and extend the limits of integration from $-\infty$ to $+\infty$. The defect density therefore scales as

$$n = \int_{-\infty}^{+\infty} dq_x dq_y \exp(-\pi\tau v_f^2(q_x^2 + q_y^2)) \sim \frac{1}{\tau} \tag{5}$$

Noting that the spectrum close to the Dirac point is of the form $\pm\sqrt{(\mu^2 + v_f^2|q|^2)}$, we find that the minimum gap vanishes as μ yielding the critical exponents $\nu z = 1$. On the other hand, the linear dispersion at the Dirac point ($\mu = 0$) yield the dynamical exponent $z = 1$. The $1/\tau$-scaling of the defect density therefore is in complete agreement with the Kibble-Zurek prediction $n \sim 1/\tau^{\nu d/\nu z+1}$. The defect density obtained via numerical integration is shown in Fig. 2.

In the optical lattices, the defect density can be measured through the total occupation of the upper band, or in more detail, by the momentum distribution function, $n(k)$. In our two-band Graphene system, $n(k)$ in the lower-band would develop a hole near the Dirac points, whose size would grow with the rate of quenching. Conversely, in the upper-band the particles will be concentrated close to the Dirac points. The resulting momentum distribution would be highly unlike a thermal broadening. A picture of the momentum distribution function in the upper band is shown in an extended Brillouin zone in Fig. 3.

We shall now concentrate on the quenching through the semi-Dirac point which provides a more general situation of the quenching dynamics. The Hamiltonian close to the semi-Dirac point with a mass term can be put in the form [27]

$$\tilde{H_k} = \begin{bmatrix} \mu & iv_f|q_y| + q_x^2/2m \\ -iv_f|q_y| + q_x^2/2m & -\mu \end{bmatrix} \tag{6}$$

If the mass term μ is quenched as t/τ and the system crosses the semi-Dirac point, the defect density scales as

$$n = \int_{-\infty}^{\infty} dq_x dq_y \exp(-\pi\tau(v_f^2 q_y^2 + \frac{q_x^4}{(2m)^2})) \sim \frac{1}{\tau^{3/4}} \tag{7}$$

The corresponding numerical result is shown in Fig. 2. Generalizing to three-dimesnions [29], where the off-diagonal term gets modified to $\Delta_q = iv_f|q_{||}| + q_\perp^2/2m$ with $\vec{q}_\perp^2 = q_x^2 + q_y^2$, the scaling of the defect density is given by

$$n = \int_{-\infty}^{\infty} dq_{||} d\vec{q}_\perp \exp(-\pi\tau(v_f^2 q_{||}^2 + \frac{|q_\perp|^4}{(2m)^2})) \sim \frac{1}{\tau} \tag{8}$$

The scaling relations derived in Eqs. (7) and (8) lead to a very interesting generic scaling relation of the defect density for quenching through a (d, m) anisotropic quantum critical point given by the modified KZ form

$$n \sim \tau^{-\left(\frac{m\nu_{||}}{\nu_{||}z_{||}+1} + \frac{(d-m)\nu_\perp}{\nu_\perp z_\perp+1)}\right)}, \tag{9}$$

where the critical exponents $(\nu_{||}, z_{||})$ and $(\nu_\perp, z_\perp)$ are conjugate to parallel and perpendicular directions, respectively. The case with $m = 0$ refers to the quenching through a band-crossing point with quadratic spectrum in d-dimensions while $m = d$ is the result for quenching through a Dirac point.

We shall generalize the modified KZ scaling (9) to the case of non-linear variation of the mass parameter $\mu = |t/\tau|^\alpha sgn(t)$. The situation is simpler as the spectrum is gapless for $\mu = 0$ (i.e., $t = 0$) [14]. Following refs. [14, 15], it is easy to show that that the probablity of excitation for the mode k, $p_k = |\tilde{C}_{1k}(t \to \infty)|^2$ must be a function of $|\Delta(k)|^2\tau^{2\alpha/\alpha+1}$ where the initial condition $|\tilde{C}_{1k}(t \to -\infty)|^2 = 1$. Expanding around the semi-Dirac

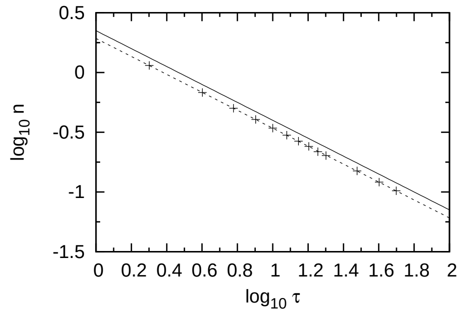

FIG. 4: The density of defect generated due to quenching through a two-dimensional semi-Dirac point obtained through the numerical integration. The $\tau^{-3/4}$-decay of the defect density is indicated with a straight line of slope -3/4.

point and extending the limits of integration form $-\infty$ to $+\infty$, we get

$$n \sim \tau^{-\left(\frac{m\alpha\nu_{||}}{\nu_{||}z_{||}+1}+\frac{(d-m)\alpha\nu_{\perp}}{\nu_{\perp}z_{\perp}+1)}\right)}, \tag{10}$$

which reduces to the form (9) for the linear case $\alpha = 1$.

In conclusion, we have proposed that Kibble-Zurek scaling can be studied in optical lattices of trapped fermions by creating geometries that have Dirac, semi-Dirac or Quadratic Band Crossings. These systems are typically interpenetrating lattices, which can be controlled independently. Thus, one can create a staggered on-site potential that leads to a gap in the spectra. As this staggered potential is varied through zero, one crosses a quantum critical point and even a slow quenching would lead to the creation of a certain density of defects. We have calculated the density of such defects for the gapped Graphene Hamiltonian and have generalized the Kibble-Zurek results to anisotropic Quantum Critical points that arise with semi-Dirac spectra.

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