

Rainbow Connection Number of Graph Power and Graph Products

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Abstract

Rainbow connection number, $rc(G)$, of a connected graph G is the minimum number of colors needed to color its edges so that every pair of vertices is connected by at least one path in which no two edges are colored the same (Note that the coloring need not be proper). In this paper we study the rainbow connection number with respect to three important *graph product* operations (namely *cartesian product*, *lexicographic product* and *strong product*) and the operation of taking the *power of a graph*. In this direction, we show that if G is a graph obtained by applying any of the operations mentioned above on non-trivial graphs, then $rc(G) \leq 2r(G) + c$, where $r(G)$ denotes the radius of G and $c \in \{0, 1, 2\}$. In general the rainbow connection number of a bridgeless graph can be as high as the square of its radius [1]. This is an attempt to identify some graph classes which have rainbow connection number very close to the obvious lower bound of *diameter* (and thus the *radius*). The bounds reported are tight upto additive constants. The proofs are constructive and hence yield polynomial time $(2 + \frac{2}{r(G)})$ -factor approximation algorithms.

Keywords: Graph Products, Graph Power, Rainbow Coloring.

1 Introduction

Edge colouring of a graph is a function from its edge set to the set of natural numbers. A path in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path*. An edge coloured graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a colouring is called a *rainbow colouring* of the graph. The minimum number of colours required to rainbow colour a connected graph is called its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [7].

The concept of rainbow colouring was introduced in [6]. It was shown in [3] that computing the rainbow connection number of a graph is NP-Hard. To rainbow colour a graph, it is enough to ensure that every edge of some spanning tree in the graph gets a distinct colour. Hence order of the graph minus one is an upper bound for rainbow connection number. Many authors view rainbow connectivity as one ‘quantifiable’ way of strengthening the connectivity property of a graph [2, 3, 12]. Hence tighter upper bounds on rainbow connection number for graphs with higher connectivity have been a subject of investigation. The following are the results in this direction reported in literature: Let G be a graph of order n . If G is 2-edge-connected (bridgeless), then $rc(G) \leq 4n/5 - 1$ and if G is 2-vertex-connected, then $rc(G) \leq \min\{2n/3, n/2 + O(\sqrt{n})\}$ [2]. This was very recently improved in [5], where it was shown that if G is 2-vertex-connected, then $rc(G) \leq \lceil n/2 \rceil$, which is the best possible upper bound for the case. It also improved the previous best known upper bound for 3-vertex connected graphs of $3(n+1)/5$ [14]. It was shown in [12] that $rc(G) \leq 20n/\delta$ where δ is the minimum degree of G . The result was improved in [4] where it was shown that $rc(G) \leq 3n/(\delta+1) + 3$. Hence it follows that $rc(G) \leq 3n/(\lambda+1) + 3$ if G is λ -edge-connected and $rc(G) \leq 3n/(\kappa+1) + 3$ if G is κ -vertex-connected. It was shown in [5] that the above bound in terms of edge connectivity is tight up to additive constants and that the bound in terms of vertex connectivity can be improved to $(2 + \epsilon)n/\kappa + 23/\epsilon^2$, for any $\epsilon > 0$.

Many, but not all, of the above bounds are increasing functions of n . Since diameter, and hence radius, are lower bounds for rainbow connection number, any upper bound which is a function of one of the lower bounds alone is of great interest.

Apart from the structural insights that it gives to the problem, it can also have applications in the design and analysis of approximation algorithms for rainbow colouring, which is known to be an NP-hard problem [3]. For a general graph, the rainbow connection number cannot be upper bounded by a function of radius or diameter alone. For instance, the star $K_{1,n}$ has a radius 1 but rainbow connection number n . Still, the question of whether such an upper bound exists for special graph classes remain.

A very general result in this direction is the one by Basavaraju et al. [1] which says that for every bridgeless graph of radius r , the rainbow connection number is upper bounded by $r(r + 2)$. They also demonstrate that the above bound, which is quadratic in the radius, is tight not just for bridgeless graphs but also for graphs of any higher connectivity. This result was extended to graphs with bridges in [8]. This throws open a few interesting questions. Which classes of graphs have upper bounds on rainbow connection number which is (1) constant factor of radius, (2) additive factor above radius, etc. It is evident that answers to these questions will help in the design and analysis of constant factor and additive factor approximation algorithms for the problem. Moreover, they can give hints to answering the still open question of characterising graphs for which the rainbow connection number is equal to the diameter. Such additive factor upper bounds were demonstrated for unit interval, interval, AT-free, circular arc, threshold and chain graphs in [4]. Basavaraju et. al [1] also showed that rainbow connection number will have a constant factor upper bound on bridgeless graphs in which the size of a maximum induced cycle (chordality) is bounded independently of radius.

In this paper, we demonstrate a large class of graphs for which the rainbow connection number is upper bounded by a linear function of its radius. We study the rainbow connection number with respect to three important *graph product* operations (namely *cartesian product*, *lexicographic product* and *strong product*) and the operation of taking the *power of a graph*. Specifically, we show that if G is a graph obtained by applying any of the operations mentioned above on non-trivial graphs, then $rc(G) \leq 2r(G) + c$, where $r(G)$ denotes the radius of G and $c \in \{0, 1, 2\}$. The bounds reported are either tight or tight upto additive constants. See Section 1.2 for the exact statements. The proofs are constructive and hence yield polynomial time $(2 + \frac{2}{r(G)})$ -factor approximation algorithms.

The rainbow connection number of some graph products has got recent attention [13, 9, 11]. One way to bound the rainbow connection number of a graph product is in terms of the rainbow connection number of the operand graphs. Such an approach was adopted by Li et al. [13] to study rainbow connection number with respect to Cartesian product and the strong product. In particular, they show that the rainbow connection number of the Cartesian product and hence the strong product of two connected graphs are upper bounded by the sum of the rainbow connection numbers of the operand graphs. Later, it was shown in [9] that the rainbow connection number of the strong product of two connected graphs is upper bounded by the larger of the rainbow connection numbers of the operand graphs. Most of the bounds mentioned above can be far from being tight when the rainbow connection number of the operand graphs is much higher than their radii. The importance of the bounds reported here is that they are independent of the rainbow connection number of the operand graphs and depends only on the radius of the resultant graph.

1.1 Preliminaries

The graphs considered in this paper are finite, simple and undirected. Given a graph G , $|G|$ denotes the number of vertices in the graph, also called the *order* of G . A *trivial* graph is a graph of order 0 or 1.

Given a graph G , a *walk* in G , from vertex u to vertex v is defined as a sequence of vertices (not necessarily distinct), starting at u and ending at v , say $(u = u_0), u_1, \dots, (u_k = v)$ such that $(u_i, u_{i+1}) \in E(G)$ for $0 \leq i \leq k - 1$. A *walk* in which all the vertices are distinct is called a *path*. The length of a path is the number of edges in that path. A single vertex is considered to be a path of length 0. The *distance* between two vertices u and v in G is the length of a shortest path between them and is denoted by $dist_G(u, v)$. Given two walks $W_1 = u_0, u_2, \dots, u_k$ and $W_2 = v_0, v_1, \dots, v_l$ such that $u_k = v_0$, we can *concatenate* W_1 and W_2 to get a longer walk, $W = W_1.W_2 = u_0, u_1, \dots, (u_k = v_0), v_1, v_2, \dots, v_l$.

Given a graph G , the *eccentricity* of a vertex, $v \in V(G)$ is given by $ecc(v) = \max\{dist_G(v, u) : u \in V(G)\}$. The *radius* of G is given by $r(G) = \min\{ecc(v) : v \in V(G)\}$ and the *diameter* of G is defined as $diam(G) = \max\{ecc(v) : v \in V(G)\}$. A *central vertex* of G is a vertex with eccentricity equal to the radius of G .

Definition 1 (The Cartesian Product). *Given two graphs G and H , the Cartesian product of G and H , denoted by $G \square H$, is defined as follows: $V(G \square H) = V(G) \times V(H)$. Two distinct vertices $[g_1, h_1]$ and $[g_2, h_2]$ of $G \square H$ are adjacent if and only if either $g_1 = g_2$ and $(h_1, h_2) \in E(H)$ or $h_1 = h_2$ and $(g_1, g_2) \in E(G)$.*

Definition 2 (The Lexicographic Product). *Given two graphs G and H , the lexicographic product of G and H , denoted by $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$. Two distinct vertices $[g_1, h_1]$ and $[g_2, h_2]$ of $G \circ H$ are adjacent if and only if either $(g_1, g_2) \in E(G)$ or $g_1 = g_2$ and $(h_1, h_2) \in E(H)$.*

Definition 3 (The Strong Product). *Given two graphs G and H , the strong product of G and H , denoted by $G \boxtimes H$, is defined as follows: $V(G \boxtimes H) = V(G) \times V(H)$. Two distinct vertices $[g_1, h_1]$ and $[g_2, h_2]$ of $G \boxtimes H$ are adjacent if and only if one of the three conditions hold:*

1. $g_1 = g_2$ and $(h_1, h_2) \in E(H)$ or
2. $h_1 = h_2$ and $(g_1, g_2) \in E(G)$ or
3. $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$.

It is easy to see from the definitions of the products above that if $G = K_1$ (respectively $H = K_1$) then the resultant graph is isomorphic to H (respectively G). The above graph products are extensively studied in graph theory. See [10] for a comprehensive treatment of the topic.

Definition 4 (Power of a graph). *The k -th Power of a graph, denoted by G^k where $k \geq 1$, is defined as follows: $V(G^k) = V(G)$. Two vertices u and v are adjacent in $V(G^k)$ if and only if the distance between vertices u and v in G , i.e., $dist_G(u, v) \leq k$.*

Given a graph G , another graph G' is called a *spanning subgraph* of G if G' is a subgraph of G and $V(G') = V(G)$. A vertex v is called *universal* if it is adjacent to all the other vertices in the graph.

Given a tree T , the unique path between any two vertices, u and v in T is denoted by $P_T(u, v)$. It is sometimes convenient to consider some vertex from the tree as special; such a vertex is then called the *root* of this tree. A tree with a fixed root is called a *rooted tree*.

Let T be a *rooted tree* with root $root(T) = v_0$. The *level number* of any vertex $v \in T$ is given by $\ell_T(v) = dist_T(v, v_0)$. If the tree in context is clear then we simply use $\ell(v)$. The *depth* of T is given by $d(T) = \max\{\ell(v) : v \in V(T)\}$. Given two vertices $u, v \in V(T)$, u is called an *ancestor* of v if $u \in P_T(v, v_0)$. It is easy to see that $\ell(v) \geq \ell(u)$. If u is an ancestor of v and $\ell(v) = \ell(u) + 1$ then u is called the *parent* of v and is denoted by $parent(v)$.

Definition 5 (Layer-wise Coloring of a Rooted Tree). *Given a rooted tree T and an ordered multi-set of colors $C = \{c_i : 1 \leq i \leq n\}$ where $n \geq d(T)$, we define the edge coloring, $f_{T,C} : E(T) \rightarrow C$ as $f_{T,C}((u, v)) = c_i$ where $i = \max\{\ell(u), \ell(v)\}$. We refer to $f_{T,C}$ as the Layer-wise Coloring of T that uses colors from the set C .*

Given an edge coloring f of a graph G using colors from the set C , let $C' \subseteq C$. Consider a path in G that is rainbow colored with respect to f . We call this path a *C' -Rainbow-Path* if every edge of the path is colored only from the set C' .

Observation 1. *Let T be a rooted tree and $C = \{c_1, c_2, \dots, c_n\}$ be an ordered set of colors such $c_i \neq c_j$ for $i \neq j$ and $n \geq d(T)$. Let $f_{T,C}$ be the Layer-wise Coloring of T using colors from C . If $u, v \in V(T)$ such that u is an ancestor of v in T , then $P_T(v, u)$ is a C -Rainbow-Path with respect to the coloring $f_{T,C}$. In particular $P_T(v, u)$ is a $\{c_{\ell(u)+1}, c_{\ell(u)+2}, \dots, c_{\ell(v)}\}$ -Rainbow-Path with respect to $f_{T,C}$.*

Recall the definition of the Cartesian Product of two graphs G and H , denoted by $G \square H$. We define a decomposition of $G \square H$ into edge disjoint subgraphs as follows:

Definition 6 ((G,H)-Decomposition of $G \square H$). *Given graphs G and H with vertex sets $V(G) = \{g_i : 0 \leq i \leq |G| - 1\}$ and $V(H) = \{h_i : 0 \leq i \leq |H| - 1\}$ respectively. We define a decomposition of $G \square H$ as follows: For $0 \leq j \leq |H| - 1$, define induced subgraphs, G_j , with vertex sets, $V(G_j) = \{[g_i, h_j] : 0 \leq i \leq |G| - 1\}$. Similarly, for $0 \leq i \leq |G| - 1$, define induced subgraphs, H_i , with vertex sets, $V(H_i) = \{[g_i, h_j] : 0 \leq j \leq |H| - 1\}$. Then we have the following:*

1. For $0 \leq j \leq |H| - 1$, G_j is isomorphic to G and for $0 \leq i \leq |G| - 1$, H_i is isomorphic to H .
2. For $0 \leq i < j \leq |H| - 1$, $V(G_i) \cap V(G_j) = \emptyset$ and hence $E(G_i) \cap E(G_j) = \emptyset$.
3. For $0 \leq k < l \leq |G| - 1$, $V(H_k) \cap V(H_l) = \emptyset$ and hence $E(H_k) \cap E(H_l) = \emptyset$.
4. For $0 \leq j \leq |H| - 1$ and $0 \leq i \leq |G| - 1$, $V(G_j) \cap V(H_i) = [g_i, h_j]$ and $E(G_j) \cap E(H_i) = \emptyset$.

We call $G_1, G_2, \dots, G_{|H|-1}, H_1, H_2, \dots, H_{|G|-1}$ as the **(G,H)-Decomposition** of $G \square H$.

1.2 Our Results

1. If G is a connected graph then $r(G^k) \leq rc(G^k) \leq 2r(G^k) + 1$ for all $k \geq 2$. The upper bound is tight up to an additive constant of 1. Note that $r(G^k) = \left\lceil \frac{r(G)}{k} \right\rceil$. [See Theorem 1, Section 2]
2. If G and H are two connected, non-trivial graphs then $r(G \square H) \leq rc(G \square H) \leq 2r(G \square H)$. The bounds are tight. Note that $r(G \square H) = r(G) + r(H)$. [See Theorem 2, Section 3]
3. Given two non-trivial graphs G and H such that G is connected we have the following:
 - (a) If $r(G \circ H) \geq 2$ then $r(G \circ H) \leq rc(G \circ H) \leq 2r(G \circ H)$. This bound is tight.
 - (b) If $r(G \circ H) = 1$ then $1 \leq rc(G \circ H) \leq 3$. This bound is tight.

[See Theorem 3, Section 4]

4. If G and H are two connected, non-trivial graphs then $r(G \boxtimes H) \leq rc(G \boxtimes H) \leq 2r(G \boxtimes H) + 2$. The upper bound is tight up to an additive constant 2. Note that $r(G \boxtimes H) = \max\{r(G), r(H)\}$. [See Theorem 4, Section 5]

Most of the bounds available in literature for graph products are in terms of rainbow connection number of the operand graphs and hence can be far from being tight when the rainbow connection number of the operand graphs is much higher than their radii. It may happen that $rc(G)$ or $rc(H)$ are very large whereas $rc(G \square H)$, $rc(G \boxtimes H)$, etc. are very small in comparison. For example let $G = K_{1,n}$ and $H = K_2$ then by the result in [13], $rc(G \square H) \leq n + 1$ and by the result in [9], $rc(G \boxtimes H) \leq n$. But our results show that $rc(G \square H) \leq 4$ and $rc(G \boxtimes H) \leq 4$. This suggests that the rainbow connection number of product of graphs may be related to the radii of the operand graphs (and hence on the radius of the resultant graph) rather than on their rainbow connection numbers. The results reported here confirm that it is indeed the case. It may be noted that a similar case is true even for graph powers. That is, $rc(G^k)$ is independent of $rc(G)$ and is upper-bound by a linear function of $r(G^k) = \left\lceil \frac{r(G)}{k} \right\rceil$.

2 Rainbow Connection Number of the k-th Power of a Graph H

For $k \geq 1$, recall that the k -th power of a graph H , denoted by H^k , as follows: $V(H^k) = V(H)$ and any two vertices u and $v \in V(H^k)$ are adjacent if and only if $dist_H(u, v) \leq k$. It is easy to verify that $r(H^k) = \left\lceil \frac{r(H)}{k} \right\rceil$ and $diam(H^k) = \left\lceil \frac{diam(H)}{k} \right\rceil$.

Since $H^1 = H$, for the remainder of this section we assume that $k \geq 2$. Let T be the *BFS-Tree* rooted at some central vertex, say h_0 , of H . Then clearly the depth of tree T , $d(T) = r(H)$. Clearly T^k is a spanning subgraph of H^k and hence $rc(H^k) \leq rc(T^k)$. So in order to derive an upper bound for $rc(H^k)$ in terms of $r(H^k)$ it is enough to derive an upper bound for $rc(T^k)$ in terms of $\left\lceil \frac{d(T)}{k} \right\rceil$ ($r(H^k) = \left\lceil \frac{d(T)}{k} \right\rceil$).

Let $V(T) = \{h_i : 0 \leq i \leq |H| - 1\}$. For $0 \leq i \leq k - 1$, let $V_i = \{u \in V(T) : \ell_T(u) > 0 \text{ and } \ell_T(u) \equiv i \pmod{k}\}$. It is easy to see that $V = \bigsqcup_{i=0}^{k-1} V_i \sqcup \{h_0\}$.

For $0 \leq i \leq k - 1$ and $0 \leq j \leq \left\lceil \frac{d(T)}{k} \right\rceil$ we define $V_i^j = \{u \in V_i \cup \{h_0\} : \left\lceil \frac{\ell_T(u)}{k} \right\rceil = j\}$. Note that if $u \in V(T) \setminus \{h_0\}$ then u belongs to exactly one V_i^j where $0 \leq i \leq k - 1$ and $1 \leq j \leq \left\lceil \frac{d(T)}{k} \right\rceil$. For all $0 \leq i \leq k - 1$, vertex h_0 is the only vertex in V_i^0 .

Now we define a function, $par: V(T) \setminus \{h_0\} \rightarrow V(T)$ as follows: $\forall u \in V(T) \setminus \{h_0\}$, $par(u) = v$ such that if $u \in V_i^j$ then $v \in V_i^{j-1}$ and $(u, v) \in E(T^k)$. Such a vertex v always exists because of the following reasons: If $1 \leq \ell_T(u) \leq k$ then $u \in V_i^1$ for some $0 \leq i \leq k - 1$; we may choose v to be h_0 since $h_0 \in V_i^0$ and $(h_0, u) \in E(T^k)$. If $\ell_T(u) > k$ then we may choose v to be the ancestor of u in T such that $\ell_T(v) = \ell_T(u) - k$. Then clearly $v \in V_i^{j-1}$ and $(u, v) \in E(T^k)$.

For $0 \leq i \leq k - 1$, define graph G_i with vertex set, $V(G_i) = V_i \cup \{h_0\}$ and edge set, $E(G_i) = \{(u, par(u)) : u \in V_i\}$. Since every vertex in G_i has a path to h_0 , the only vertex in V_i^0 , G_i is connected. Moreover using the definition of

the function par , it is easy to verify that G_i does not contain any cycle. Hence G_i is a tree. For $0 \leq i \leq k-1$ let $root(G_i) = h_0$. For $i \neq j$ we have $V(G_i) \cap V(G_j) = \{h_0\}$, a singleton set and hence $E(G_i) \cap E(G_j) = \emptyset$.

We define an edge coloring, $f: E(T^k) \rightarrow A \uplus B \uplus \{c\}$ where $A = \{a_i: 1 \leq i \leq \lceil d(T)/k \rceil\}$, $B = \{b_i: 1 \leq i \leq \lceil d(T)/k \rceil\}$ and $\{c\}$ are ordered sets of colors. Since $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$, in order to define the edge coloring f it is sufficient to define an edge coloring of G_i , for $0 \leq i \leq k-1$ and an edge coloring of all the remaining edges of T^k , separately. For $0 \leq i \leq k-1$, if $i \equiv 0 \pmod{2}$ then we choose the *Layer-wise Coloring* $f_{G_i,A}$ to color the edges of G_i else we choose *Layer-wise Coloring* $f_{G_i,B}$ to color the edges of G_i . All the remaining edges of T^k are colored c .

Claim 1. *The edge coloring f is a rainbow coloring of T^k*

Proof. Let u and v be two distinct vertices of T^k . Without loss of generality let $u \neq h_0$. Then $u \in G_i$ where $0 \leq i \leq k-1$. By *Observation 1* there is an *A-Rainbow-Path* (*B-Rainbow-Path*) from u to h_0 if i is *even* (*odd*). Now we can assume that $u, v \neq h_0$. Let $u \in V(G_i)$ and $v \in V(G_j)$. To illustrate a rainbow path between u and v we consider the following two cases.

Case 1: [When $|i-j| \equiv 1 \pmod{2}$]

Without loss of generality let $i \equiv 0 \pmod{2}$ and $j \equiv 1 \pmod{2}$.

Let $Q_1 = P_{G_i}(u, h_0)$ and $Q_2 = P_{G_j}(h_0, v)$ be the *A* and *B-Rainbow-Paths* in G_i and G_j with respect to the *Layer-wise Colorings* $f_{G_i,A}$ and $f_{G_j,B}$ respectively (See *Observation 1*). It follows that Q_1 and Q_2 are *A* and *B-Rainbow-Paths* in T^k with respect to edge coloring f . Clearly $Q = Q_1, Q_2$ is a $(A \cup B)$ -*Rainbow-Path* from vertex u to vertex v .

Case 2: [When $|i-j| \equiv 0 \pmod{2}$]

Without loss of generality we may assume that $\ell_T(v) \geq \ell_T(u)$.

If $(u, v) \in E(T^k)$ then there is a trivial rainbow path between them. If $\ell_T(u_1) \leq 1$ and $\ell_T(u_2) \leq 1$ then $(u_1, u_2) \in E(T^k)$ (since $k \geq 2$). We consider the case when $(u, v) \notin E(T^k)$. This happens when the level number of one of the vertices is ≥ 2 i.e. $\ell_T(v) \geq 2$. Let $v_1 \in V(T^k)$ be the parent of v in T . Since $\ell_T(v) \geq 2$, $v_1 \neq h_0$. Let $v_1 \in G_l$ where $\ell_T(v_1) = \ell_T(v) - 1 \equiv l \pmod{k}$. From *Case 1* we know that there is a $(A \cup B)$ -*Rainbow-Path*, say P , between vertices u and v_1 since $|i-l| \equiv 1 \pmod{2}$. Edge (v, v_1) is colored c since $(v, v_1) \notin E(G_i)$ for any $0 \leq i \leq k-1$. Extending P by edge (v, v_1) we get the required rainbow path between vertices u and v .

We have thus proved that f is a rainbow coloring of T^k . □

Theorem 1. *If H is any connected, non-trivial graph then for all $k \geq 2$, $r(H^k) \leq rc(H^k) \leq 2r(H^k) + 1$.*

Proof. The edge coloring f uses $|A| + |B| + |\{c\}| = 2r(H^k) + 1$ colors. The upper bound follows from *Claim 1*. The lower bound is trivial. □

Tight Example:

Let H be a path on $2kr + 1$ vertices. It is easy to see that $rc(H^k) \geq diam(H^k) = 2r(H^k)$.

3 Rainbow Connection Number of the Cartesian Product of Two Non-trivial Graphs G' and H'

Recall that the Cartesian product, $G' \square H'$, of two graphs G' and H' is defined as follows: $V(G' \square H') = V(G') \times V(H')$. Two distinct vertices $[g_1, h_1]$ and $[g_2, h_2]$ of $G' \square H'$ are adjacent if **either** $g_1 = g_2$ and $(h_1, h_2) \in E(H')$ **or** $(g_1, g_2) \in E(G')$ and $h_1 = h_2$. It is easy to verify that $diam(G' \square H') = diam(G') + diam(H')$ and that $r(G' \square H') = r(G') + r(H')$. See [10] for proof.

Let G be the *Breadth-First-Search-Tree* (*BFS-Tree*) rooted at some central vertex, say g_0 , of G' . Similarly let H be the *BFS-Tree* rooted at some central vertex, say h_0 , of H' . We have that $d(G) = r(G')$ and $d(H) = r(H')$ where $d(G)$ and $d(H)$ are the depths of trees G and H respectively. Clearly $G \square H$ is a connected spanning subgraph of $G' \square H'$ and therefore $rc(G' \square H') \leq rc(G \square H)$. So in order to derive an upper bound for $rc(G' \square H')$ in terms of $r(G' \square H')$ it is sufficient to derive an upper bound for $rc(G \square H)$ in terms of $r(G' \square H')$.

Let $V(G) = \{g_0, g_1, \dots, g_{|G|-1}\}$ and $V(H) = \{h_0, h_1, \dots, h_{|H|-1}\}$. Let $G_1, \dots, G_{|H|-1}, H_1, \dots, H_{|G|-1}$ be the (G, H) -Decomposition of $G \square H$ as defined in Definition-6. For $0 \leq i \leq |H| - 1$ define $root(G_i) = [g_0, h_i]$ and for $0 \leq j \leq |G| - 1$ define $root(H_j) = [g_j, h_0]$.

Recall the following simple observations.

Observation 2. $V(G_i) \cap V(H_j) = \{[g_j, h_i]\}$, $V(G_i) \cap V(G_j) = \emptyset$ and $V(H_i) \cap V(H_j) = \emptyset$, for all $i \neq j$.

Observation 3. $E(G \square H) = \bigsqcup_{i=0}^{|H|-1} E(G_i) \bigsqcup_{j=0}^{|G|-1} E(H_j)$

We now define an edge coloring, $f: E(G \square H) \rightarrow A \uplus B \uplus C \uplus D$ where $A = \{a_i : 1 \leq i \leq d(G)\}$, $B = \{b_i : 1 \leq i \leq d(G)\}$, $C = \{c_i : 1 \leq i \leq d(H)\}$ and $D = \{d_i : 1 \leq i \leq d(H)\}$ are ordered sets of colors. In view of Observation-3 it is clear that in order to define the coloring f , it is sufficient to describe separately, an edge coloring for each G_i , $0 \leq i \leq |H| - 1$ and an edge coloring for each H_j , $0 \leq j \leq |G| - 1$. We choose Layer-wise Coloring $f_{G_0, A}$ to be the edge coloring of G_0 and $f_{G_i, B}$ to be the edge coloring of G_i for $1 \leq i \leq |H| - 1$. Similarly we choose Layer-wise Coloring $f_{H_0, C}$ to be the edge coloring of H_0 and $f_{H_i, D}$ to be the edge coloring of H_i for $1 \leq i \leq |G| - 1$.

Claim 2. The edge coloring, f , is a rainbow coloring of $G \square H$.

Proof. Let $u = [g_i, h_j]$ and $v = [g_k, h_l]$ be two distinct vertices of $G \square H$. We demonstrate a rainbow path between u and v , by considering the following cases:

Case 1: [At least one of the vertices belong to $V(G \square H) \setminus (V(G_0) \cup V(H_0))$]

Without loss of generality let $v \in V(G \square H) \setminus (V(G_0) \cup V(H_0))$ i.e. $l \neq 0$ and $k \neq 0$. We now consider the following two sub-cases.

Case 1.a: [Vertex $u \notin V(G_0)$, hence $j \neq 0$]

Vertex $v = [g_k, h_l] \in V(H_k)$ and $root(H_k) = [g_k, h_0]$. Let $Q_1 = P_{H_k}(v, [g_k, h_0])$, is a D -Rainbow-Path in H_k with respect to the coloring $f_{H_k, D}$, by observation 1. Similarly let $Q_2 = P_{G_0}([g_k, h_0], [g_0, h_0])$, $Q_3 = P_{H_0}([g_0, h_0], [g_0, h_j])$ and $Q_4 = P_{G_j}([g_0, h_j], [g_i, h_j])$ be A -, C - and B -Rainbow-Paths in G_0 , H_0 and G_j ($j \neq 0$) respectively. It follows that Q_1, Q_2, Q_3 and Q_4 are D -, A -, C - and B -Rainbow-Paths in $G \square H$ with respect to the coloring f . Clearly $Q = Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4$ is a rainbow walk from v to u in $G \square H$ that contains a rainbow path between them.

Case 1.b: [Vertex $u \in V(G_0)$, hence $u = [g_i, h_0]$]

Vertex $v \in V_{G_l}$, let $Q_1 = P_{G_l}(v, [g_0, h_l])$, is a B -Rainbow-Path in G_l with respect to edge coloring $f_{G_l, B}$, by Observation-1. Similarly let $Q_2 = P_{H_0}([g_0, h_l], [g_0, h_0])$ and $Q_3 = P_{G_0}([g_0, h_0], [g_i, h_0])$ be C - and A -Rainbow-Paths in H_0 and G_0 respectively. It follows that Q_1, Q_2 and Q_3 are B -, C - and A -Rainbow-Paths in $G \square H$ with respect to the coloring f . Clearly $Q = Q_1 \cdot Q_2 \cdot Q_3$ is a rainbow walk from v to u in $G \square H$ that contains a rainbow path between them.

Case 2: [Both the vertices belong to $V(G_0) \cup V(H_0)$]

Without loss of generality let $v \neq [g_0, h_0]$. We consider the following 3 sub-cases:

Case 2.a: [Both the vertices belong to $V(H_0)$, hence $u = [g_0, h_j]$ and $v = [g_0, h_l]$]

Vertex $v = [g_0, h_l] \in V(G_l)$. Let $v' = [g_{k'}, h_l]$ be another vertex in G_l such that $(v, v') \in E(G_l)$. The existence of v' is guaranteed since $G' \neq K_1$. Let $Q_1 = P_{G_l}(v, v')$ i.e. the single edge (v, v') is a B -Rainbow-Path in G_l with respect to the coloring $f_{G_l, B}$, noting that $l \neq 0$ by the assumption that $v \neq [g_0, h_0]$. Similarly let $Q_2 = P_{H_{k'}}(v', [g_{k'}, h_0])$, $Q_3 = P_{G_0}([g_{k'}, h_0], [g_0, h_0])$ and $Q_4 = P_{H_0}([g_0, h_0], [g_0, h_j])$ be D -, A - and C -Rainbow-Paths in $H_{k'}$, G_0 and H_0 respectively. It follows that Q_1, Q_2, Q_3 and Q_4 are B -, D -, A - and C -Rainbow-Paths in $G \square H$ with respect to coloring f . Clearly $Q = Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4$ is a rainbow walk from v to u in $G \square H$ that contains a rainbow path between them.

Case 2.b: [Both the vertices belong to $V(G_0)$, hence $u = [g_i, h_0]$ and $v = [g_k, h_0]$]

Vertex $v \in V(H_k)$. Let $v' = [g_{k'}, h_l]$ be another vertex in H_k such that $(v, v') \in E(H_k)$. The existence of v' is guaranteed since $H' \neq K_1$. Let $Q_1 = P_{H_k}(v, v')$ i.e. the single edge (v, v') is a D -Rainbow-Path in H_k with respect to the coloring $f_{H_k, D}$, noting that $l \neq 0$ by the assumption that $v \neq [g_0, h_0]$. Similarly let $Q_2 = P_{G_{l'}}(v', [g_0, h_l])$, $Q_3 = P_{H_0}([g_0, h_l], [g_0, h_0])$ and $Q_4 = P_{G_0}([g_0, h_0], [g_i, h_0])$ be B -, C - and A -Rainbow-Paths in $G_{l'}$, H_0 and G_0 respectively.

It follows that Q_1, Q_2, Q_3 and Q_4 are D -, B -, C - and A -Rainbow-Paths in $G \square H$ with respect to coloring f . Clearly $Q = Q_1 \cdot Q_2 \cdot Q_3 \cdot Q_4$ is a rainbow walk from v to u in $G \square H$ that contains a rainbow path between them.

Case 2.c: [One vertex belongs to $V(G_0)$ and the other to $V(H_0)$]

Without loss of generality let $u \in V(G_0)$, $v \in V(H_0)$ then $j = 0$ and $l = 0$. In view of *Cases 2.a* and *2.b* we can assume that $u, v \neq [g_0, h_0]$.

Let $Q_1 = P_{H_0}(v, [g_0, h_0])$ and $Q_2 = P_{G_0}([g_0, h_0], u)$ is a C - and A -Rainbow-Paths in H_0 and G_0 respectively. It follows that Q_1 and Q_2 are C - and A -Rainbow-Paths in $G \square H$ with respect to the coloring f . Clearly $Q = Q_1 \cdot Q_2$ is a rainbow walk from vertex v to vertex u in $G \square H$ that contains a rainbow path between them.

It follows that f is a rainbow coloring of $G \square H$. □

Theorem 2. *If G' and H' are two non-trivial, connected graphs then $r(G' \square H') \leq rc(G' \square H') \leq 2r(G' \square H')$*

Proof. The edge coloring f uses $|A| + |B| + |C| + |D| = 2(d(G) + d(H)) = 2(r(G') + r(H')) = 2r(G' \square H')$ number of colors. The upper bound follows from *Claim-2* and the lower bound is obvious. □

Tight Example:

Consider two graphs G_1 and G_2 such that $diam(G_1) = 2r(G_1)$ and $diam(G_2) = 2r(G_2)$. For example G_1 and G_2 may be taken as paths with odd number of vertices. Then $diam(G_1 \square G_2) = diam(G_1) + diam(G_2) = 2(r(G_1) + r(G_2))$.

4 Rainbow Connection Number of the Lexicographic Product of Two Non-trivial Graphs G' and H

Recall that the lexicographic product, $G' \circ H$, of two graphs G' and H is defined as follows: $V(G' \circ H) = V(G') \times V(H)$. Two distinct vertices $[g_1, h_1]$ and $[g_2, h_2]$ of $G' \circ H$ are adjacent if either $(g_1, g_2) \in E(G')$ or $g_1 = g_2$ and $(h_1, h_2) \in E(H)$. Note that unlike the *Cartesian Product* and the *Strong Product*, the *Lexicographic Product* is a non-commutative product. Thus $G' \circ H$ need not be isomorphic to $H \circ G'$. Also note that if G' and H are non-trivial graphs then $r(G' \circ H) = 1$ if and only if $r(G') = 1$ and $r(H) = 1$.

Theorem 3. *Given two non-trivial graphs G' and H such that G' is connected we have the following:*

1. *If $r(G' \circ H) \geq 2$ then $r(G' \circ H) \leq rc(G' \circ H) \leq 2r(G' \circ H)$. This bound is tight.*
2. *If $r(G' \circ H) = 1$ then $1 \leq rc(G' \circ H) \leq 3$. This bound is tight.*

Part 1: $r(G' \circ H) \geq 2$

Since $r(G' \circ H) \geq 2$, either $r(G') \geq 2$ or $r(H) \geq 2$. In either case it can be shown that $r(G' \circ H) \geq r(G')$. Let G be the *BFS-Tree* rooted at some central vertex, say g_0 , of graph G' . It is easy to see that the depth of G , $d(G) = r(G')$. Since $G \circ H$ is a connected spanning subgraph of $G' \circ H$, $rc(G' \circ H) \leq rc(G \circ H)$. In order to derive an upper bound for $rc(G' \circ H)$ in terms of $r(G' \circ H)$ it is sufficient to derive an upper bound for $rc(G \circ H)$ in terms of $r(G' \circ H)$.

Let $V(G) = \{g_i: 0 \leq i \leq |G| - 1\}$ and $V(H) = \{h_i: 0 \leq i \leq |H| - 1\}$. Since G is connected and non-trivial, vertex g_0 has at least one neighbor. We label this neighbor as g_1 i.e. $(g_0, g_1) \in E(G)$. Since H is a non-trivial graph, there are at least two vertices in $H - h_0$ and h_1 . Note that (h_0, h_1) need not be an edge in H . It is easy to see that $G \square H$ is a spanning subgraph of $G \circ H$.

It is easy to see that $G \square H$ is a spanning subgraph of $G \circ H$. Let $G_0, G_1, \dots, G_{|H|-1}, H_0, H_1, \dots, H_{|G|-1}$ be the (G, H) -Decomposition of the subgraph of $G \circ H$ that is isomorphic to $G \square H$ (See *Definition 6*). Recall that every G_i is isomorphic to G and every H_j is isomorphic to H . We define $root(G_i) = [g_0, h_i]$ and $root(H_j) = [g_j, h_0]$. From *Observation 2* we know that any vertex $[g_i, h_j]$ belongs to both G_j and H_i .

Special note on notation:

In the rest of this section for any vertex $v = [g_i, h_j] \in V(G_j)$, we abuse the notation and simply use $\ell(v) / \ell([g_i, h_j])$ instead $\ell_{G_j}(v) / \ell_{G_j}([g_i, h_j])$. Note that $\ell_H(v)$ need not make sense as H need not be a tree.

Definition 7. Let $E_1 = \uplus_{i=0}^{|H|-1} E(G_i) \uplus_{j=0}^{|G|-1} E(H_j)$ and $E_2 = E(G \circ H) \setminus E_1$.

We now define an edge coloring, $f : E(G \circ H) \rightarrow A \uplus B$ where $A = \{a_i : 1 \leq i \leq r(G' \circ H)\}$ and $B = \{b_i : 1 \leq i \leq r(G' \circ H)\}$ are ordered sets of colors. Since $r(G' \circ H) \geq 2$, both the sets A and B are of cardinality at least 2. Since $E(G \circ H) = E_1 \uplus E_2$, it is enough to define separately a coloring for E_1 and a coloring for E_2 .

Coloring the edges of E_1 :

To define a coloring of E_1 it is enough to define an edge colorings for each G_i , $0 \leq i \leq |H| - 1$ and an edge coloring for each H_j , $0 \leq j \leq |G| - 1$. We choose the *Layer-wise Coloring*, $f_{G_0, A}$ (as defined in *Definition 5*) to color the edges of G_0 .

We define a new ordered set, $B' = \{b'_i : 1 \leq i \leq r(G' \circ H)\}$ where $b'_1 = a_{r(G' \circ H)} \in A$ and for $2 \leq i \leq r(G' \circ H)$, $b'_i = b_i \in B$. For $1 \leq i \leq |H| - 1$, we choose the *Layer-wise Coloring* $f_{G_i, B'}$ to be the edge coloring of G_i . For $0 \leq j \leq |G| - 1$, we color all the edges of H_j using the color b_1 .

Coloring the edges of E_2 :

For any vertex $v \in V(G \circ H)$ let $\mathcal{E}(v)$ be the set of edges from E_2 that are incident on v . We partition $\mathcal{E}(v)$ into two sets $\mathcal{E}_L(v)$ and $\mathcal{E}_U(v)$. Consider some edge $(v, u) \in \mathcal{E}(v)$, then $(v, u) \in \mathcal{E}_L(v)$ if and only if $\ell(u) > \ell(v)$ and $(v, u) \in \mathcal{E}_U(v)$ if and only if $\ell(u) < \ell(v)$. For two vertices v_1 and $v_2 \in V(G \circ H)$ we have that $(v_1, v_2) \in \mathcal{E}_L(v_1)$ if and only if $(v_1, v_2) \in \mathcal{E}_U(v_2)$.

To color the edges of E_2 we have the following set of rules:

Rule #1 : All the edges of $\mathcal{E}_L([g_0, h_0])$ are colored b_1 .

Rule #2 : For all $v \in V(G_0) \setminus [g_0, h_0]$, all the edges of $\mathcal{E}_L(v)$ are colored $a_{\ell(v)+1}$.

Rule #3 : All the edges of $\mathcal{E}_U([g_i, h_0])$, where $\ell([g_i, h_0]) = 1$, are colored $b_{r(G' \circ H)}$.

Rule #4 : All the edges of $\mathcal{E}_L([g_0, h_1]) \setminus \{([g_0, h_1], [g_i, h_0]) : \ell([g_i, h_0]) = 1\}$ are colored $a_{r(G' \circ H)}$.

Rule #5 : For all $v \in V(G_1) \setminus [g_0, h_1]$, all the edges from $\mathcal{E}_L(v)$ are colored $b_{\ell(v)+1}$.

Rule #6 : All the edges of $\mathcal{E}_U([g_i, h_1]) \setminus \{([g_i, h_1], [g_0, h_0])\}$, where $\ell_G(g_i) = 1$, are colored $a_{r(G' \circ H)}$.

Rule #7 : All the remaining edges of E_2 are colored b_1 .

Claim 3. The coloring f is a rainbow coloring of the edges of $G \circ H$.

Proof. Let $u = [g_i, h_j]$ and $v = [g_k, h_l]$ be two distinct vertices of $G \circ H$ such that $\ell(v) \geq \ell(u)$. We demonstrate a rainbow path between them by considering the following cases.

Case I: [When $\ell(v) \geq 2$]

First we make the following 3 observations.

(a): There exists an *A-Rainbow-Path* from v to the vertex $[g_0, h_0]$:

If $v \in V(G_0)$, then the path $P_{G_0}(v, [g_0, h_0])$ is an *A-Rainbow-Path* in G_0 with respect to the edge coloring $f_{G_0, A}$ (See *Observation 1*). If $v \notin V(G_0)$, then $\exists v_1 \in V(G_0)$ such that $\ell(v_1) \geq 1$, $\ell(v_1) = \ell(v) - 1$ and $(v_1, v) \in \mathcal{E}_L(v_1)$. Such a vertex always exists since we have assumed that $\ell(v) \geq 2$; G, H are non-trivial graphs and G is connected. Since $v_1 \in V(G_0)$ there is an *A-Rainbow-Path* from v_1 to $[g_0, h_0]$ as explained earlier, let this path be P . Specifically P is a $\{a_1, a_2, \dots, a_{\ell(v_1)}\}$ -*Rainbow-Path*. Since edge (v_1, v) is colored $a_{\ell(v_1)+1}$ by *Rule #2*, we can extend path P by (v_1, v) to get the required *A-Rainbow-Path* from v to $[g_0, h_0]$.

(b): There exists a *B-Rainbow-Path* from v to the vertex $[g_0, h_0]$:

If $v \in V(G_1)$ then there exists an ancestor of v , say v_2 , in G_1 such that $\ell(v_2) = 1$. The path $P_1 = P_{G_1}(v, v_2)$ is a $\{b_{\ell(v)}, b_{\ell(v)-1}, \dots, b_2\}$ -*Rainbow-Path* from v to v_2 with respect to the edge coloring $f_{G_1, B'}$. The edge $(v_2, [g_0, h_0])$ is colored b_1 by *Rule #1*. We can extend P_1 by edge $(v_2, [g_0, h_0])$ to get the required *B-Rainbow-Path* from vertex v to $[g_0, h_0]$. If $v \notin V(G_1)$, then there exists $v_3 = [g_{i'}, h_1] \in V(G_1)$ such that $(v, v_3) \in \mathcal{E}_L(v_3)$. Since $v_3 \in V(G_1)$ as explained earlier there is a $\{b_{\ell(v)}, b_{\ell(v)-1}, \dots, b_2, b_1\}$ -*Rainbow-Path*, say P_2 , from v_3 to $[g_0, h_0]$. Since the edge (v_3, v) is colored $b_{\ell(v_3)+1}$ by *Rule #5*, we can extend P_2 by (v_3, v) to get the required *B-Rainbow-Path* from v to $[g_0, h_0]$.

(c): There exist both $\{b_{\ell(v)}, b_{\ell(v)-1}, \dots, b_2, a_{r(G' \circ H)}\}$ and

$\{a_{\ell(v)}, a_{\ell(v)-1}, \dots, a_2, b_{r(G' \circ H)}\}$ -*Rainbow-Paths* from v to any vertex in $V(H_0) \setminus \{[g_0, h_0]\}$:

Recall that $\ell(v) \geq 2$. From observation (a) it can be inferred that there is a $\{b_{\ell(v)}, b_{\ell(v)-1}, \dots, b_2\}$ -*Rainbow-Path* from v to some vertex $v_4 \in V(G_1)$ such that $\ell(v_4) = 1$. For any $v_5 \in V(H_0) \setminus [g_0, h_0]$, the edge (v_4, v_5) is colored $a_{r(G' \circ H)}$ by *Rule #6* or by the *Layer-wise Coloring* $f_{G_1, B'}$ (whatever is applicable). This implies that there is a $\{b_{\ell(v)}, b_{\ell(v)-1}, \dots, b_2, a_{r(G' \circ H)}\}$ -*Rainbow-Path* from vertex v to any vertex in $V(H_0) \setminus \{[g_0, h_0]\}$.

Similarly from observation (b) it can be inferred that there is a $\{a_{\ell(v)}, a_{\ell(v)-1}, \dots, a_2\}$ -*Rainbow-Path* from vertex v to some vertex $v_6 \in V(G_0)$ such that $\ell(v_6) = 1$. By *Rule #3* any vertex in $V(H_0) \setminus \{[g_0, h_0]\}$ is adjacent to v_6 and is colored $b_{r(G' \circ H)}$.

Now consider the different cases involving vertex u . If $\ell(u) \geq 2$ then from observations (a) and (b) it follows that u and v are rainbow connected. If $\ell(u) = 0$ then from observation (c) it follows that u and v are rainbow connected. Finally if $\ell(u) = 1$ then we know that $(u, [g_0, h_0]) \in E(G \circ H)$ and is colored either a_1 or b_1 . Since v has both an *A* and a *B-Rainbow-Path* to $[g_0, h_0]$. It follows that u and v are rainbow connected.

Case 2: [When $\ell(v) \leq 1$]

Without loss of generality we assume that vertex $u \neq [g_0, h_0]$.

Case 2.a: [When $\ell(v) \neq \ell(u)$]

Vertices u and v are connected by an edge which is a trivial *rainbow path* between them.

Case 2.b: [When $\ell(v) = \ell(u) = 0$, hence $u = [g_0, h_j]$ and $v = [g_0, h_i]$]

If $v = [g_0, h_0]$ then we claim that the two length path, $P = \{v = [g_0, h_0], [g_1, h_0], \{[g_0, h_j] = u\}\}$ is a rainbow path from v to u . The edges of P are colored $a_1, b_{r(G' \circ H)}$ in that order. To see this: edge $(v, [g_1, h_0]) \in E(G_0)$ and G_0 is edge colored using the *Layer-wise Coloring*, $f_{G_0, A}$. It follows that the edge is colored a_1 (See *Observation 1*). The edge $([g_1, h_0], u) \in \mathcal{E}_U([g_1, h_0])$ and is colored $b_{r(G' \circ H)}$ by *Rule #3*. Note that edge $(v, [g_1, h_0]) \in E(G \circ H)$ since G is non-trivial and it is assumed that edge $(g_0, g_1) \in E(G)$.

If $v \in V(H_0) \setminus \{[g_0, h_0]\}$ then we claim that the four length path, $P = \{u = [g_0, h_j], [g_1, h_0], [g_0, h_0], [g_1, h_1], \{[g_0, h_i] = v\}\}$ is a rainbow path from u to v . The edges of P are colored $b_{r(G' \circ H)}, a_1, b_1, a_{r(G' \circ H)}$ in that order. To see this: edge $(u, [g_1, h_0]) \in \mathcal{E}_U([g_1, h_0])$ and is colored $b_{r(G' \circ H)}$ by *Rule #3*; edge $([g_1, h_0], [g_0, h_0]) \in E(G_0)$ and is colored a_1 by the *Layer-wise Coloring* $f_{G_0, A}$; edge $([g_0, h_0], [g_1, h_1]) \in \mathcal{E}_L([g_0, h_0])$ and is colored b_1 by *Rule #1*; finally edge $([g_1, h_1], v)$ is colored $a_{r(G' \circ H)}$ by one of the two applicable rules: (a): Edge $([g_1, h_1], v) \in E(G_1)$ and G_1 is edge colored using the *Layer-wise Coloring* $f_{G_1, B'}$ or (b): Edge $([g_1, h_1], v) \in \mathcal{E}_U([g_1, h_1]) \setminus \{([g_0, h_0], [g_1, h_1])\}$ and is colored $a_{r(G' \circ H)}$ by *Rule #4*.

Case 2.c: [When $\ell(v) = \ell(u) = 1$]

If exactly one of the vertices is in G_0 . Without loss of generality let $u \in V(G_0)$ and $v \notin V(G_0)$ then $u = [g_i, h_0]$ and $v = [g_k, h_{i \neq 0}]$. We claim that the two length path $P = \{u = [g_i, h_0], [g_0, h_0], \{[g_k, h_i] = v\}\}$ is a rainbow path from vertex u to vertex v . The edges of P are colored a_1, b_1 in that order.

If $u, v \in V(G_0)$ then $u = [g_i, h_0]$ and $v = [g_k, h_0]$. We claim that the four length path $P = \{u = [g_i, h_0], [g_0, h_0], [g_1, h_1], [g_0, h_1], \{v = [g_k, h_0]\}\}$ is a rainbow path from vertex u to vertex v . The edges are colored $a_1, b_1, a_{r(G' \circ H)}, b_{r(G' \circ H)}$ in that order.

If $u, v \notin V(G_0)$ then $u = [g_i, h_{j \neq 0}]$ and $v = [g_k, h_{i \neq 0}]$. We claim that the four length path $P = \{u = [g_i, h_j], [g_0, h_0], [g_1, h_0], [g_0, h_1], \{v = [g_k, h_i]\}\}$ is a rainbow path from u to v . The edges of P are colored $b_1, a_1, b_{r(G' \circ H)}, a_{r(G' \circ H)}$ in that order.

We have thus proved that f is a rainbow coloring of $G \circ H$. Since f uses $2r(G \circ H)$ colors, we have $rc(G \circ H) \leq 2r(G \circ H)$. Since it is assumed that $r(G \circ H) \geq 2$ we have proved the upper-bound in *Part 1* of *Theorem 3*. \square

Tight Example:

Let G be a connected graph such that $r(G) \geq 2$ and $diam(G) = 2r(G)$; let H be any non-trivial graph. It is easy to see that $diam(G \circ H) = diam(G)$ and $r(G \circ H) = r(G)$. Hence we can conclude that $diam(G \circ H) = 2r(G \circ H)$. We know that $rc(G \circ H) \geq diam(G \circ H)$ and $rc(G \circ H) \leq 2r(G \circ H)$ (*Part 1* from *Theorem 3*). It follows that $rc(G \circ H) = 2r(G \circ H)$.

Part 2: $r(G' \circ H) = 1$

We know that if $r(G' \circ H) = 1$ then $r(G') = r(G) = 1$ and $r(H) = 1$.

Claim 4. *If G' and H are two non-trivial graphs such that $r(G' \circ H) = 1$ then $rc(G' \circ H) \leq 3$.*

Proof. Since $r(G' \circ H) = 1$ there exists an universal vertex, say $u \in V(G' \circ H)$. It is easy to verify that $G' \circ H$ is 2 vertex connected. Now consider the following theorem:

Theorem Chandran et al.[4]: *If D is a connected two-way dominating set in a graph G , then $rc(G) \leq rc(G[D]) + 3$.*

The proof and definitions involved are given in [4].

The universal vertex, u , is a trivial dominating set. Moreover since $G' \circ H$ is two vertex connected and consequently two edge connected, it follows that $\{u\}$ is a two-way dominating set in $G' \circ H$. As a result $rc(G' \circ H) \leq rc(\{u\}) + 3$. Since $rc(\{u\}) = 0$ we have $rc(G' \circ H) \leq 3$. We have thus proved the *claim* and the upper-bound in *Part 2* of *Theorem 3*. \square

Tight Example:

Consider two non-trivial graphs G and H such that $G = K_{1,n}$ (a *star graph*) where $n \geq 2^m + 1$ and H is a graph such that $r(H) = 1$ and $|H| = m$. We claim that $rc(G \circ H) = 3$.

Proof. We prove the claim by contradiction.

Let f be a rainbow coloring of $G \circ H$ using at most 2 colors, say a_1 and a_2 . Let $V(G) = \{g_0, g_1, \dots, g_n\}$ where g_0 is the central vertex of G . Similarly let $V(H) = \{h_0, h_1, \dots, h_{m-1}\}$. Let H_0 be the induced subgraph of $G \circ H$ with vertex set $V(H_0) = \{[g_0, h_i] : 0 \leq i \leq m-1\}$. Graph H_0 is isomorphic to H .

For $1 \leq i \leq n$ define the function $f_i : \{[g_i, h_0]\} \times V(H_0) \rightarrow \{a_1, a_2\}$ as $f_i([g_i, h_0], [g_0, h_j]) = f([g_i, h_0], [g_0, h_j])$. Each of the functions, f_i , are one among $2^{|H|}$ possible functions. Since $n > 2^{|H|}$, by *pigeon hole principle* there must exist some f_i and f_k such that $i \neq k$ and $f_i = f_k$. If so there is *no rainbow path* between the vertices $[g_i, h_0]$ and $[g_k, h_0]$ with respect to the edge coloring f . This is because any rainbow path with respect to f between the two vertices is of length 2. Now any two length path between the vertices is of the form $[g_i, h_0], v, [g_k, h_0]$ where v is the intermediate vertex. It is easy to see that $v \in V(H_0)$. We know that $f_i([g_i, h_0], v) = f_k([g_k, h_0], v) = f([g_i, h_0], v) = f([g_k, h_0], v)$ for all $v \in V(H_0)$. This is a contradiction. Hence f is not a *rainbow coloring* of $G \circ H$.

Therefore any rainbow coloring of $G \circ H$ uses at least 3 colors. It follows from *Claim 4* that $rc(G \circ H) = 3$. \square

Proof of Theorem 3: The upper bounds follow from *Claim 3* and *Claim 4*. The lower bounds are trivial.

5 Rainbow Connection Number of the Strong Product of Two Non-Trivial, Connected Graphs G' and H'

Recall that the *strong product* of two graphs G' and H' , denoted by $G' \boxtimes H'$, is defined as follows: $V(G' \boxtimes H') = V(G') \times V(H')$. The edge set of $G' \boxtimes H'$ consists of two types of edges. An edge $([g_1, h_1], [g_2, h_2])$ is *Type-1* if and only if **either** $g_1 = g_2$ and $(h_1, h_2) \in E(H')$ **or** $h_1 = h_2$ and $(g_1, g_2) \in E(G')$. The edge is of *Type-2* if and only if $(g_1, g_2) \in E(G')$ and $(h_1, h_2) \in E(H')$. Let $r_{max} = \max\{r(G'), r(H')\}$. It is easy to see that $r(G' \boxtimes H') = r_{max}$ and $diam(G' \boxtimes H') = \max\{diam(G'), diam(H')\}$. See [10] for proof.

We assume without loss of generality that $r(G') \geq r(H')$ as $G' \boxtimes H'$ is isomorphic to $H' \boxtimes G'$. Let G and H be *BFS-Trees* rooted at some central vertices, g_0 and h_0 respectively of G' and H' . It is easy to see that the depths of G and H are $d(G) = r(G')$ and $d(H) = r(H')$ respectively. Let $V(G) = \{g_i : 0 \leq i \leq |G| - 1\}$ and $V(H) = \{h_i : 0 \leq i \leq |H| - 1\}$. Since G and H are non-trivial connected trees there is atleast one neighbor for g_0 and h_0 in G and H respectively. In the remainder of the section we always let these vertices be g_1 and h_1 respectively. Therefore $(g_0, g_1) \in E(G)$ and $(h_0, h_1) \in E(H)$.

Let $L_w(G) = \{g_i \in V(G) : \ell_G(g_i) = w\}$ for $0 \leq w \leq d(G)$ and $L_x(H) = \{h_i \in V(H) : \ell_H(h_i) = x\}$ for $0 \leq x \leq d(H)$. We define $V_{w,x} = L_w(G) \times L_x(H)$ for $0 \leq w \leq d(G)$ and $0 \leq x \leq d(H)$.

Since $G \boxtimes H$ is a spanning subgraph of $G' \boxtimes H'$, $rc(G' \boxtimes H') \leq rc(G \boxtimes H)$. So in order to derive an upper bound for $rc(G' \boxtimes H')$ in terms of $r(G' \boxtimes H')$ it is enough to derive an upper bound for $rc(G \boxtimes H)$ in terms of $d(G) = r_{max} = r(G')$. Recall that we have assumed that $r(G') \geq r(H')$ and therefore $r(G' \boxtimes H') = r(G')$.

We define an edge coloring, $f : E(G \boxtimes H) \rightarrow A \uplus B \uplus \{c, d\}$ where $A = \{a_i : 1 \leq i \leq d(G)\}$ and $B = \{b_i : 1 \leq i \leq d(G)\}$ are ordered sets of colors; and c and d are colors that are not in $A \uplus B$. Since $E(G \boxtimes H)$ is the disjoint union of *Type-1* and *Type-2* edges, we can define the coloring for *Type-1* and *Type-2* edges separately.

Coloring the Type-1 edges

Note that if we restrict the edge set of $G \boxtimes H$ to *Type-1* edges alone then the subgraph thus obtained is isomorphic to $G \square H$, the Cartesian Product of G and H . Let $G_1, G_2, \dots, G_{|H|-1}, H_1, H_2, \dots, H_{|G|-1}$ be the $(G-H)$ -Decomposition of $G \square H$ (*Type-1* edges) as defined in *Definition 6*. For $0 \leq j \leq |H| - 1$, define $root(G_j) = [g_0, h_j]$ and for $0 \leq i \leq |G| - 1$, define $root(H_i) = [g_i, h_0]$

Recall that $A = \{a_i : 1 \leq i \leq d(G)\}$ and $B = \{b_i : 1 \leq i \leq d(G)\}$ are ordered sets of colors. We define several new ordered (multi) sets of colors by slightly modifying the sets A and B . First we define the ordered set, $A_0 = \{a_i^0 : 1 \leq i \leq d(G)\}$ where $a_1^0 = c$ and $a_i^0 = a_i \in A$ for $2 \leq i \leq d(G)$. Also for $1 \leq w \leq d(H)$, we define ordered multi-sets, $A_w = \{a_i^w : 1 \leq i \leq d(G)\}$ and $B_w = \{b_i^w : 1 \leq i \leq d(G)\}$ where $a_i^w = d$ and $b_i^w = d$ for $1 \leq i \leq w$ and $a_i^w = a_i \in A$ and $b_i^w = b_i \in B$ for $w + 1 \leq i \leq d(G)$.

Rules to colors the Type-1 edges:

T1-R1: We choose the *Layer-wise Coloring* $f_{H_0,A}$ to color the edges of H_0 .

T1-R2: For each H_i such that $\ell_G(g_i) = 1$, we choose the *Layer-wise coloring* $f_{H_i,B}$ to color the edges of H_i .

T1-R3: For each H_i such that $\ell_G(g_i) \geq 2$, we color all the edges of H_i using d .

T1-R4: For $0 \leq w \leq d(H)$ we choose f_{G_i,A_w} to color the edges of G_i if w is *even* and we choose f_{G_i,B_w} to color the edges of G_i if w is *odd*.

Coloring the Type-2 edges

Observation 4. If an edge $([g_i, h_j], [g_k, h_l]) \in E(G \boxtimes H)$ is of *Type-2* such that $[g_i, h_j] \in V_{w,x}$ and $[g_k, h_l] \in V_{y,z}$ then we have $|w - y| = 1$ and $|x - z| = 1$.

Proof. Since the edge $([g_i, h_j], [g_k, h_l])$ is of *Type-2*, edges (g_i, g_k) and (h_j, h_l) are edges of trees G and H respectively. Therefore $|w - y| = |\ell_G(g_i) - \ell_G(g_k)| = 1$ and $|x - z| = |\ell_H(h_j) - \ell_H(h_l)| = 1$. \square

Rules to colors the Type-2 edges:

T2-R1: Let $([g_i, h_j], [g_k, h_l]) \in E(G \boxtimes H)$ be an edge of *Type-2* such that $[g_i, h_j] \in V_{y,z}$ and $[g_k, h_l] \in V_{y+1,z+1}$, then define

$$f(([g_i, h_j], [g_k, h_l])) = \begin{cases} a_{z+1} & \text{if } |z - y| \text{ is even} \\ b_{z+1} & \text{if } |z - y| \text{ is odd} \end{cases}$$

Note that $z + 1 = \ell_H(h_l) \leq d(H) \leq d(G)$ and therefore a_{z+1} and b_{z+1} exist.

T2-R2: Let $([g_i, h_j], [g_k, h_l]) \in E(G \boxtimes H)$ such that $[g_i, h_j] \in V_{1,1}$ and $[g_k, h_l] \in V_{2,0}$ then we choose $f(([g_i, h_j], [g_k, h_l])) = a_2$.

Note that if $[g_k, h_l] \in V_{2,0}$ then $\ell_G(g_k) = 2$ and thus $d(G) \geq 2$ and a_2 exists.

T2-R3: All the remaining edges of *Type-2* are colored d .

A-Reachable and B-Reachable Vertices:

We define the following 2 concepts with respect to the edge coloring f . We define a vertex $[g_i, h_j] \in V(G \boxtimes H)$ to be *A-Reachable* if there exists an *A-Rainbow-Path* from $[g_i, h_j]$ to the vertex $[g_0, h_0]$. We define $[g_i, h_j]$ to be *B-Reachable* if there exists a *B-Rainbow-Path* from $[g_i, h_j]$ to some vertex in $V_{1,0}$.

We define two subsets, R_A and R_B of $V(G \boxtimes H)$:

$$R_A = \bigcup_{0 \leq z \leq d(H)} V_{0,z} \quad \bigcup_{1 \leq y \leq z, |y-z| \text{ is even}} V_{y,z} \quad \bigcup_{2 \leq y \leq d(G)} V_{y,0} \quad \bigcup_{2 \leq z < y, z \text{ is even}} V_{y,z}$$

$$R_B = \bigcup_{0 \leq z \leq d(H)} V_{1,z} \quad \bigcup_{2 \leq y \leq z, |y-z| \text{ is odd}} V_{y,z} \quad \bigcup_{z < y, z \text{ is odd}} V_{y,z}$$

It is easy to verify that $R_A \cup R_B = V(G \boxtimes H)$, but $R_A \cap R_B$ is non-empty.

Claim 5. *If $u \in R_A$, then u is A-Reachable with respect to the edge coloring f .*

Proof. Let $u = [g_i, h_j] \in V_{y,z}$. We consider the following 4 cases.

Case 1: [When $u \in V_{0,z}$ where $0 \leq z \leq d(H)$]

From *Rule T1-R1* we know that the edges of H_0 are colored using the *Layer-wise Coloring*, $f_{H_0,A}$. Hence by *Observation 1* there is an *A-Rainbow-Path* from vertex u to $\text{root}(H_0) = [g_0, h_0]$. It follows that u is *A-Reachable*.

Case 2: [When $u \in V_{y,z}$ where $1 \leq y \leq z$ and $|y - z|$ is even]

Since $\ell_G(g_i) = y$, the path from g_i to g_0 in G has $y + 1$ vertices. Let this path be $g_i = g_{i_0}, g_{i_1}, \dots, g_{i_y} = g_0$. Let $h_{j'}$ be the ancestor of h_j in H such that $\ell_H(h_{j'}) = z - y$. Let $h_j = h_{j_0}, h_{j_1}, \dots, h_{j'} = h_{j_y}$ be the path from h_j to $h_{j'}$ in H . It has $y + 1$ vertices. Clearly $P_1 = \{[g_i, h_j] = [g_{i_0}, h_{j_0}], [g_{i_1}, h_{j_1}], \dots, [g_0, h_{j'}]\}$ is a path in $G \boxtimes H$ whose edges are colored $a_z, a_{z-1}, \dots, a_{z-y+1}$ in that order (By *Rule T2-R1*). Note that if $y = z$ then $h_{j'} = h_0$ and P_1 is the required *A-Rainbow-Path* from u to $[g_0, h_0]$. If $z < y$ then since $[g_0, h_{j'}] \in V(H_0)$, by *Case 1* there is a *A-Rainbow-Path*, say P_2 , from $[g_0, h_{j'}]$ to $[g_0, h_0]$. In particular P_2 is a $\{a_{z-y}, a_{z-y-1}, \dots, a_1\}$ *Rainbow Path*. Clearly $P = P_1 \cdot P_2$ is a $\{a_1, a_2, \dots, a_z\}$ -*Rainbow-Path* from vertex u to $[g_0, h_0]$ with respect to coloring f . Hence u is *A-Reachable*.

Case 3: [When $u \in V_{y,0}$ where $2 \leq y \leq d(G)$, hence $u = [g_i, h_0] \in V(G_0)$]

Let $u_1 = [g_{i'}, h_0]$ be an ancestor of u in G_0 such that $\ell_{G_0}(u_1) = 2$. By *Rule T1-R4* G_0 is edge colored using the *Layer-wise Coloring* f_{G_0,A_0} . The path from vertex u to u_1 in G_0 , say P_1 , is rainbow colored using colors from the set $\{a_y, a_{y-1}, \dots, a_3\}$. Let $g_{i''}$ be the parent of $g_{i'}$ in G . Since H is non-trivial h_1 exists and $(h_0, h_1) \in E(H)$. Therefore $([g_{i'}, h_0], [g_{i''}, h_1]) \in E(G \boxtimes H)$ and is colored a_2 by *Rule T2-R2*. Since $\ell_G(g_{i''}) = 1$, $(g_{i''}, g_0) \in E(G)$ and therefore $([g_{i''}, h_1], [g_0, h_0]) \in E(G \boxtimes H)$ and is colored a_1 by *Rule T2-R1*. Hence the path $P = P_1 \cdot ([g_{i'}, h_0], [g_{i''}, h_1], [g_0, h_0])$ is an *A-Rainbow-Path* from vertex u to $[g_0, h_0]$. Hence u is *A-Reachable*.

Case 4: [When $u \in V_{y,z}$ where $y > z \geq 2$ and z is even]

Vertex $u = [g_i, h_j] \in V(G_j)$. Let $u_1 = [g_{i'}, h_j]$ be an ancestor of u in G_j such that $\ell_{G_j}(u_1) = z$. Let P_1 be the path in G_j from vertex u to u_1 . Since $\ell_H(h_j) = z$ is even, by *Rule T1-R4*, G_j is edge colored using the *Layer-wise Coloring* f_{G_j,A_z} . The edges of P_1 are colored $a_y, a_{y-1}, \dots, a_{z+1}$ in that order. Since $u_1 = [g_{i'}, h_j] \in V_{z,z}$ and $z \geq 2$, by *Case 2* we have a $\{a_z, a_{z-1}, \dots, a_1\}$ -*Rainbow-Path*, say P_2 , from vertex u_1 to vertex $[g_0, h_0]$. Clearly $P = P_1 \cdot P_2$ is an *A-Rainbow-Path* from vertex u to $[g_0, h_0]$. Hence vertex u is *A-Reachable*. \square

Claim 6. *If $u \in R_B$, then u is B-Reachable with respect to the edge coloring f .*

Proof. Let $u = [g_i, h_j] \in V_{y,z}$. We consider the following 3 cases.

Case 1: [When $u \in V_{1,z}$ for $0 \leq z \leq d(G)$]

Vertex $u \in V(H_i)$ with $\text{root}(H_i) = [g_i, h_0]$. Since $\ell_G(g_i) = 1$, H_i is edge colored using the *Layer-wise Coloring* $f_{H_i,B}$ by *Rule T1-R2*. From *Observation 1* we infer that there is a $\{b_1, b_2, \dots, b_z\}$ -*Rainbow-Path* from vertex u to $[g_i, h_0] \in V_{1,0}$ in H_i . It follows that u is *B-Reachable* with respect to the edge coloring f .

Case 2: [When $u \in V_{y,z}$ where $2 \leq y \leq z$ and $|y - z|$ is odd]

Let $u = [g_i, h_j] \in V_{y,z}$. In G let $g_{i'}$ be the ancestor of g_i with $\ell_G(g_{i'}) = 1$. Since $\ell_G(g_i) = y$, the path in G from g_i to $g_{i'}$ in G has y vertices. Let $g_i = g_{i_0}, g_{i_1}, \dots, g_{i_{y-1}} = g_{i'}$ be that path. Similarly in H let $h_{j'}$ be the ancestor of h_j with $\ell_H(h_{j'}) = z - y + 1$. Then the path in H from h_j to $h_{j'}$ has y vertices. Let $h_j = h_{j_0}, h_{j_1}, \dots, h_{j_{y-1}} = h_{j'}$ be that path. Clearly $P_1 = [g_i, h_j], [g_{i_1}, h_{j_1}], \dots, [g_{i'}, h_{j'}]$ is a path in $G \boxtimes H$ and its edges are colored $b_z, b_{z-1}, \dots, b_{z-y+2}$ in that order (By *Rule T2-R1*). Now $[g_{i'}, h_{j'}] \in V_{1,z-y+1}$ and by *Case 1* there is a $\{b_1, b_2, \dots, b_{z-y+1}\}$ -*Rainbow-Path*, say P_2 , from $[g_{i'}, h_{j'}]$ to $[g_{i'}, h_0] \in V_{1,0}$. Clearly $P = P_1 \cdot P_2$ is a *B-Rainbow-Path* from u to $[g_{i'}, h_0] \in V_{1,0}$. It follows that u is *B-Reachable* with respect to the edge coloring f .

Case 3: [When $u \in V_{y,z}$ where $y > z$ and z is odd]

Let $u = [g_i, h_j] \in V_{y,z}$. We consider the following two sub-cases.

Case 3.a: [When $y = z + 1$]

Since $\ell_H(h_j) = z$, the path from h_j to h_0 in H has $z + 1$ vertices. Let this path be $h_j = h_{j_0}, h_{j_1}, \dots, h_{j_z} = h_0$. Similarly let $g_{i'}$ be the ancestor of g_i in G such that $\ell_G(g_{i'}) = 1$. Since $\ell_G(g_i) = z + 1$ the path from g_i to $g_{i'}$ in G has $z + 1$ vertices. Let this path be $g_i = g_{i_0}, g_{i_1}, \dots, g_{i_z} = g_{i'}$. Clearly $u = [g_i, h_j], [g_{i_1}, h_{j_1}], \dots, [g_{i'}, h_0]$ is a path in $G \boxtimes H$ and is colored b_z, b_{z-1}, \dots, b_1 in that order (By *Rule T2-R1*). Since $[g_{i'}, h_0] \in V_{1,0}$ vertex u is *B-Reachable*.

Case 3.b: [When $y > z + 1$]

Vertex $u \in G_j$. Let $u_1 = [g_{i'}, h_j]$ be an ancestor of u in G_j such that $\ell_{G_j}(u_1) = z + 1$. Since z is odd, by *Rule T1-R4* we know that G_j is edge colored using the *Layer-wise Coloring* f_{G_j, B_z} . The edges of path, $P_1 = P_{G_j}(u, u_1)$ are colored $b_y, b_{y-1}, \dots, b_{z+2}$ in that order and is a rainbow path. Since $u_1 \in V_{z+1,z}$ by *Case 3.a* there is a $\{b_z, b_{z-1}, \dots, b_1\}$ -*Rainbow-Path*, say P_2 , from vertex u_1 to some vertex, say u_2 in $V_{1,0}$. Clearly $P = P_1 \cdot P_2$ is a *B-Rainbow-Path* from vertex u to $u_2 \in V_{1,0}$. It follows that u is *B-Reachable* with respect to the coloring f . \square

Claim 7. Let $u \in V(G \boxtimes H) \setminus \{[g_0, h_0]\}$ then we have the following:

- (a) If $u \in R_A \setminus R_B$ then there exists $u_1 \in R_B$ such that $(u, u_1) \in E(G \boxtimes H)$ and is colored d .
- (b) If $u \in R_B \setminus R_A$ then there exists $u_1 \in R_A$ such that $(u, u_1) \in E(G \boxtimes H)$ and is colored d .

Proof. We consider the following cases.

Case 1: [When $u \in V_{0,z}$ where $0 \leq z \leq d(H)$, i.e $u \in V(H_0)$]

In this case $u = [g_0, h_j] \in R_A \setminus R_B$. We take $u_1 = [g_1, h_j]$. Since G is non-trivial, vertex g_1 exists and $(g_0, g_1) \in E(G)$. Since $\ell_G(g_i) = 1$, we have $u_1 \in V_{1,z} \subseteq R_B$, where $1 \leq z = \ell_H(h_j) \leq d(H)$. Note that $z \neq 0$ since $u \neq [g_0, h_0]$. Now the edge $(u, u_1) = ([g_0, h_j], [g_1, h_j]) \in E(G_j)$. By *Rule T1-R4*, G_j is edge colored using the *Layer-wise Coloring* f_{G_j, A_z} or f_{G_j, B_z} , where $z = \ell_H(h_j)$, depending on whether z is even or odd. Recalling that $A_z = \{a_1^z, a_2^z, \dots, a_{d(G)}^z\}$ and $B_z = \{b_1^z, b_2^z, \dots, b_{d(G)}^z\}$ the edge (u, u_1) is colored either a_1^z or b_1^z . Since $z \geq 1$, $a_1^z = b_1^z = d$ and hence the edge (u, u_1) is colored either $a_1^z = d$ or $b_1^z = d$.

Case 2: [When $u \in V_{1,z}$ where $0 \leq z \leq d(H)$]

In this case $u \in R_B$. Note that if z is odd then $V_{1,z} \subseteq R_A \cap R_B$. So we can assume that z is even.

Case 2.a: [When $u \in V_{1,0}$]

Let $u = [g_i, h_0] \in V_{1,0}$ with $\ell_G(g_i) = 1$. We take $u_1 = [g_0, h_1]$. Since H is non-trivial, h_1 exists and $(h_0, h_1) \in E(H)$. Also edge $(g_0, g_i) \in E(G)$. Therefore the edge $(u, u_1) = ([g_i, h_0], [g_0, h_1]) \in E(G \boxtimes H)$. It is easy to see that $u_1 \in V_{0,1} \subseteq V(H_0) \subseteq R_A$. The edge (u, u_1) is colored d by *Rule T2-R3*.

Case 2.b: [When $u \in V_{1,z}$ where $2 \leq z \leq d(H)$ and z is even]

Let $u = [g_i, h_j] \in V_{1,z}$ with $\ell_G(g_i) = 1$. Then $(g_0, g_1) \in E(G)$. We take $u_1 = [g_0, h_j] \in V_{0,z} \subseteq R_A$, then $(u, u_1) = ([g_i, h_j], [g_0, h_j]) \in E(G_j)$. By Rule T1-R4, G_j is edge colored using the *Layer-wise Coloring* f_{G_j, A_z} , since $z = \ell_H(h_j)$ is even. Since $z \geq 2$, $a_1^z = b_1^z = d$ and the edge (u, u_1) is colored d .

Case 3: [When $u \in V_{y,z}$ where $2 \leq y \leq z$]

Let $u = [g_i, h_j] \in G_j$. Let $u_1 = [g_{i'}, h_j]$ be the parent of u in G_j . Since $\ell_G(g_{i'}) = \ell_G(g_i) - 1 = y - 1$, $u_1 \in V_{y-1,z}$. We claim that if $u \in V_{y,z} \subseteq R_A \setminus R_B$ then $u_1 \in V_{y-1,z} \subseteq R_B$ and if $u \in V_{y,z} \subseteq R_B \setminus R_A$ then $u_1 \in V_{y-1,z} \subseteq R_A$. To see this first note that $\bigcup_{1 \leq y \leq z, |y-z| \text{ is even}} V_{y,z} \subseteq R_A$ and $\bigcup_{1 \leq y \leq z, |y-z| \text{ is odd}} V_{y,z} \subseteq R_B$. Now the following is easy to see: if $2 \leq y \leq z$ and $V_{y,z} \subseteq R_B \setminus R_A$ (respectively $R_A \setminus R_B$) then $1 \leq y-1 < z$ and $V_{y-1,z} \subseteq R_A$ (respectively R_B) since the parity of $|y-z|$ is different from the parity of $|(y-1)-z|$. By Rule T1-R4, G_j is edge colored using the *Layer-wise Coloring* f_{G_j, A_z} or f_{G_j, B_z} depending on whether $z = \ell_H(h_j)$ is even or odd. From the definition of the sets A_z and B_z we have that, for $1 \leq i \leq z$, $a_i^z = b_i^z = d$. Since $2 \leq y \leq z$, edge (u, u_1) is colored $a_y^z = d$ or $b_y^z = d$.

Case 4: [When $u \in V_{y,0}$ where $2 \leq y \leq d(G)$]

In this case $u \in R_A \setminus R_B$. Let $u = [g_i, h_0] \in V(H_i)$. Let $u_1 = [g_i, h_1] \in V(H_i)$. Since $(h_0, h_1) \in E(H)$, $(u, u_1) = ([g_i, h_0], [g_i, h_1]) \in E(H_i)$. Vertex $u_1 \in V_{y,1} \subseteq R_B$ as $(z=1) < 2 \leq y$ and 1 is odd. Since $\ell_G(g_i) = y \geq 2$, by Rule T1-R3 all the edges of H_i are colored d . Hence (u, u_1) is colored d .

Case 5: [When $u \in V_{y,z}$ where $1 \leq z < y$]

Let $u = [g_i, h_j] \in V(H_i)$. Let $u_1 = [g_i, h_{j'}]$ be the parent of u in H_i . Then $(u, u_1) = ([g_i, h_j], [g_i, h_{j'}]) \in E(H_i)$ and $\ell_H(h_{j'}) = \ell_H(h_j) - 1 = z - 1 \geq 0$. Since $y > z - 1$ if $u \in V_{y,z} \subseteq R_A \setminus R_B$ (respectively $R_B \setminus R_A$) then $z - 1$ is odd (even) and $u \in V_{y,z-1} \subseteq R_B$ (respectively R_A). Also since $y \geq 2$ by Rule T1-R3, all the edges of H_i are colored d . \square

Lemma 1. *The edge coloring f is a rainbow coloring of $G \boxtimes H$.*

Proof. We show that any distinct pair of vertices, u and v from $G \boxtimes H$ have a rainbow path between them with respect to the edge coloring f . Since $V(G \boxtimes H) = R_A \cup R_B$, vertex $u \in R_A$ or $u \in R_B$. The same applies to vertex v . Let $u = [g_0, h_0]$. If $v \in R_A$ then by Claim 5 there is an *A-Rainbow-Path* from v to $u = [g_0, h_0]$. If $v \in R_B$ then by Claim 6 there is a *B-Rainbow-Path* from v to some vertex $v' \in V_{1,0}$. We know that $(v', [g_0, h_0]) \in E(G_0)$ and is colored c by the *Layer-wise Coloring* f_{G_0, A_0} . Hence there is a $(\{c\} \uplus B)$ -*Rainbow-Path* from vertex v to $u = [g_0, h_0]$.

We may now assume that $u, v \neq [g_0, h_0]$. We have the following two cases:

Case 1: [When one of the vertices is in R_A and the other is in R_B]

Without loss of generality let $u \in R_A$ and $v \in R_B$. By Claim 5 there is an *A-Rainbow-Path* between vertex u and vertex $[g_0, h_0]$, let this path be P_1 . Similarly by Claim 6 there is a *B-Rainbow-Path* between vertex v and some vertex $v_1 = [g_i, h_0] \in V_{1,0}$, let this path be P_2 . Now $v_1 \in V(G_0)$ and $\ell_G(g_i) = 1$, hence $(g_0, g_1) \in E(G)$ and $(v_1, [g_0, h_0]) \in E(G_0)$. By Rule T1-R4 G_0 is edge colored using the *Layer-wise Coloring* f_{G_0, A_0} . The edge $(v_1, [g_0, h_0])$ is colored $a_1^0 = c$. Clearly the path $P = P_1 \cdot ([g_0, h_0], v_1) \cdot P_2$ is a $(A \uplus B \uplus \{c\})$ -*Rainbow-Path* between vertices u and v .

Case 2: [When both the vertices are in $R_A \setminus R_B$]

By Claim 7 there exists a vertex $u_1 \in R_B \subset V(G \boxtimes H)$ such that $(u, u_1) \in E(G \boxtimes H)$ and is colored d . Since $v \in R_A$ and $u_1 \in R_B$ by Case 1 there is a $(A \uplus B \uplus \{c\})$ -*Rainbow-Path* from vertex v to u_1 , say P_1 . Clearly $P = P_1 \cdot (u_1, u)$ is a rainbow path from vertex v to vertex u .

Case 3: [When both the vertices are in $R_B \setminus R_A$]

By Claim 7 there exists a vertex $u_2 \in R_A \subset V(G \boxtimes H)$ such that $(u, u_2) \in E(G \boxtimes H)$ and is colored d . Now using arguments similar to Case 2 we can prove that there exists a rainbow path between vertices u and v . \square

Theorem 4. $r(G' \boxtimes H') \leq rc(G' \boxtimes H') \leq 2r(G' \boxtimes H') + 2$

Proof. The rainbow coloring f uses $|A| + |B| + |\{c, d\}| = 2d(G) + 2 = 2r(G \boxtimes H) + 2$ colors. Since $d(G) = r(G') = r(G' \boxtimes H')$ From of Claim 1 the upper bound follows. The lower bound is trivial. \square

Tight Example:

Consider two graphs G_1 and G_2 such that $\text{diam}(G_1) = 2r(G_1) \geq \text{diam}(G_2)$. For example G_1 may be taken as a path with odd number of vertices. Then $rc(G_1 \boxtimes G_2) \geq \text{diam}(G_1 \boxtimes G_2) = 2r(G_1 \boxtimes G_2)$.

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