

REES' THEOREM FOR FILTRATIONS, MULTIPLICITY FUNCTION AND REDUCTION CRITERIA

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ABSTRACT. Let $J \subset I$ be ideals in a formally equidimensional local ring with $\lambda(I/J) < \infty$. Rees proved that for all $n \gg 0$, $\lambda(I^n/J^n)$ is a polynomial $P(I/J)(X)$ in n of degree at most $\dim R$ and J is a reduction of I if and only if $\deg P(I/J)(X) \leq \dim R - 1$. We extend this result for all Noetherian filtrations of ideals in a formally equidimensional local ring and for (not necessarily Noetherian) filtrations of ideals in analytically irreducible rings. We provide certain classes of ideals such that $\deg P(I/J)$ achieves its maximal degree. On the other hand, for ideals $J \subset I$ in a formally equidimensional local ring, we consider the multiplicity function $e(I^n/J^n)$ which is a polynomial in n for all large n . We explicitly determine the $\deg e(I^n/J^n)$ in some special cases. For an ideal J of analytic deviation one, we give characterization of reductions in terms of $\deg e(I^n/J^n)$ under some additional conditions.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d and I be an ideal in R . If I is \mathfrak{m} -primary then Samuel [24] showed that for all large n , the *Hilbert-Samuel function of I* , $H_I(n) = \lambda(R/I^n)$ (here $\lambda(M)$ denotes the length of an R -module M) coincides with a polynomial

$$P_I(n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

of degree d , called the *Hilbert-Samuel polynomial of I* . The coefficient $e(I) := e_0(I)$ is called the *multiplicity of I* . Rees used this numerical invariant $e(-)$ to study the numerical characterization of reductions. He showed that if R is a formally equidimensional local ring (i.e. R is a Noetherian local ring and its completion in the topology defined by the maximal ideal is equidimensional) and $J \subset I$ are \mathfrak{m} -primary ideals in R then J is a reduction of I if and only if $e(I) = e(J)$ [20]. In literature this result has been generalized in several directions. Amao [1], considered more general case where I, J are not necessarily \mathfrak{m} -primary ideals and proved that if M is a finitely generated R -module and $J \subset I$ are ideals of R such that IM/JM has finite length then $\mu(n) := \lambda(I^n M/J^n M)$ is a polynomial $P(IM/JM)$ in n for $n \gg 0$. In [22], Rees showed that in a Noetherian local ring (R, \mathfrak{m}) , the degree of $P(I/J)$ is at most $\dim R$ and he further proved the following theorem.

Theorem 1.1. (Rees, [22, Theorem 2.1]) *Let R be a formally equidimensional Noetherian local ring and $J \subset I$ be ideals in R with $\lambda(I/J) < \infty$. Then J is a reduction of I if and only if $\deg P(I/J)$ is at most $\dim R - 1$.*

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Since for some n , J^n is a reduction of I^n implies J is a reduction of I , we can interpret Rees' result as follows:

Theorem 1.2. *Let R be a formally equidimensional Noetherian local ring of dimension d and $J \subset I$ be ideals in R with $\lambda(I/J) < \infty$. Then the following are true.*

- (1) *The function $F(n) = \lim_{m \rightarrow \infty} \lambda(I^{mn}/J^{mn})/m^d$ is a polynomial in n of degree at most d .*
- (2) *Fix $n \in \mathbb{Z}_+$, then J is a reduction of I if and only if $F(n) = 0$.*

In [6, Theorem 6.1], Cutkosky proved the following result.

Theorem 1.3. (Cutkosky, [6, Theorem 6.1]) *Let R be an analytically unramified local ring of dimension $d > 0$. Suppose $\{I_i\}$ and $\{J_i\}$ are two graded families of ideals such that for all i , $J_i \subset I_i$ and there exists $c \in \mathbb{Z}_+$, such that $\mathfrak{m}^{ci} \cap I_i = \mathfrak{m}^{ci} \cap J_i$ for all i . Assume that if P is a minimal prime of R then $I_1 \subset P$ implies $I_i \subset P$ for all $i > 0$. Then the limit $\lim_{m \rightarrow \infty} \lambda(I_m/J_m)/m^d$ exists.*

Therefore it is natural to ask whether Rees' result (Theorem 1.2) holds true for two (not necessarily Noetherian) filtrations of ideals $\{I_i\}$ and $\{J_i\}$ in an analytically unramified local ring R of dimension d . In this direction we first prove Rees' theorem for Noetherian filtrations of ideals.

Theorem 1.4. (=Theorem 2.3) *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d > 0$ and for $1 \leq l \leq r$, $\mathcal{I}(l) = \{I(l)_n\}$, $\mathcal{J}(l) = \{J(l)_n\}$ be Noetherian filtrations of ideals in R such that $J(l)_n \subset I(l)_n$ and $\lambda(I(l)_n/J(l)_n) < \infty$ for all $n \in \mathbb{N}$ and $1 \leq l \leq r$. Let*

$$R_1 = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{m_1} \cdots I(r)_{m_r} \text{ and } R_2 = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{m_1} \cdots J(r)_{m_r}.$$

Fix $n_1, \dots, n_r \in \mathbb{Z}_+$. If R_1 is integral over R_2 then

$$\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0.$$

Converse holds if R is formally equidimensional and $\text{grade}(J(1)_1 \cdots J(r)_1) \geq 1$.

We prove that the “only if” part of Rees' theorem 1.2(2) holds true for (not necessarily Noetherian) filtrations of ideals in analytically irreducible local rings (i.e. R is a Noetherian local ring and its completion in the topology defined by the maximal ideal is domain). We also give an example to show that the “if” part of Rees' theorem 1.2(2) is not true for non-Noetherian filtrations of ideals in general (See example 2.7).

Theorem 1.5. (=Theorem 2.5) *Let (R, \mathfrak{m}) be an analytically irreducible local ring of dimension $d > 0$ and for $1 \leq l \leq r$, $\mathcal{I}(l) = \{I(l)_n\}$, $\mathcal{J}(l) = \{J(l)_n\}$ be (not necessarily Noetherian) filtrations of ideals in R such that $J(l)_n \subset I(l)_n$ for all $n \in \mathbb{N}$ and $1 \leq l \leq r$. Fix $n_1, \dots, n_r \in \mathbb{Z}_+$. Suppose there exists an integer $c \in \mathbb{Z}_+$ such that for all $i \in \mathbb{N}$,*

$$\mathfrak{m}^{ci} \cap I(1)_{in_1} \cdots I(r)_{in_r} = \mathfrak{m}^{ci} \cap J(1)_{in_1} \cdots J(r)_{in_r}.$$

Let $R_1 = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{m_1} \cdots I(r)_{m_r}$ be integral over $R_2 = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{m_1} \cdots J(r)_{m_r}$. Then

$$\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0.$$

Rees' theorem [22, Theorem 2.1] shows that J is a reduction of I implies $P(I/J)$ is a polynomial of degree at most $\dim R - 1$. In this direction one can ask that if J is a reduction of I when the degree of the polynomial $P(I/J)$ attains the maximal. We use

the notation (I, J) to denote J is a reduction of I . We produce a lower bound of degree of $P(I/J)$ and show that if J is a complete intersection ideal then $\deg P(I/J) = l(J) - 1$ where $l(J)$ is the analytic spread of J (See Theorem 3.7). We also improve the lower bound of $\deg P(I/J)$ for the following class of ideals.

Theorem 1.6. (=Theorem 3.9) *Let (R, \mathfrak{m}) be a Noetherian local ring, $J \subsetneq I$ be ideals in R such that $\lambda(I/J) < \infty$ and $\text{grade}(J) = s$. Let $a_1, \dots, a_s, b_1, \dots, b_t$ be a d -sequence in R which minimally generates the ideal J and $t \geq 1$. Suppose $I \cap (J' : b_1) = J'$ where $J' = (a_1, \dots, a_s)$ ($J' = (0)$ if $s = 0$). Then the following are true.*

- (1) $\text{grade}(J) \leq \deg P(I/J)$.
- (2) *If J is a reduction of I then $\text{grade}(J) \leq \deg P(I/J) \leq l(J) - 1$.*

This helps us to provide a large class of ideals (I, J) which attain the maximal degree of $P(I/J)$, i.e. $l(J) - 1$.

Corollary 1.7. (=Corollary 3.11) *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field, I be an ideal of R and J be any minimal reduction of I with $\lambda(I/J) < \infty$. Suppose I satisfies $G_{l(I)}$ and $AN_{l(I)-2}^-$ conditions. Then for any ideal K with $J \subsetneq K \subseteq I$, $\deg P(K/J) = l(J) - 1$.*

In the other direction, Herzog, Puthenpurakal and Verma [12] proved that, for ideals $J \subset I$ in a formally equidimensional local ring, $e(I^n/J^n)$ is eventually a polynomial function. In the same vein, in a recent paper, Ciupercă [2], showed that $e(I^n/J^n)$ is a polynomial function of degree at most $\dim R - t$ where t is a constant equal to $\dim Q$ where $Q = R/J^n : I^n$ for all $n \gg 0$ and in [3], he remarked that if J is reduction of I then $\deg e(I^n/J^n) \leq l(J) - 1$. He also studied the function $e(I^n/J^n)$ where J is an equimultiple ideal (i.e. $\text{ht } J = l(J)$) and proved the following characterization of reductions for equimultiple ideals in terms of $\deg e(I^n/J^n)$ [3, Theorem 2.6].

Theorem 1.8. (Ciupercă, [3, Theorem 2.6]) *Let (R, \mathfrak{m}) be a formally equidimensional local ring and $J \subseteq I$ proper ideals of R with J equimultiple. Let $f(n) = e(I^n/J^n)$. The following are true.*

- (1) *If $J \subseteq I$ is a reduction then $\deg f(n) \leq l(J) - 1$.*
- (2) *If $J \subseteq I$ is not a reduction then $\deg f(n) = l(J)$.*

In this situation one may ask if J has analytic deviation one (i.e. $l(J) = \text{ht}(J) + 1$) can we give characterization of reduction in terms of $\deg e(I^n/J^n)$?

Motivated by the result of Ciupercă (Theorem 1.8), we provide upper and lower bounds of $\deg e(I^n/J^n)$ and show that if J is a complete intersection ideal then J is reduction of I if and only if $\deg e(I^n/J^n) = l(J) - 1$ (See Proposition 4.1). In the case J has analytic deviation one, we characterize when J is a reduction of I in terms of $\deg e(I^n/J^n)$ under some additional hypotheses.

Theorem 1.9. (=Theorem 4.4) *Let (R, \mathfrak{m}) be a formally equidimensional local ring of dimension $d \geq 2$, $J \subsetneq I$ be ideals in R and J has analytic deviation one. Suppose $l(J_p) < l(J)$ for all prime ideals p in R such that $\text{ht } p = l(J)$. Then the following are true.*

- (1) *If J is not a reduction of I then $\deg e(I^n/J^n) = l(J) - 1$.*
- (2) *If $l(J) = d - 1$, $\text{depth}(R/J) > 0$ and for all $n \geq 1$, $\sqrt{J : I} = \sqrt{J^n : I^n}$ then J is a reduction of I if and only if $\deg e(I^n/J^n) \leq l(J) - 2$.*

We can not omit the condition $\text{depth}(R/J) > 0$ from Theorem 4.4 (2) (see Example 4.5). We give sufficient conditions on the ideal J for the equality $\sqrt{J : I} = \sqrt{J^n : I^n}$ for all $n \geq 1$ (see Proposition 4.3).

2. REES' THEOREM FOR FILTRATIONS

In this section we prove Rees' theorem for (not necessarily Noetherian) filtrations of ideals. A family $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$ of ideals in R is called *filtration of ideals* if $I_0 = R$, $I_m \subseteq I_n$ for all $m \geq n$ and $I_m I_n \subseteq I_{m+n}$ for all $m, n \in \mathbb{N}$. A filtration $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$ of ideals in R is said to be *Noetherian filtration* if $\bigoplus_{n \geq 0} I_n$ is a finitely generated R -algebra. We first prove Rees' result for Noetherian filtrations of ideals and then using iterated Noetherian filtrations, we prove the "only if" part for (not necessarily Noetherian) filtrations of ideals in analytically irreducible rings. We conclude this section with an example which shows that for non-Noetherian filtrations the "if" part of Rees' theorem is not true in general.

Lemma 2.1. *Let (R, \mathfrak{m}) be a local ring of dimension $d > 0$ and $A = \bigoplus_{m_1, \dots, m_r \geq 0} A_{m_1, \dots, m_r}$ be a \mathbb{N}^r -graded finitely generated R -algebra where $A_{0, \dots, 0} = R$. Let $u_1, \dots, u_r \in \mathbb{Z}_+$. Consider the graded subring $A^{(u_1, \dots, u_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} A_{u_1 m_1, \dots, u_r m_r}$ of A . Then A is finitely generated $A^{(u_1, \dots, u_r)}$ -module.*

Proof. Let $\{a_1, \dots, a_p\}$ be a generating set of A as R algebra. Define $\mathcal{S} = \{a_1^{r_1} \cdots a_p^{r_p} \mid 0 \leq r_i < u_1 \cdots u_r\}$. Let $h_1, \dots, h_p \in \mathbb{N}$ such that $h_i = u_1 \cdots u_r q_i + t_i$ where $0 \leq t_i < u_1 \cdots u_r$ and $1 \leq i \leq p$. Then $a_1^{h_1} \cdots a_p^{h_p} = (a_1^{q_1} \cdots a_p^{q_p})^{u_1 \cdots u_r} a_1^{t_1} \cdots a_p^{t_p}$. Thus \mathcal{S} is a generating set of A as $A^{(u_1, \dots, u_r)}$ -module. \square

Lemma 2.2. *Let (R, \mathfrak{m}) be a Noetherian local ring and for $i = 1, \dots, s$, $J_i \subset I_i$ be ideals in R and $\text{grade}(J_1 \cdots J_s) \geq 1$. Suppose J_i is not a reduction of I_i for some $i \in \{1, \dots, s\}$. Then $J_1 \cdots J_s$ is not a reduction of $I_1 \cdots I_s$.*

Proof. Without loss of generality, assume that J_1 is not a reduction of I_1 . Suppose for some $n \geq 0$, $(J_1 \cdots J_s)(I_1 \cdots I_s)^n = (I_1 \cdots I_s)^{n+1}$. Then

$$J_1 I_1^n (I_2 \cdots I_s)^{n+1} \supset (J_1 I_1^n) \cdots (J_s I_s^n) = I_1 I_1^n (I_2 \cdots I_s)^{n+1} \supset J_1 I_1^n (I_2 \cdots I_s)^{n+1}.$$

Hence $J_1 M = I_1 M$ where $M = I_1^n (I_2 \cdots I_s)^{n+1}$. Therefore by [21, Lemma 1.5], J_1 is a reduction of I_1 which is a contradiction. \square

Theorem 2.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d > 0$ and for $1 \leq l \leq r$, $\mathcal{I}(l) = \{I(l)_n\}$, $\mathcal{J}(l) = \{J(l)_n\}$ be Noetherian filtrations of ideals in R such that $J(l)_n \subset I(l)_n$ and $\lambda(I(l)_n/J(l)_n) < \infty$ for all $n \in \mathbb{N}$ and $1 \leq l \leq r$. Let*

$$R_1 = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{m_1} \cdots I(r)_{m_r} \text{ and } R_2 = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{m_1} \cdots J(r)_{m_r}.$$

Fix $n_1, \dots, n_r \in \mathbb{Z}_+$. If R_1 is integral over R_2 then

$$\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0.$$

Converse holds if R is formally equidimensional and $\text{grade}(J(1)_1 \cdots J(r)_1) \geq 1$.

Proof. Since for all $1 \leq l \leq r$, $\mathcal{J}(l) = \{J(l)_n\}$ are Noetherian filtrations of ideals in R , there exists an integer $\alpha \in \mathbb{Z}_+$, such that $G_l^\alpha = \bigoplus_{n \geq 0} J(l)_{\alpha n}$ are Noetherian standard graded R -algebras for all $1 \leq l \leq r$. Hence $T = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{\alpha m_1} \cdots J(r)_{\alpha m_r}$ is a Noetherian standard graded R -algebra and graded subring of R_2 .

Using Lemma 2.1 for $u_1 = \dots = u_r = \alpha$, we get R_2 is finitely generated T -module. Since R_1 is finitely generated R_2 -module, we have R_1 is finitely generated T -module. Now for all integers $0 \leq b_j \leq \alpha - 1$ with $1 \leq j \leq r$, $S^{(\alpha, b_1, \dots, b_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{\alpha m_1 + b_1} \cdots I(r)_{\alpha m_r + b_r}$

are T -submodules of R_1 . Therefore $S^{(\alpha, b_1, \dots, b_r)}$ are finitely generated T -modules for all integers $0 \leq b_j \leq \alpha - 1$ with $1 \leq j \leq r$ and hence

$$G^{(b_1, \dots, b_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{\alpha m_1 + b_1} \cdots I(r)_{\alpha m_r + b_r} / J(1)_{\alpha m_1 + b_1} \cdots J(r)_{\alpha m_r + b_r}$$

are finitely generated T -modules for all integers $0 \leq b_j \leq \alpha - 1$ with $1 \leq j \leq r$ where we consider the grading

$$G_{m_1, \dots, m_r}^{(b_1, \dots, b_r)} = I(1)_{\alpha m_1 + b_1} \cdots I(r)_{\alpha m_r + b_r} / J(1)_{\alpha m_1 + b_1} \cdots J(r)_{\alpha m_r + b_r}$$

and $T_{m_1, \dots, m_r} = J(1)_{\alpha m_1} \cdots J(r)_{\alpha m_r}$. Since $\lambda(I(l)_n / J(l)_n) < \infty$ for all $n \in \mathbb{N}$ and $1 \leq l \leq r$, there exists an integer $t \in \mathbb{Z}_+$, such that $G^{(b_1, \dots, b_r)}$ is finitely generated $T/\mathfrak{m}^t T$ -module for all $0 \leq b_1, \dots, b_r \leq \alpha - 1$.

By [11, Theorem 4.1], for all $1 \leq j \leq r$ and integers $0 \leq b_j \leq \alpha - 1$, there exist polynomials $P_{(b_1, \dots, b_r)}(X_1, \dots, X_r) \in \mathbb{Q}[X_1, \dots, X_r]$ of total degree at most $\dim \text{Supp}_{++} T/\mathfrak{m}^t T$ and an integer $f \in \mathbb{Z}_+$ such that for all $m_1, \dots, m_r \geq f$,

$$P_{(b_1, \dots, b_r)}(m_1, \dots, m_r) = \lambda(G_{m_1, \dots, m_r}^{(b_1, \dots, b_r)}).$$

Since $J(1)_{\alpha m_1} \cdots J(r)_{\alpha m_r} = J(1)_{\alpha}^{m_1} \cdots J(r)_{\alpha}^{m_r}$, by [16], [10, Corollary 3.3 and Corollary 5.3], we get

$$\dim \text{Supp}_{++} T/\mathfrak{m}^t T \leq \dim \text{Supp}_{++} T/\mathfrak{m} T = l(J(1)_{\alpha} \cdots J(r)_{\alpha}) - 1 < d \quad (2.3.1)$$

where $l(J(1)_{\alpha} \cdots J(r)_{\alpha})$ is the analytic spread of $J(1)_{\alpha} \cdots J(r)_{\alpha}$.

Let $i_1, \dots, i_r \in \mathbb{N}$ with $i_1 + \dots + i_r < d$ and

$$P_{(b_1, \dots, b_r)}(X_1, \dots, X_r) = \sum_{i_1 + \dots + i_r < d} z_{i_1, \dots, i_r}(b_1, \dots, b_r) X_1^{i_1} X_2^{i_2} \cdots X_r^{i_r}$$

where $z_{i_1, \dots, i_r}(b_1, \dots, b_r) \in \mathbb{Q}$. Let

$$C := \max\{|z_{i_1, \dots, i_r}(b_1, \dots, b_r)|(n_1 \cdots n_r)^d, \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) : 0 \leq b_1, \dots, b_r \leq \alpha - 1, \sum_{j=1}^r i_j < d \text{ and } 0 \leq m \leq \alpha(f + 1)\}.$$

Then for any $m \in \mathbb{Z}_+$, by equation (2.3.1), we get

$$\lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d \leq C/m < C.$$

Therefore

$$\begin{aligned} 0 &\leq \liminf_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d \\ &\leq \limsup_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0 \end{aligned}$$

implies

$$\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0.$$

Now we prove the converse. Let R be a formally equidimensional local ring, $\text{grade}(J(1)_1 \cdots J(r)_1) \geq 1$ and $\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0$.

Suppose R_1 is not integral over R_2 .

Claim : For any $u_1, \dots, u_r \in \mathbb{Z}_+$, $R_1^{(u_1, \dots, u_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{u_1 m_1} \cdots I(r)_{u_r m_r}$ is not integral over $R_2^{(u_1, \dots, u_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{u_1 m_1} \cdots J(r)_{u_r m_r}$.

Proof of the claim: Suppose there exist $u_1, \dots, u_r \in \mathbb{Z}_+$ such that $R_1^{(u_1, \dots, u_r)}$ is integral

over $R_2^{(u_1, \dots, u_r)}$. By Lemma 2.1, R_1 is finitely generated $R_1^{(u_1, \dots, u_r)}$ -module. Hence R_1 is integral over $R_2^{(u_1, \dots, u_r)}$ as well as over R_2 which is a contradiction.

Since for all $1 \leq l \leq r$, $\mathcal{I}(l) = \{I(l)_n\}$, $\mathcal{J}(l) = \{J(l)_n\}$ are Noetherian filtrations of ideals in R , there exists an integer $\alpha \in \mathbb{Z}_+$, such that $\bigoplus_{n \geq 0} I(l)_{\alpha n}$, $\bigoplus_{n \geq 0} J(l)_{\alpha n}$ are Noetherian standard graded R -algebras for all $1 \leq l \leq r$. Now by the claim,

$$R_1^{(\alpha n_1, \dots, \alpha n_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{\alpha n_1 m_1} \cdots I(r)_{\alpha n_r m_r} = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{\alpha n_1}^{m_1} \cdots I(r)_{\alpha n_r}^{m_r}$$

is not integral over

$$R_2^{(\alpha n_1, \dots, \alpha n_r)} = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{\alpha n_1 m_1} \cdots J(r)_{\alpha n_r m_r} = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{\alpha n_1}^{m_1} \cdots J(r)_{\alpha n_r}^{m_r}.$$

Therefore there exists $t \in \{1, \dots, r\}$ such that $\bigoplus_{m_t \geq 0} I(t)_{\alpha n_t}^{m_t}$ is not finitely generated over $\bigoplus_{m_t \geq 0} J(t)_{\alpha n_t}^{m_t}$, i.e., $J(t)_{\alpha n_t}$ is not a reduction of $I(t)_{\alpha n_t}$.

Let $n = \max\{n_1, \dots, n_r\}$. Since $(J(1)_1 \cdots J(r)_1)^{n\alpha} \subset J(1)_{\alpha n_1} \cdots J(r)_{\alpha n_r}$, by Lemma 2.2, $J(1)_{\alpha n_1} \cdots J(r)_{\alpha n_r}$ is not a reduction of $I(1)_{\alpha n_1} \cdots I(r)_{\alpha n_r}$. Therefore

$$\mathcal{R}(I(1)_{\alpha n_1} \cdots I(r)_{\alpha n_r}) = \bigoplus_{m \geq 0} (I(1)_{\alpha n_1} \cdots I(r)_{\alpha n_r})^m = \bigoplus_{m \geq 0} I(1)_{\alpha n_1 m} \cdots I(r)_{\alpha n_r m}$$

is not finitely generated module over

$$\mathcal{R}(J(1)_{\alpha n_1} \cdots J(r)_{\alpha n_r}) = \bigoplus_{m \geq 0} (J(1)_{\alpha n_1} \cdots J(r)_{\alpha n_r})^m = \bigoplus_{m \geq 0} J(1)_{\alpha n_1 m} \cdots J(r)_{\alpha n_r m}.$$

Thus by [22, Theorem 2.1], $\lim_{m \rightarrow \infty} \lambda(I(1)_{\alpha m n_1} \cdots I(r)_{\alpha m n_r} / J(1)_{\alpha m n_1} \cdots J(r)_{\alpha m n_r}) / m^d \neq 0$, which is a contradiction. \square

Now we prove the ‘‘only if’’ part of Rees’ theorem 1.2 for (not necessarily Noetherian) filtrations. Let (R, \mathfrak{m}) be a complete local domain of dimension $d > 0$. Then by [7, Lemma 4.2], [6], there exists a regular local ring S of dimension d which birationally dominates R . Let $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ be rationally independent real numbers such that $\lambda_i \geq 1$ for all i and $y_1, \dots, y_d \in S$ be a regular system of parameters in S . We consider a valuation ν on the quotient field of R which dominates S as mentioned in [6], i.e., $\nu(y_1^{a_1} \cdots y_d^{a_d}) = a_1 \lambda_1 + \cdots + a_d \lambda_d$ for $a_1, \dots, a_d \in \mathbb{N}$ and $\nu(\gamma) = 0$ if $\gamma \in S$ is a unit. Let $k = R/\mathfrak{m}_R$ and $k' = S/\mathfrak{m}_S$. Then by [6] (Page 10), we get $k' = S/\mathfrak{m}_S = V_\nu/\mathfrak{m}_\nu$ where V_ν is the valuation ring of ν and $[k' : k] < \infty$.

For $\mu \in \mathbb{R}_{>0}$, define ideals K_μ and K_μ^+ in the valuation ring V_ν by

$$K_\mu = \{f \in \mathbb{Q}(R) \mid \nu(f) \geq \mu\},$$

$$K_\mu^+ = \{f \in \mathbb{Q}(R) \mid \nu(f) > \mu\}.$$

Let $\mathcal{T}(m) = \{T(m)_n\}$ be filtrations of ideals in R for all $1 \leq m \leq r$ and $a \in \mathbb{Z}_+$. We recall the definition of a th truncated filtration of ideals defined in [7]. The a -th truncated filtration of ideals $\mathcal{T}_a(m) = \{T_a(m)_n\}$ of $\mathcal{T}(m)$ is defined by $T_a(m)_n = T(m)_n$ if $n \leq a$ and if $n > a$, then $T_a(m)_n = \sum T_a(m)_i T_a(m)_j$ where the sum is over $i, j > 0$ such that $i + j = n$.

Fix $n_1, \dots, n_r \in \mathbb{Z}_+$. Define filtrations of ideals

$$\mathcal{T} = \{\mathcal{T}_m := T(1)_{m n_1} \cdots T(r)_{m n_r}\} \text{ and } \mathcal{T}(a) = \{\mathcal{T}(a)_m := T_a(1)_{m n_1} \cdots T_a(r)_{m n_r}\}.$$

By [5, Lemma 4.3], there exists $\alpha \in \mathbb{Z}_+$ such that $K_{\alpha m} \cap R \subset \mathfrak{m}^m$ for all $m \in \mathbb{N}$.

Note that $\mathfrak{m}^{\alpha b m d} \mathcal{T}_m \subset \mathcal{T}_m \cap K_{\alpha m b}$ and $\mathfrak{m}^{\alpha b m d} \mathcal{T}(a)_m \subset \mathcal{T}(a)_m \cap K_{\alpha m b}$ for all $m \in \mathbb{N}$ and $b \in \mathbb{Z}_+$.

Proposition 2.4. *Suppose (R, \mathfrak{m}) , \mathcal{T} , $\mathcal{T}(a)$, α and K_μ as above. Fix $b' \in \mathbb{Z}_+$. For each $a \in \mathbb{Z}_+$, let $\mathcal{F}(a) = \{\mathcal{F}(a)_n\}$ be a filtration of ideals such that for all n ,*

$$\mathcal{T}(a)_n \subset \mathcal{F}(a)_n \subset \mathcal{T}_n.$$

Then there exists an integer b such that $b \geq b'$ and

$$\begin{aligned} & \lim_{a \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \lambda(\mathcal{T}(a)_m / \mathcal{T}(a)_m \cap K_{\alpha m b}) \right) / m^d = \lim_{m \rightarrow \infty} \lambda(\mathcal{T}_m / \mathcal{T}_m \cap K_{\alpha m b}) / m^d \\ & = \lim_{a \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \lambda(\mathcal{F}(a)_m / \mathcal{F}(a)_m \cap K_{\alpha m b}) \right) / m^d. \end{aligned}$$

Proof. We follow the argument given in [6], [7, Proposition 4.3]. For $t \geq 1$, define the semigroups

$$\begin{aligned} \Gamma^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \dim_k \mathcal{T}_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{T}_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b' i\}, \\ \Gamma(a)^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \\ &\quad \dim_k \mathcal{T}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{T}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b' i\} \\ \Gamma(\mathcal{F}(a))^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \\ &\quad \dim_k \mathcal{F}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{F}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b' i\}. \end{aligned}$$

By [6, Lemma 4.5], there exists an integer $b \geq b'$ such that the semigroups

$$\begin{aligned} \Gamma^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \dim_k \mathcal{T}_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{T}_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b i\}, \\ \Gamma(a)^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \\ &\quad \dim_k \mathcal{T}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{T}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b i\} \\ \Gamma(\mathcal{F}(a))^{(t)} &= \{(m_1, \dots, m_d, i) \in \mathbb{N}^{d+1} \mid \\ &\quad \dim_k \mathcal{F}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d} / \mathcal{F}(a)_i \cap K_{m_1 \lambda_1 + \dots + m_d \lambda_d}^+ \geq t \\ &\quad \text{and } m_1 + \dots + m_d \leq \alpha b i\}. \end{aligned}$$

satisfy equations (5) and (6) of [6].

Define $\Gamma_m^{(t)} = \Gamma^{(t)} \cap (\mathbb{N}^d \times \{m\})$, $\Gamma(a)_m^{(t)} = \Gamma(a)^{(t)} \cap (\mathbb{N}^d \times \{m\})$ and $\Gamma(\mathcal{F}(a))_m^{(t)} = \Gamma(\mathcal{F}(a))^{(t)} \cap (\mathbb{N}^d \times \{m\})$ for $m \in \mathbb{N}$. For a (strongly nonnegative) sub semigroup S of $\mathbb{Z}^d \times \mathbb{N}$, $\text{con}(S)$ is defined as the closed convex cone which is the closure of the set of all linear combinations $\sum \lambda_i s_i$ with $s_i \in S$ and λ_i a nonnegative real number and the *Newton-Okounkov body* is defined as

$$\Delta(S) = \text{con}(S) \cap (\mathbb{R}^d \times \{1\}).$$

This theory is developed in [19], [17] and [15] and is summarized in [6, Section 3]. By [6, Lemma 4.5] and [6, Theorem 3.2],

$$\lim_{m \rightarrow \infty} \frac{\#\Gamma_m^{(t)}}{m^d} = \text{Vol}(\Delta(\Gamma^{(t)})), \quad (2.4.1)$$

$$\lim_{m \rightarrow \infty} \frac{\#\Gamma(a)_m^{(t)}}{m^d} = \text{Vol}(\Delta(\Gamma(a)^{(t)})), \quad (2.4.2)$$

$$\lim_{m \rightarrow \infty} \frac{\#\Gamma(\mathcal{F}(a))_m^{(t)}}{m^d} = \text{Vol}(\Delta(\Gamma(\mathcal{F}(a))^{(t)})). \quad (2.4.3)$$

Therefore

$$\begin{aligned} \lim_{m \rightarrow \infty} \lambda(\mathcal{T}_m / \mathcal{T}_m \cap K_{\alpha mb}) / m^d &= \lim_{m \rightarrow \infty} \dim_k \left(\bigoplus_{0 \leq \mu < \alpha mb} \mathcal{T}_m \cap K_\mu / \mathcal{T}_m \cap K_\mu^+ \right) / m^d \\ &= \sum_{t=1}^{[k':k]} \lim_{m \rightarrow \infty} \frac{\#\Gamma_m^{(t)}}{m^d}. \end{aligned} \quad (2.4.4)$$

Similarly

$$\begin{aligned} &\lim_{m \rightarrow \infty} \lambda(\mathcal{T}(a)_m / \mathcal{T}(a)_m \cap K_{\alpha mb}) / m^d \\ &= \lim_{m \rightarrow \infty} \dim_k \left(\bigoplus_{0 \leq \mu < \alpha mb} \mathcal{T}(a)_m \cap K_\mu / \mathcal{T}(a)_m \cap K_\mu^+ \right) / m^d \\ &= \sum_{t=1}^{[k':k]} \lim_{m \rightarrow \infty} \frac{\#\Gamma(a)_m^{(t)}}{m^d} \end{aligned} \quad (2.4.5)$$

and

$$\begin{aligned} &\lim_{m \rightarrow \infty} \lambda(\mathcal{F}(a)_m / \mathcal{F}(a)_m \cap K_{\alpha mb}) / m^d \\ &= \lim_{m \rightarrow \infty} \dim_k \left(\bigoplus_{0 \leq \mu < \alpha mb} \mathcal{F}(a)_m \cap K_\mu / \mathcal{F}(a)_m \cap K_\mu^+ \right) / m^d \\ &= \sum_{t=1}^{[k':k]} \lim_{m \rightarrow \infty} \frac{\#\Gamma(\mathcal{F}(a))_m^{(t)}}{m^d}. \end{aligned} \quad (2.4.6)$$

Let $\hat{a} = \lfloor a / \max\{n_1, \dots, n_r\} \rfloor$ where $\lfloor x \rfloor$ be the largest integer smaller than or equal to the real number x . Now

$$\Gamma_i^{(t)} = \Gamma(\mathcal{F}(a))_i^{(t)} = \Gamma(a)_i^{(t)} \text{ for } i \leq \hat{a}. \quad (2.4.7)$$

Hence

$$n * \Gamma_{\hat{a}}^{(t)} := \{x_1 + \dots + x_n \mid x_1, \dots, x_n \in \Gamma_{\hat{a}}^{(t)}\} \subset \Gamma(a)_{n\hat{a}}^{(t)} \subset \Gamma(\mathcal{F}(a))_{n\hat{a}}^{(t)} \text{ for all } n \geq 1.$$

By [6, Theorem 3.3] and since $\hat{a} \rightarrow \infty$ as $a \rightarrow \infty$, given $\epsilon > 0$, there exists $a_0 > 0$ such that for all $a \geq a_0$, we have

$$\begin{aligned} \text{Vol}(\Delta(\Gamma^{(t)})) &\geq \text{Vol}(\Delta(\Gamma(a)^{(t)})) = \lim_{n \rightarrow \infty} \frac{\#\Gamma(a)_n^{(t)}}{n^d} = \lim_{n \rightarrow \infty} \frac{\#\Gamma(a)_{n\hat{a}}^{(t)}}{(n\hat{a})^d} \\ &\geq \lim_{n \rightarrow \infty} \frac{\#(n * \Gamma_{\hat{a}}^{(t)})}{(n\hat{a})^d} \geq \text{Vol}(\Delta(\Gamma^{(t)})) - \epsilon \end{aligned} \quad (2.4.8)$$

and

$$\begin{aligned} \text{Vol}(\Delta(\Gamma^{(t)})) &\geq \text{Vol}(\Delta(\Gamma(\mathcal{F}(a))^{(t)})) = \lim_{n \rightarrow \infty} \frac{\#\Gamma(\mathcal{F}(a))_n^{(t)}}{n^d} = \lim_{n \rightarrow \infty} \frac{\#\Gamma(\mathcal{F}(a))_{n\hat{a}}^{(t)}}{(n\hat{a})^d} \\ &\geq \lim_{n \rightarrow \infty} \frac{\#(n * \Gamma_{\hat{a}}^{(t)})}{(n\hat{a})^d} \geq \text{Vol}(\Delta(\Gamma^{(t)})) - \epsilon. \end{aligned} \quad (2.4.9)$$

By equations (2.4.4)-(2.4.9), we get the desired result. \square

Theorem 2.5. *Let (R, \mathfrak{m}) be an analytically irreducible local ring of dimension $d > 0$ and for $1 \leq l \leq r$, $\mathcal{I}(l) = \{I(l)_n\}$, $\mathcal{J}(l) = \{J(l)_n\}$ be (not necessarily Noetherian) filtrations of ideals in R such that $J(l)_n \subset I(l)_n$ for all $n \in \mathbb{N}$ and $1 \leq l \leq r$. Fix $n_1, \dots, n_r \in \mathbb{Z}_+$. Suppose there exists an integer $c \in \mathbb{Z}_+$ such that for all $i \in \mathbb{N}$,*

$$\mathfrak{m}^{ci} \cap I(1)_{in_1} \cdots I(r)_{in_r} = \mathfrak{m}^{ci} \cap J(1)_{in_1} \cdots J(r)_{in_r}.$$

Let $R_1 = \bigoplus_{m_1, \dots, m_r \geq 0} I(1)_{m_1} \cdots I(r)_{m_r}$ be integral over $R_2 = \bigoplus_{m_1, \dots, m_r \geq 0} J(1)_{m_1} \cdots J(r)_{m_r}$. Then

$$\lim_{m \rightarrow \infty} \lambda(I(1)_{mn_1} \cdots I(r)_{mn_r} / J(1)_{mn_1} \cdots J(r)_{mn_r}) / m^d = 0.$$

Proof. Let \hat{R} denote the \mathfrak{m} -adic completion of R . Without loss of generality, we may replace R by \hat{R} and $I(l)_n, J(l)_n$ by $I(l)_n \hat{R}, J(l)_n \hat{R}$ respectively for all n and $1 \leq l \leq r$.

For $a \in \mathbb{Z}_+$, we define filtrations of ideals

$$\mathcal{I} = \{\mathcal{I}_i := I(1)_{in_1} \cdots I(r)_{in_r}\}, \quad \mathcal{I}(a) = \{\mathcal{I}(a)_i := I_a(1)_{in_1} \cdots I_a(r)_{in_r}\},$$

and

$$\mathcal{J} = \{\mathcal{J}_i := J(1)_{in_1} \cdots J(r)_{in_r}\}.$$

Now there exists an integer $\underline{a} \in \mathbb{Z}_+$, such that $a \leq \underline{a}$ and every element of $\oplus_{i \geq 0} \mathcal{I}(a)_i$ is integral over $\oplus_{i \geq 0} \mathcal{J}(\underline{a})_i$ where

$$\mathcal{J}(\underline{a}) = \{\mathcal{J}(\underline{a})_i := J_{\underline{a}}(1)_{in_1} \cdots J_{\underline{a}}(r)_{in_r}\}.$$

Define the following filtrations of ideals

$$\mathcal{A}_a = \{\mathcal{A}_{a,m} := \sum_{\substack{\alpha+\beta=m \\ \alpha, \beta \in \mathbb{N}}} \mathcal{I}(a)_\alpha \mathcal{J}(\underline{a})_\beta\}$$

and

$$\mathcal{B}_{\underline{a}} = \{\mathcal{B}_{\underline{a},m} := \mathcal{A}_{a,m} \cap \mathcal{J}_m\}.$$

Note that $\oplus_{n \geq 0} \mathcal{A}_{a,n}$ is finitely generated $\oplus_{n \geq 0} \mathcal{B}_{\underline{a},n}$ -module. Thus by Theorem 2.3,

$$\lim_{m \rightarrow \infty} \lambda(\mathcal{A}_{a,m}/\mathcal{B}_{\underline{a},m})/m^d = 0. \quad (2.5.1)$$

Now for all n ,

$$\mathcal{A}_{a,n} \cap \mathfrak{m}^{cn} = \mathcal{A}_{a,n} \cap \mathcal{I}_n \cap \mathfrak{m}^{cn} = \mathcal{A}_{a,n} \cap \mathcal{J}_n \cap \mathfrak{m}^{cn} = \mathcal{B}_{\underline{a},n} \cap \mathfrak{m}^{cn}.$$

Let $\alpha \in \mathbb{Z}_+$ be the integer such that $K_{\alpha n} \cap R \subset \mathfrak{m}^n$ for all $n \in \mathbb{N}$ [5, Lemma 4.3]. Therefore

$$\mathcal{A}_{a,n} \cap K_{\alpha n} = \mathcal{A}_{a,n} \cap K_{\alpha n} \cap R \subset \mathcal{A}_{a,n} \cap \mathfrak{m}^{cn} \subset \mathcal{B}_{\underline{a},n}$$

and hence $\mathcal{A}_{a,n} \cap K_{\alpha n} = \mathcal{B}_{\underline{a},n} \cap K_{\alpha n}$ for all n .

Now for all $n \in \mathbb{N}$,

$$\mathcal{I}(a)_n \subset \mathcal{A}_{a,n} \subset \mathcal{I}_n, \quad \mathcal{J}(\underline{a})_n \subset \mathcal{B}_{\underline{a},n} \subset \mathcal{J}_n.$$

Choose an integer $c' \geq c$ such that Proposition 2.4 holds for the above two equations. Also note that $i \in \mathbb{N}$,

$$\mathfrak{m}^{c'i} \cap I(1)_{in_1} \cdots I(r)_{in_r} = \mathfrak{m}^{c'i} \cap J(1)_{in_1} \cdots J(r)_{in_r}.$$

Replace c by c' . Since $\underline{a} \rightarrow \infty$ as $a \rightarrow \infty$, by equation (2.5.1) and Proposition 2.4, we get

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lambda(\mathcal{I}_m/\mathcal{J}_m)/m^d \\ &= \lim_{m \rightarrow \infty} \lambda(\mathcal{I}_m/\mathcal{I}_m \cap K_{\alpha m})/m^d - \lim_{m \rightarrow \infty} \lambda(\mathcal{J}_m/\mathcal{J}_m \cap K_{\alpha m})/m^d \\ &= \lim_{a \rightarrow \infty} \lim_{m \rightarrow \infty} \lambda(\mathcal{A}_{a,m}/\mathcal{A}_{a,m} \cap K_{\alpha m})/m^d \\ &\quad - \lim_{a \rightarrow \infty} \lim_{m \rightarrow \infty} \lambda(\mathcal{B}_{\underline{a},m}/\mathcal{B}_{\underline{a},m} \cap K_{\alpha m})/m^d \\ &= \lim_{a \rightarrow \infty} \lim_{m \rightarrow \infty} \lambda(\mathcal{A}_{a,m}/\mathcal{B}_{\underline{a},m})/m^d = 0. \end{aligned}$$

□

It is natural to ask the following.

Question 2.6. *Does Theorem 2.5 hold true for any analytically unramified local ring?*

The author believes the answer is positive but is unable to prove.

The following example shows that the “if” part of Rees’ theorem 1.2 (2) is not true for non-Noetherian filtrations of ideals in general.

Example 2.7. *Let $R = k[[x, y]]$ be a formal power series ring of dimension two over a field k . Let $\mathcal{I} = \{I_m := x^m\}$ and $\mathcal{J} = \{J_m := (x^{m+1}, x^m y)\}$. Note that $I_m \cap \mathfrak{m}^{2m} = J_m \cap \mathfrak{m}^{2m}$ for all $m \geq 0$ and $\lim_{m \rightarrow \infty} \lambda(I_{mn}/J_{mn})/m^2 = 0$ but $\oplus_{m \geq 0} I_m$ is not integral over $\oplus_{m \geq 0} J_m$.*

Let $r \geq 2$. We define a *multigraded filtration* $\mathcal{I} = \{I_{\mathbf{n}}\}$ where $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ of ideals in a ring R to be a collection of ideals of R such that $R = I_{\mathbf{0}}$, $I_{\mathbf{n}+e_i} \subset I_{\mathbf{n}}$ for all $i = 1, \dots, r$ where $e_i = (0, \dots, 1, \dots, 0)$ with 1 at i th position and $I_{\mathbf{n}}I_{\mathbf{m}} \subset I_{\mathbf{n}+\mathbf{m}}$ whenever $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$. As a generalization of Rees' theorem 1.2, we pose the following question for multigraded filtration of ideals.

Question 2.8. *Let $\{I_{\mathbf{n}}\}$ and $\{J_{\mathbf{n}}\}$ be multigraded filtrations of ideals in a Noetherian local ring such that $J_{\mathbf{n}} \subset I_{\mathbf{n}}$ and $\lambda(I_{\mathbf{n}}/J_{\mathbf{n}}) < \infty$ for all $\mathbf{n} \in \mathbb{N}^r$. Fix $n_1, \dots, n_r \in \mathbb{Z}_+$.*

- (1) *Is it true that if $\bigoplus_{\mathbf{n} \in \mathbb{N}^r} I_{\mathbf{n}}$ is integral over $\bigoplus_{\mathbf{n} \in \mathbb{N}^r} J_{\mathbf{n}}$ then*

$$\lim_{m \rightarrow \infty} \lambda(I_{mn_1, \dots, mn_r} / J_{mn_1, \dots, mn_r}) / m^d = 0?$$

- (2) *Does the converse of (1) hold true if $\bigoplus_{\mathbf{n} \in \mathbb{N}^r} I_{\mathbf{n}}$ and $\bigoplus_{\mathbf{n} \in \mathbb{N}^r} J_{\mathbf{n}}$ both are Noetherian?*

3. THE DEGREE OF THE REES POLYNOMIAL

In this section, we provide bounds for the degree of the function $\lambda(I^n/J^n)$ where $J \subsetneq I$ are ideals in a Noetherian local ring with $\lambda(I/J) < \infty$ and J is reduction of I . We concentrate on the ideals generated by d -sequences. The notion of d -sequence was introduced by Huneke [13]. Let $\underline{x} = x_1, \dots, x_n$ be a sequence of elements in R . Then \underline{x} is called a *d -sequence* if

- (1) $x_i \notin (x_1, \dots, \widehat{x}_i, \dots, x_n)$ for all $i = 1, \dots, n$,
(2) for all $k \geq i + 1$ and all $i \geq 0$, $(x_0 = 0)$

$$((x_0, \dots, x_i) : x_{i+1}x_k) = ((x_0, \dots, x_i) : x_k).$$

Every regular sequence is a d -sequence but the converse is not true. Every system of parameters in a Buchsbaum local ring is a d -sequence. Examples of d -sequences are abundant. Ideals generated by d -sequences have nice properties, e.g. they are of linear type [13], Castelnuovo-Mumford regularity of Rees algebras of such ideals is zero [25]. For our convenience we omit the condition (1) in the definition of d -sequence. We say, a sequence of elements $\underline{x} = x_1, \dots, x_n$ in R is a *d -sequence* if condition (2) is satisfied.

Remark 3.1. Let $J \subsetneq I$ be ideals in a Noetherian local ring (R, \mathfrak{m}) with the property that $\lambda(I/J) < \infty$. Suppose J is a reduction of I and $JI^m = I^{m+1}$ for all $m \geq l$. Consider the Veronese subring $A = \bigoplus_{n \geq 0} J^{nl}$ of $\mathcal{R}(J)$. Then $M = \bigoplus_{n \geq 0} I^{nl}/J^{nl}$ is a finite A -module. Therefore there exists an integer $t \in \mathbb{Z}_+$, such that $\mathfrak{m}^t I^{nl} \subseteq J^{nl}$ for all $n \geq 1$. Hence M is $A/\mathfrak{m}^t A$ -module. Thus $\dim M \leq \dim A/\mathfrak{m}^t A = \dim \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J) = l(J)$. Therefore $\lambda(I^{nl}/J^{nl})$ is a polynomial type function of degree at most $l(J) - 1$. Hence $\lambda(I^n/J^n)$ is a polynomial type function of degree at most $l(J) - 1$.

Let $J \subset I$ be ideals in a Noetherian local ring (R, \mathfrak{m}) and a_1, \dots, a_s be a sequence of elements in J . Throughout this section we use the following notation: $R_i = R/(a_0, a_1, \dots, a_i)$ for all $i = 0, \dots, s$ where $a_0 = 0$ and I_i, J_i denote the images of I, J in R_i respectively for all $i = 0, \dots, s$. The next two results are required to prove our main results.

Lemma 3.2. *Let (R, \mathfrak{m}) be a Noetherian local ring, $J \subsetneq I$ be ideals in R and $\lambda(I/J) < \infty$. Let a_1, \dots, a_s be a sequence of elements in J . Suppose for some $i \in \{0, 1, \dots, s-1\}$ and all $n \gg 0$, $(J_i^{n+1} : (a_{i+1}^{(i)})) \cap I_i^n = J_i^n$ where $a_{i+1}^{(i)}$ denotes the image of a_{i+1} in R_i . Then for all $n \gg 0$,*

$$\lambda(I_{i+1}^{n+1}/J_{i+1}^{n+1}) \leq \lambda(I_i^{n+1}/J_i^{n+1}) - \lambda(I_i^n/J_i^n).$$

Proof. For all $n \gg 0$, consider the following exact sequence

$$0 \longrightarrow (J_i^{n+1} : (a_{i+1}^{(i)})) \cap I_i^n / J_i^n \longrightarrow I_i^n / J_i^n \xrightarrow{a_{i+1}^{(i)}} I_i^{n+1} / J_i^{n+1} \longrightarrow I_i^{n+1} / a_{i+1}^{(i)} I_i^n + J_i^n \longrightarrow 0.$$

Since

$$I_{i+1}^{n+1} / J_{i+1}^{n+1} = I_i^{n+1} + (a_{i+1}^{(i)}) / J_i^{n+1} + (a_{i+1}^{(i)}) \simeq I_i^{n+1} / (a_{i+1}^{(i)}) \cap I_i^{n+1} + J_i^{n+1}$$

and $a_{i+1}^{(i)} I_i^n + J_i^{n+1} \subseteq (a_{i+1}^{(i)}) \cap I_i^{n+1} + J_i^{n+1}$, we have the required inequality. \square

Proposition 3.3. *Let x_1, \dots, x_n be a d -sequence in a Noetherian local ring (R, \mathfrak{m}) and $J = (x_1, \dots, x_n)$. Then $J^n \cap (x_1) = x_1 J^{n-1}$ for all $n \geq 1$.*

Proof. Note that images of x_2, \dots, x_n is a d -sequence in $R/(x_1)$. Let $x_1 r \in J^n \cap (x_1)$. Then $x_1 r = x_1 a + b$ for some $a \in J^{n-1}$ and $b \in J_1^n$ where $J_1 = (x_2, \dots, x_n)$. By [13, Theorem 2.1], for all $n \geq 1$,

$$x_1(r - a) \in J_1^n \cap (x_1) \subset x_1 J_1^{n-1}.$$

Therefore $x_1 r \in x_1 J^{n-1}$. \square

Remark 3.4. Note that if $J = (x_1, \dots, x_n)$ where x_1, \dots, x_n is a d -sequence and $\text{grade}(J) = r$ then x_1, \dots, x_r is a regular sequence.

Next we state some well-known properties of d -sequence. Here $G(J) = \bigoplus_{n \geq 0} J^n / J^{n+1}$ denotes the graded associated ring of J .

Proposition 3.5. *Let (R, \mathfrak{m}) be a Noetherian local ring and J an ideal in R with $\text{grade}(J) = s$. Suppose J is generated by a d -sequence $a_1, \dots, a_s, \dots, a_t$. Then $\text{grade } G(J)_+ = \text{grade}(J)$.*

Proposition 3.6. *Let x_1, \dots, x_s be a d -sequence in a Noetherian local ring (R, \mathfrak{m}) and J be an ideal minimally generated by x_1, \dots, x_s . Suppose x_1 is a nonzerodivisor on R . Then $l(J/(x_1)) = l(J) - 1$.*

The following theorem gives a lower bound for $\deg P(I/J)$ and using the lower bound of $\deg P(I/J)$ we show that if J is a complete intersection ideal then $\deg P(I/J) = l(J) - 1$.

Theorem 3.7. *Let (R, \mathfrak{m}) be a Noetherian local ring, $J \subsetneq I$ be ideals in R such that $\lambda(I/J) < \infty$. Then the following are true.*

- (1) $\text{grade } G(J)_+ - 1 \leq \deg P(I/J)$.
- (2) *If J is a reduction of I then $\text{grade } G(J)_+ - 1 \leq \deg P(I/J) \leq l(J) - 1$.*
- (3) *If J is a complete intersection ideal and reduction of I then $\deg P(I/J) = l(J) - 1$.*

Proof. (1) Let $S = R[X]_{\mathfrak{m}[X]}$. Then S is faithfully flat extension of R , S has infinite residue field, $\lambda_S(I^n S / J^n S) = \lambda_R(I^n / J^n) \lambda_S(S / \mathfrak{m} S) = \lambda_R(I^n / J^n)$ whenever $\lambda_R(I^n / J^n) < \infty$ and $\text{grade } G(JS)_+ \geq \text{grade } G(J)_+$. Therefore without loss of generality we assume that R has infinite residue field. Let $\text{grade } G(J)_+ = s$. We may assume $s \geq 1$. Then there exist $a_1^*, \dots, a_s^* \in G(J)_1$ such that a_1^*, \dots, a_s^* is $G(J)$ -sequence. Therefore for all $i = 0, \dots, s-1$ and $n \geq 0$, we have $(J_i^{n+1} : (a_{i+1}^{(i)})) = J_i^n$ and hence $(J_i^{n+1} : (a_{i+1}^{(i)})) \cap I_i^n = J_i^n$ where $a_{i+1}^{(i)}$ denotes the image of a_{i+1} in R_i (R_i, I_i, J_i are defined before Lemma 3.2). Thus by Lemma 3.2, for all $i = 0, \dots, s-2$ and $n \gg 0$, we have

$$\lambda(I_{i+1}^{n+1} / J_{i+1}^{n+1}) \leq \lambda(I_i^{n+1} / J_i^{n+1}) - \lambda(I_i^n / J_i^n). \quad (3.7.1)$$

Suppose $\deg P(I/J) < s - 1$. Then using the inequality (3.7.1), for all $i = 0, \dots, s-2$, we get that $\lambda(I_{s-1}^n / J_{s-1}^n)$ is a polynomial type function of degree less than zero and hence there exists an integer k such that $I_{s-1}^n = J_{s-1}^n$ for all $n \geq k$.

We show that if $I_{s-1}^m = J_{s-1}^m$ for some integer m then $I_{s-1}^{m-1} = J_{s-1}^{m-1}$. Let $x' \in I_{s-1}^{m-1}$ where x' denotes the image of x in R_{s-1} . Then $x'a_s^{(s-1)} \in I_{s-1}^m = J_{s-1}^m$. Thus $x' \in J_{s-1}^m : a_s^{(s-1)} = J_{s-1}^{m-1}$. Using this technique $m-1$ times, we get $I_{s-1} = J_{s-1}$. This implies $I = J$ which is a contradiction. Therefore $\text{grade } G(J)_+ - 1 \leq \deg P(I/J)$.

(2) This follows from part (1) and Remark 3.1.

(3) since J is a complete intersection ideal, J is generated by a regular sequence and by Proposition 3.5, we have $\text{grade } G(J)_+ = \text{grade}(J) = \text{ht } J = l(J)$. Hence the result follows from part (2). \square

Example 3.8. Let $R = K[X, Y]_{(X, Y)}$ where K is a field. Let $J = (XY^2, X^4)$ and $I = (X^4, XY^2, X^3Y)$. Then $\lambda(I/J) < \infty$ and J is a reduction of I . Now $\text{grade } G(J)_+ \geq 1$, J is generated by a d -sequence. For all $n \gg 0$,

$$I^n = (X^{4n-1}Y, X^{4n-4}Y^3, X^{4n-7}Y^5, \dots, X^{4n-3n+2}Y^{2(n-1)+1}) + J^n$$

where $J^n = (X^{4n}, X^{4n-3}Y^2, X^{4n-6}Y^4, \dots, X^{4n-3n}Y^{2n})$. Then $\lambda(I^n/J^n)$ is a polynomial in n of degree one.

The following theorem provides a better lower bound for $\deg P(I/J)$ and as a consequence of that we obtain a class of ideals for which the polynomial $P(I/J)(X)$ attains the maximal degree.

Theorem 3.9. *Let (R, \mathfrak{m}) be a Noetherian local ring, $J \subsetneq I$ be ideals in R such that $\lambda(I/J) < \infty$ and $\text{grade}(J) = s$. Let $a_1, \dots, a_s, b_1, \dots, b_t$ be a d -sequence in R which minimally generates the ideal J and $t \geq 1$. Suppose $I \cap (J' : b_1) = J'$ where $J' = (a_1, \dots, a_s)$ ($J' = (0)$ if $s = 0$). Then the following are true.*

- (1) $\text{grade}(J) \leq \deg P(I/J)$.
- (2) If J is a reduction of I then $\text{grade}(J) \leq \deg P(I/J) \leq l(J) - 1$.

Proof. (1) (R_i, I_i, J_i) are defined before Lemma 3.2.

First we show that if $s = 0$ then $H_{\mathcal{R}(J)_+}^0(\mathcal{R}(I))_1 = 0$. Let $at \in H_{\mathcal{R}(J)_+}^0(\mathcal{R}(I))_1$. Then for some $m > 0$, we have $(b_1^m t^m)(at) = 0$ in $\mathcal{R}(I)$. Hence $ab_1^m = 0$. If $m \geq 2$, since b_1, \dots, b_t is a d -sequence in R , we have

$$ab_1^{m-2} \in ((0) : b_1^2) = ((0) : b_1).$$

Thus $ab_1^{m-1} = 0$. Using the same method $m-1$ -times, we get $ab_1 = 0$. Therefore $a \in I \cap ((0) : b_1) = (0)$.

Note that images of $a_i, \dots, a_s, b_1, \dots, b_t$ in R_{i-1} is a d -sequence for all $i = 1, \dots, s$. Now $\text{grade}(J) = s$ implies a_1, \dots, a_s is a regular sequence in R . Thus by Proposition 3.3, for all $i = 0, \dots, s-1$ and $n \gg 0$, we have

$$(J_i^{n+1} : (a_{i+1}^{(i)})) \cap I_i^n = J_i^n.$$

Hence by Lemma 3.2, for all $i = 0, \dots, s-1$ and $n \gg 0$, we have

$$\lambda(I_{i+1}^{n+1}/J_{i+1}^{n+1}) \leq \lambda(I_i^{n+1}/J_i^{n+1}) - \lambda(I_i^n/J_i^n). \quad (3.9.1)$$

Suppose $\deg P(I/J) \leq s-1$. Then using the inequality (3.9.1), for all $i = 0, \dots, s-1$, we get that $\lambda(I_s^n/J_s^n)$ is a polynomial type function of degree less than zero and hence there exists an integer k such that $I_s^n = J_s^n$ for all $n \geq k$.

Now replace R by R_s and I, J by I_s, J_s respectively. Hence $J \subsetneq I$, $J = (c_1, \dots, c_t)$ where c_1, \dots, c_t is a d -sequence in R (c_1, \dots, c_t are images of b_1, \dots, b_t in R_s respectively),

$\text{grade}(J) = 0$, $J' = (0)$ and $I \cap ((0) : c_1) = (0)$. Hence $H_{\mathcal{R}(J)_+}^0(\mathcal{R}(I))_1 = 0$. Consider the exact sequence of $\mathcal{R}(J)$ -modules

$$0 \longrightarrow \mathcal{R}(J) \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{R}(I)/\mathcal{R}(J) \longrightarrow 0$$

which induces a long exact sequence of local cohomology modules whose n -graded component is

$$\cdots \longrightarrow H_{\mathcal{R}(J)_+}^i(\mathcal{R}(I))_n \longrightarrow H_{\mathcal{R}(J)_+}^i(\mathcal{R}(I)/\mathcal{R}(J))_n \longrightarrow H_{\mathcal{R}(J)_+}^{i+1}(\mathcal{R}(J))_n \longrightarrow \cdots.$$

Consider the case $i = 0$ and $n = 1$. Since $I^n = J^n$ for all $n \gg 0$, $\mathcal{R}(I)/\mathcal{R}(J)$ is $\mathcal{R}(J)_+$ -torsion. Thus $H_{\mathcal{R}(J)_+}^0(\mathcal{R}(I)/\mathcal{R}(J))_1 = I/J$. Since J is generated by d -sequence, by Theorem [25, Corollary 5.2], $\text{reg}(\mathcal{R}(J)) = 0$ and hence $H_{\mathcal{R}(J)_+}^1(\mathcal{R}(J))_1 = 0$. Therefore from the long exact sequence, for $i = 0$ and $n = 1$, we get $I = J$ which is a contradiction. Hence $s \leq \deg P(I/J)$.

(2) Suppose J is a reduction of I . Then by part (1) and Remark 3.1, we get the required result. \square

Example 3.10. Let $R = K[X, Y, Z, W]_{(X, Y, Z, W)}$ where K is a field. Let

$$I = (XZ, XW, YZ, YW) \text{ and } J = (XZ, YW, XW + YZ).$$

Then J is a reduction of I and $\lambda(I/J) < \infty$. By computations on Macaulay2 [9], we get $\text{grade}(J) = 2$, $I \cap ((XZ, YW) : (XW + YZ)) = (XZ, YW)$ and $XZ, YW, XW + YZ$ is a d -sequence. Consider the ideals $\bar{J} = J/(XZ, YW)$ and $\bar{I} = I/(XZ, YW)$ in $\bar{R} = R/(XZ, YW)$. Let x, y, z, w denote images of X, Y, Z, W in \bar{R} . Then $\bar{J}^n = (x^n w^n + y^n z^n)$ and $\bar{I}^n = (x^n w^n, y^n z^n)$. Hence $\lambda(\bar{I}^n/\bar{J}^n)$ is nonzero constant for all $n \gg 0$. Thus $2 \leq \deg P(I/J)$. Since $l(J) = l(I) = \dim K[XZ, XW, YZ, YW] = 3$, we get $\deg P(I/J) = 2$.

Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. We say an ideal I satisfies G_s if $\mu(I_p) \leq \text{ht } p$ for every $p \in V(I)$ such that $\text{ht}(p) < s$. A proper ideal J is an s -residual intersection of I if there exist s elements x_1, \dots, x_s of I such that $J = (x_1, \dots, x_s) : I$ and $\text{ht } J \geq s$. We say J is a *geometric s -residual intersection* of I if in addition we have $\text{ht } I + J > s$. The ideal I is said to satisfy the *Artin-Nagata property AN_s^-* if the ring R/K is Cohen-Macaulay for every $0 \leq i \leq s$ and for every geometric i -residual intersection K of I .

Corollary 3.11. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field, I be an ideal of R and J be any minimal reduction of I with $\lambda(I/J) < \infty$. Suppose I satisfies $G_{l(I)}$ and $AN_{l(I)-2}^-$ conditions. Then for any ideal K with $J \subsetneq K \subseteq I$, $\deg P(K/J) = l(J) - 1$.*

Proof. By [26] and [14], $\text{grade}(J) \geq l(I) - 1$, $x_1, \dots, x_{l(I)}$ is a d -sequence and

$$K \cap ((x_1, \dots, x_{l(I)-1}) : x_{l(I)}) \subseteq I \cap ((x_1, \dots, x_{l(I)-1}) : x_{l(I)}) = (x_1, \dots, x_{l(I)-1}).$$

If $\text{grade}(J) = l(I)$ then by Theorem 3.7, $\deg P(K/J) = l(J) - 1$. Suppose $\text{grade}(J) = l(I) - 1$. Then by Theorem 3.9, we obtain $\deg P(K/J) = l(J) - 1$. \square

4. DEGREE OF THE MULTIPLICITY FUNCTION $e(I^n/J^n)$

Let (R, \mathfrak{m}) be a formally equidimensional local ring and $J \subsetneq I$ be ideals in R . Then by [2, Lemma 2.2], the chain of ideals

$$\sqrt{J : I} \subseteq \cdots \subseteq \sqrt{J^n : I^n} \subseteq \cdots$$

stabilizes. Choose an integer $r \geq 1$ such that $K := \sqrt{J^n : I^n} = \sqrt{J^{n+1} : I^{n+1}}$ for all $n \geq r$. Put $\dim R/K = t$. Then using associativity formula, for all $n \geq r$, we have

$$e(I^n/J^n) = \sum_{K \subseteq p, \dim R/p=t} e(R/p)\lambda(I_p^n/J_p^n). \quad (4.0.1)$$

By [2, Proposition 2.4], $e(I^n/J^n)$ is a function of polynomial type of degree at most $\dim R - t$.

Throughout this section we use the following notation: $K = \sqrt{J^n : I^n} = \sqrt{J^{n+1} : I^{n+1}}$ for all $n \geq r$, $t = \dim R/K$ and $S = \{p \in \text{Spec } R : K \subseteq p, \dim R/p = t\}$.

Next we show that if J is a complete intersection ideal we can explicitly detect the $\deg e(I^n/J^n)$.

Proposition 4.1. *Let (R, \mathfrak{m}) be a formally equidimensional local ring, $J \subsetneq I$ be ideals in R .*

- (1) *Then $\text{grade } G(J)_+ - 1 \leq \deg e(I^n/J^n) \leq l(J)$.*
- (2) *If J is a reduction of I then $\text{grade } G(J)_+ - 1 \leq \deg e(I^n/J^n) \leq l(J) - 1$.*
- (3) *If J is a complete intersection ideal then*
 - (a) *J is a reduction of I implies $\deg e(I^n/J^n) = l(J) - 1$.*
 - (b) *J is not a reduction of I implies $\deg e(I^n/J^n) = l(J)$.*

Proof. (1) (K, S are defined at the beginning of the Section 3). Suppose J_p is reduction of I_p for all $p \in S$ then by Theorem 3.7, for all $p \in S$,

$$\text{grade } G(J)_+ - 1 \leq \text{grade } G(J_p)_+ - 1 \leq \deg P(I_p^n/J_p^n) \leq l(J_p) - 1 \leq l(J) - 1.$$

Hence from the equation (4.0.1), $\text{grade } G(J)_+ - 1 \leq \deg e(I^n/J^n) \leq l(J) - 1$.

Suppose J_q is not a reduction of I_q for some $q \in S$. Therefore J is not a reduction of I and by [8, Proposition 3.6.3], for all $n \gg 0$, we have $\text{ht } K \leq \text{ht } \sqrt{JI^{n-1} : I^n} \leq l(J)$. Then by [22, Theorem 2.1], $\text{ht } J \leq \deg P(I_q/J_q) = \text{ht } q \leq l(J)$. Therefore the result follows from the equation (4.0.1).

(2) If J is a reduction of I then J_p is reduction of I_p for all $p \in S$ and the result follows from the argument in (1).

(3) Since J is a complete intersection ideal, we have $\text{grade}(J) = l(J)$. Therefore by Proposition 3.5 and part (2), we get that J is a reduction of I implies $\deg e(I^n/J^n) = l(J) - 1$. If J is not a reduction of I , then the result follows from [3, Theorem 2.6]. \square

Lemma 4.2. *Let (R, \mathfrak{m}) be a formally equidimensional local ring, $J \subsetneq I$ be ideals in R . Suppose there exists a prime ideal $Q \supseteq J$ such that $\dim R/J = \dim R/Q$ and $I \not\subseteq Q$. Then $\deg e(I^n/J^n) = \text{ht } J$. In particular, if J is unmixed (i.e. all associated primes of J have same height) and $\sqrt{J} \subsetneq \sqrt{I}$, then $\deg e(I^n/J^n) = \text{ht } J$.*

Proof. Let $\text{Min}(R/K) = \{Q_1, \dots, Q_k\}$ (K is defined at the beginning of the Section 3). Put $T = Q_1 \cdots Q_k$. Since $K = \sqrt{J^r : I^r}$, there exists an integer $c > 0$ such that $T^c I^r \subseteq J^r$. Let $J = \bigcap_{P \in \text{Ass}(R/J)} q(P)$ be an irredundant primary decomposition of J . Then

$$I^r \subseteq J : T^\infty \subseteq \bigcap_{P \not\supseteq T, P \in \text{Ass}(R/J)} q(P).$$

Since $I \not\subseteq Q$, we have $T \subseteq Q$. Therefore $Q_j \subseteq Q$ for some j and hence $Q = Q_j$. Thus J_{Q_j} is not a reduction of $I_{Q_j} = R_{Q_j}$ and $\dim R/Q = \dim R/K$. Hence by [22, Theorem 2.1] and the equation (4.0.1), we get $\deg e(I^n/J^n) = \deg \lambda(I_{Q_j}^n/J_{Q_j}^n) = \dim R_{Q_j} = \text{ht } J$. \square

We provide some sufficient conditions on J such that for any $J \subsetneq I$, we have $\sqrt{J : I} = \sqrt{J^n : I^n}$ for all $n \geq 1$.

Proposition 4.3. *Let (R, \mathfrak{m}) be a Noetherian local ring and $J \subsetneq I$ be ideals in R such that one of the following conditions hold,*

- (1) *the residue field of R is infinite and $\text{grade } G(J)_+ \geq 1$,*
- (2) *$\text{grade}(J) \geq 1$ and J is generated by d -sequence,*
- (3) *$J^{k+1} : J = J^k$ for all $k \geq 0$.*

Then $\sqrt{J : I} = \sqrt{J^n : I^n}$ for all $n \geq 1$.

Proof. By [2, Lemma 2.2], we know $\sqrt{J : I} \subseteq \sqrt{J^n : I^n}$ for all $n \geq 1$. Suppose $x \in \sqrt{J^n : I^n}$ for any $n > 1$. Then $x^r I^n \subseteq J^n$ for some $r \geq 1$.

First consider the case that $\text{grade } G(J)_+ \geq 1$ and the residue field of R is infinite. Then there exists an element $y \in J$ such that $J^n : (y) = J^{n-1}$ for all $n \geq 1$. Then $x^r y I^{n-1} \subseteq x^r I I^{n-1} \subseteq J^n$ implies $x^r I^{n-1} \subseteq J^n : (y) = J^{n-1}$. Using this technique $n - 1$ times we get $x^r I \subseteq J$.

Note that in condition (1), we require infinite residue field just to get a $G(J)$ -regular element of degree 1. If $\text{grade}(J) \geq 1$ and J is generated by a d -sequence, say $J = (a_1, \dots, a_s)$, by Propositions 3.3 and 3.5, we get a $G(J)$ -regular element of degree 1. Hence the result follows from the previous paragraph.

If (3) holds then for all $n \geq 1$, $x^r J I^{n-1} \subseteq x^r I I^{n-1} \subseteq J^n$ implies $x^r I^{n-1} \subseteq J^n : J = J^{n-1}$. \square

The following theorem gives characterization of reduction in terms of $\deg e(I^n/J^n)$.

Theorem 4.4. *Let (R, \mathfrak{m}) be a formally equidimensional local ring of dimension $d \geq 2$, $J \subsetneq I$ be ideals in R and J has analytic deviation one. Suppose $l(J_p) < l(J)$ for all prime ideals p in R such that $\text{ht } p = l(J)$. Then the following are true.*

- (1) *If J is not a reduction of I then $\deg e(I^n/J^n) = l(J) - 1$.*
- (2) *If $l(J) = d - 1$, $\text{depth}(R/J) > 0$ and for all $n \geq 1$, $\sqrt{J : I} = \sqrt{J^n : I^n}$ then J is a reduction of I if and only if $\deg e(I^n/J^n) \leq l(J) - 2$.*

Proof. (Note that the condition $l(J_p) < l(J)$ for all prime ideals p in R such that $\text{ht } p = l(J)$ implies $l(J) \leq d - 1$.) First we show that if $\text{ht } K = l(J)$ then J is a reduction of I (where $K = \sqrt{J^r : I^r}$, defined at the beginning of the section 3). Without loss of generality we may assume that $\text{Ass}(R/\overline{J^n}) = \text{Ass}(R/\overline{J^{n+1}})$ for all $n \geq r$. Let $q \in \text{Ass}(R/\overline{J^r})$. We show that $(J^r : I^r) \not\subseteq q$. If possible suppose $(J^r : I^r) \subseteq q$. Then either $\text{ht } q > l(J) \geq l(J_q) = l(J_q^r)$ or $\text{ht } q = l(J) > l(J_q) = l(J_q^r)$. Therefore by [18], [22, Lemma 3.1], $q \notin \text{Ass}(R/\overline{J^{nr}}) = \text{Ass}(R/\overline{J^r})$ for any $n \geq 1$, which is a contradiction. Thus by prime avoidance lemma, we have an element

$$x \in (J^r : I^r) \setminus \bigcup_{Q \in \text{Ass}(R/\overline{J^r})} Q.$$

Therefore x' is a nonzerodivisor in $R/\overline{J^r}$ where \prime denotes the image in $R/\overline{J^r}$, i.e. $\overline{J^r} : (x) = \overline{J^r}$. This implies $I^r \subseteq J^r : (x) \subseteq \overline{J^r} : (x) = \overline{J^r}$. Thus J^r is a reduction of I^r and hence J is a reduction of I .

(1) Since J is not a reduction of I , by [8, Proposition 3.6.3] and the previous paragraph, $\text{ht } K = \text{ht } J$ (where $K = \sqrt{J^r : I^r}$, defined at the beginning of the section 3). Now J is not a reduction of I implies J^r is not a reduction of I^r . Therefore, by [18],[23],[27, Theorem 8.21], we have J_p^r is not a reduction of I_p^r for some prime ideal p such that $K \subset p$ and

$\text{ht } p = l(J_p)$.

Let $K \subset p \in \text{Spec}(R)$ such that $\text{ht } p \geq l(J) = \text{ht } K + 1$. Then either $\text{ht } p > l(J) \geq l(J_p)$ or $\text{ht } p = l(J) > l(J_p)$. Therefore J_p^r is not a reduction of I_p^r for some prime ideal $p \in S$ (S is defined at the beginning of the section 3). Hence J_p is not a reduction of I_p for some prime ideal $p \in S$ and thus by [22, Theorem 2.1], $\deg e(I^n/J^n) = \text{ht } p = \text{ht } J = l(J) - 1$.

(2) It is enough to show that if J is a reduction of I then $\deg e(I^n/J^n) \leq l(J) - 2$. First we show that if J is a reduction of I then $\text{ht } K \leq l(J)$. If $\text{ht } K > l(J)$ then K is \mathfrak{m} -primary and hence there exists an integer $n > 0$ such that $\mathfrak{m}^n I \subseteq J$. Since $\mathfrak{m} \notin \text{Ass}(R/J)$, we have $I \subseteq J$ which is a contradiction. Suppose $\text{ht } K = \text{ht } J = l(J) - 1$. Since J is a reduction of I , I_p, J_p are pR_p -primary for all $p \in S$ and hence we get $\deg e(I^n/J^n) \leq l(J) - 2$. Now suppose $\text{ht } K = l(J)$. Since J is a reduction of I , for any $p \in S$, we have $\text{ht } p = l(J)$ and hence $\deg P(I_p/J_p) \leq l(J_p) - 1 \leq l(J) - 2$. Thus $\deg e(I^n/J^n) \leq l(J) - 2$. \square

Example 4.5. We can not omit the condition $\text{depth}(R/J) > 0$ from Theorem 4.4 (2).

Let $R = \mathbb{Q}[x, y, z, w]_{(x, y, z, w)}$ and $J = (xz, yw, xw + yz)$. Then $3 = l(J) = \text{ht } J + 1$, $\mathfrak{m} \in \text{Ass}(R/J)$ and $l(J_p) \leq l(J) - 1$ for all prime ideals p in R such that $\text{ht } p = l(J)$. Let $I = (xz, xw, yz, yw)$. Then J is a reduction of I and $\sqrt{J : I} = (x, y, z, w)$. Since $xz, yw, xw + yz$ is a d -sequence, by Proposition 4.3, $\sqrt{J : I} = \sqrt{J^n : I^n}$ for all $n \geq 1$. Then $\deg e(I^n/J^n) = \deg \lambda(I^n/J^n) = 2$.

Example 4.6. This example shows existence of an ideal J which satisfies conditions of Theorem 4.4. Let $R = \mathbb{Q}[x, y, z, w]_{(x, y, z, w)}$ and $J = (xyw^2, xyz^2, xw^2 + yz^2)$. Then by computations on Macaulay2 [9], we get $l(J) = 3$, J is an ideal of linear type (hence basic) and $\text{Ass}(R/J) = \{p_1 = (w, z), p_2 = (w, y), p_3 = (x, z), p_4 = (x, y)\}$. Since $xyzw \in R \setminus J$ is integral over J , J is not integrally closed. Note that $J_{p_1} = (w^2, z^2)R_{p_1}$, $J_{p_2} = (y, w^2)R_{p_2}$, $J_{p_3} = (x, z^2)R_{p_3}$, $J_{p_4} = (xy, xw^2 + yz^2)R_{p_4}$. Hence J is generically complete intersection. Therefore by [4, Lemma 2.5], $\text{grade } G(J)_+ \geq 1$. Thus by Proposition 4.3, $\sqrt{J : I} = \sqrt{J^n : I^n}$ for all $n \geq 1$ and any ideal I containing J . Suppose p is a prime ideal in R of height 3. Then either $x \notin p$ (hence $J_p = (yz^2, w^2)R_p$) or $y \notin p$ (hence $J_p = (xw^2, z^2)R_p$) or $z \notin p$ (hence $J_p = (xy, xw^2 + yz^2)R_p$) or $w \notin p$ (hence $J_p = (xy, xw^2 + yz^2)R_p$). Thus $l(J_p) \leq l(J) - 1$.

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