PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

REPRESENTATION OF A REAL B^*-ALGEBRA ON A QUATERNIONIC HILBERT SPACE

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let A be a real B^*-algebra containing a $*$-subalgebra that is $*$-isomorphic to the real quaternion algebra $\mathbb{H}$. Suppose the spectrum of every self-adjoint element in A is contained in the real line. Then it is proved that there exists a quaternionic Hilbert space X and an isometric $*$-isomorphism π of A onto a closed $*$-subalgebra of $BL(X)$, the algebra of all bounded linear operators on X. If, in addition to the above hypotheses, every element in A is normal, then A is also proved to be isometrically $*$-isomorphic to $C(Y, \mathbb{H})$, the algebra of all continuous $\mathbb{H}$-valued functions on a compact Hausdorff space Y.

INTRODUCTION

Let A be a real B^*-algebra containing a $*$-subalgebra that is $*$-isomorphic to the real quaternion algebra. Soffer and Horwitz [7] have shown that the Gelfand-Naimark-Segal (GNS) construction can be generalized to such an algebra by making use of quaternion linear states. This construction was then used to obtain a representation of A as an algebra of operators on a quaternionic Hilbert space (module). However, they seem to have overlooked the possibility that there may not be sufficiently many quaternion linear states on A. We give an example of an algebra A satisfying the above properties which does not have a separating family of quaternion linear states. Thus the claim of Soffer and Horwitz that it is possible to construct a separating family of quaternion linear states and hence the weakest topology for which all these functionals are continuous is Hausdorff, is incorrect. (See Corollary 4.1 of [7].) It is necessary to have a separating family of states to obtain a faithful representation.

We show that this situation can be remedied by making an additional hypothesis, namely that the spectrum of every selfadjoint element of A is contained in the real line. The necessity of this hypothesis was recognized by Palmer and others while developing analogues of the classical Gelfand-Naimark theorem to a real B^*-algebra [2, 4, 5, 6]. A $*$-algebra satisfying this condition is called *hermitian*. Every complex B^*-algebra is hermitian. A real B^*-algebra is a C^*-algebra if and only if it is hermitian. Proofs of all these facts can be found

Received by the editors September 20, 1992.
1991 *Mathematics Subject Classification*. Primary 46L05.

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in [2, 5, 6]. Reference [10] contains some equivalent conditions guaranteeing hermitianness.

We further show that, once this hypothesis is made, the conclusions of Soffer and Horwitz can be derived more directly from the above analogue of the Gelfand-Naimark theorem. This is achieved by showing that the real Hilbert space obtained by this theorem can be converted into (regarded as) a quaternionic Hilbert space by defining a suitable quaternionic inner product on it.

We refer the reader to the paper of Soffer and Horwitz [7] for an excellent introduction highlighting the relevance of quaternionic Hilbert spaces (and representation on such spaces) to quantum mechanics and in particular to quantum field theories with nonabelian gauge fields. The paper also contains a selected list of references on this topic.

Preliminaries

$\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ stand for the set of all real numbers, the set of all complex numbers, and the set of all real quaternions respectively. Let A be a real algebra with a unit element 1 and $a \in A$. The *spectrum of a in A* is defined by

$$\mathrm{Sp}(a, A) := \{s + it \in \mathbb{C}\colon (s-a)^2 + t^2 \text{ is singular in } A\}.$$

An *involution* $*$ *on a real algebra* A is a map $a \to a^*$ such that for all $a, b \in A$ and $s \in \mathbb{R}$, (i) $(a+b)^* = a^* + b^*$, (ii) $(sa)^* = sa^*$, (iii) $(ab)^* = b^*a^*$, and (iv) $(a^*)^* = a$. A $*$-algebra is an algebra A with an involution $*$. A subalgebra (of a $*$-algebra) that is closed under the involution $*$ is called a $*$-subalgebra. A homomorphism ϕ (respectively, isomorphism) of A to a $*$-algebra B is called a $*$-homomorphism (respectively $*$-isomorphism) if $\phi(a^*) = (\phi(a))^*$ for all $a \in A$. A B^*-algebra is a Banach algebra A with an involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$.

For $q = q_0 + q_i i + q_2 j + q_3 k \in \mathbb{H}$, q^* is defined as

$$q^* = q_0 - q_i i - q_2 j - q_3 k \quad \text{and} \quad |q|^2 = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}.$$

Note that $|q|^2 = q^*q = qq^*$. Thus $\mathbb{H}$ is a real B^*-algebra.

Definiton 1 [7]. Let A be a real B^*-algebra containing a $*$-subalgebra B that is $*$-isomorphic to the real quaternion algebra $\mathbb{H}$. We shall not distinguish between the elements of B and $\mathbb{H}$. A linear mapping $\rho\colon A \to \mathbb{H}$ is called a *two-sided quaternion linear functional* (relative to B) if $\rho(qaq') = q\rho(a)q'$ for all $a \in A$, $q, q' \in \mathbb{H}$. (Note that $\mathbb{H}$ is identified with B.) A two-sided quaternion linear functional ρ is called *positive* if $\rho(a^*a) \geq 0$ for all $a \in A$. A *state* is a positive two-sided quaternion linear functional ρ satisfying $\rho(1) = 1$.

Example 2. Let $A = \mathbb{H} \times \mathbb{C}$. Define the algebraic operations componentwise, and for $(q, z) \in A$ define $(q, z)^* = (q^*, z)$ and $\|(q, z)\| = \max\{|q|, |z|\}$. Then A is a real B^*-algebra and it contains a $*$-subalgebra $\mathbb{H} \times \{0\}$ that is $*$-isomorphic to $\mathbb{H}$. Now let ρ be a state on A. Then

$$\rho(0, 1) = \rho((0, 1)(0, 1)) = \rho((0, 1)^*(0, 1)) \geq 0$$

and

$$-\rho(0, 1) = \rho(0, -1) = \rho((0, i)^*(0, i)) \geq 0.$$

Thus $\rho(0, 1) = 0$. This shows that the elements $(0, 1)$ and $(0, 0)$ are not separated by any state on A. Thus there is no separating family of states on A. Note that A is not hermitian.

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We now show that with the additional hypothesis that A is hermitian, it is possible to obtain a representation of A on a quaternionic Hilbert space. We include a definition of a quaternionic Hilbert space, for the sake of completeness.

Definition 3. A real vector space X is said to be a (left) *quaternionic vector space* if it is a left $\mathbb{H}$ module, that is, there is a mapping $(q, x) \to qx$ of $\mathbb{H} \times X$ into X satisfying

1. $(q_1 + q_2)x = q_1x + q_2x$ for each $q_1, q_2 \in \mathbb{H}$, $x \in X$.
2. $q(x_1 + x_2) = qx_1 + qx_2$ for all $q \in \mathbb{H}$, $x_1, x_2 \in X$.
3. $q_1(q_2x) = (q_1q_2)x$ for all $q_1, q_2 \in \mathbb{H}$, $x \in X$.

A *quaternionic inner product* on a quaternionic vector space X is a mapping $\langle\ ,\ \rangle: X \times X \to \mathbb{H}$, satisfying

4. $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
5. $\langle y, x\rangle = \langle x, y\rangle^*$ for all $x, y \in X$.
6. $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$ for all $x_1, x_2, y \in X$.
7. $\langle qx, y\rangle = q\langle x, y\rangle$ for all $q \in \mathbb{H}$, $x, y \in X$.

A *quaternionic inner product space* is a quaternionic vector space X with a quaternionic inner product $\langle\ ,\ \rangle$ defined on it. For such a space X, $\|x\| := \langle x, x\rangle^{1/2}$ is a norm on X that makes X a real normed linear space and $\|qx\| = |q|\,\|x\|$ for all $q \in \mathbb{H}$, $x \in X$. In particular, X is a normed left $\mathbb{H}$-module. If X is complete with respect to this norm, it is called a *quaternionic Hilbert space*.

The theory of a quaternionic Hilbert space has been developed in a similar fashion to that of a real or a complex Hilbert space. (See [7, 9] and the references listed therein.) In [7] a quaternionic Hilbert space is called a quaternionic Hilbert module. We have avoided this nomenclature, as the term "Hilbert module" has been used elsewhere with a different meaning. (See, for example, [1].)

We are now in a position to present our main theorem.

Theorem 4. *Let A be a real B^*-algebra containing a $*$-subalgebra that is $*$-isomorphic to $\mathbb{H}$ and suppose $Sp(a, A) \subset \mathbb{R}$ for all $a \in A$ with $a^* = a$. Then there exist a quaternionic Hilbert space X and an isometric $*$-isomorphism π of A onto a closed $*$-subalgebra of $BL(X)$, the algebra of all bounded (real) linear operators on X.*

Proof. Since A is a real B^*-algebra and $Sp(a, A) \subset \mathbb{R}$ whenever $a = a^*$, it follows by a theorem of Palmer that there exist a real Hilbert space X and an isometric $*$-isomorphism π of A onto a closed $*$-subalgebra of $BL(X)$. (See [2, 5, 6] for a proof of this.) We shall show that X is, in fact, a quaternionic Hilbert space when A contains a $*$-subalgebra that is $*$-isomorphic to $\mathbb{H}$. It is easy to see that this $*$-isomorphism between $\mathbb{H}$ and the subalgebra of A is also an isometry. (See Lemma 2.1 of [7].) Hence, identifying $\mathbb{H}$ with this $*$-subalgebra, we may regard $\mathbb{H}$ as a subset of A. Thus for $q \in \mathbb{H} \subset A$, $\pi(q) \in BL(X)$. For $q \in \mathbb{H}$ and $x \in X$, define

$$qx := \pi(q)x. \tag{1}$$

Since $\pi(q) \in BL(X)$, we have, for $x_1, x_2 \in X$

$$\pi(q)(x_1 + x_2) = \pi(q)x_1 + \pi(q)x_2.$$

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Also, since π is an isomorphism,

$$\pi(q_1 + q_2) = \pi(q_1) + \pi(q_2) \quad \text{and} \quad \pi(q_1 q_2) = \pi(q_1)\pi(q_2).$$

Thus, Properties 1, 2, and 3 of Definition 3 are satisfied and X is a quaternionic vector space. We now define a quaternionic inner product on X as follows.

Let $\langle\ ,\ \rangle$ denote the original real inner product on X. Define a map $\langle\ ,\ \rangle_{\mathbb{H}}: X \times X \to \mathbb{H}$ by

$$\langle x, y\rangle_{\mathbb{H}} := \langle x, y\rangle + i\langle x, \pi(i)y\rangle + j\langle x, \pi(j)y\rangle + k\langle x, \pi(k)y\rangle \tag{2}$$

for $x, y \in X$.

We shall use the properties of the real inner product $\langle\ ,\ \rangle$ and the $*$-isomorphism π to show that $\langle x, y\rangle_{\mathbb{H}}$ is a quaternioinic inner product. First note that in view of (1), (2) can be written as

$$\langle x, y\rangle_{\mathbb{H}} := \langle x, y\rangle + i\langle x, iy\rangle + j\langle x, jy\rangle + k\langle x, ky\rangle. \tag{3}$$

For $q \in \mathbb{H}$ and $x, y \in X$ we have

$$\langle qx, y\rangle := \langle \pi(q)x, y\rangle = \langle x, \pi(q)^* y\rangle = \langle x, \pi(q^*)y\rangle = \langle x, q^* y\rangle. \tag{4}$$

In particular, if $q^* = -q$, then

$$\langle qx, y\rangle = -\langle x, qy\rangle \quad \text{for all } x, y \in X, \tag{5}$$

and hence

$$\langle qx, x\rangle = 0 \quad \text{for all } x \in X. \tag{6}$$

Now, using (6) for $q = i, j, k$, we get from (3)

$$\langle x, x\rangle_{\mathbb{H}} = \langle x, x\rangle \quad \text{for all } x \in X. \tag{7}$$

Hence

$$\langle x, x\rangle_{\mathbb{H}} \geq 0 \quad \text{for all } x \in X \qquad \text{and} \qquad \langle x, x\rangle_{\mathbb{H}} = 0 \quad \text{iff } x = 0. \tag{8}$$

Next using (5) for $q = i, j, k$, we get, for $x, y \in X$

$$\begin{aligned} \langle y, x\rangle_{\mathbb{H}} &:= \langle y, x\rangle + i\langle y, ix\rangle + j\langle y, jx\rangle + k\langle y, kx\rangle \\ &= \langle y, x\rangle - i\langle iy, x\rangle - j\langle jy, x\rangle - k\langle ky, x\rangle = (\langle x, y\rangle_{\mathbb{H}})^*. \end{aligned} \tag{9}$$

It is straightforward to check that

$$\langle x_1 + x_2, y\rangle_{\mathbb{H}} = \langle x_1, y\rangle_{\mathbb{H}} + \langle x_2, y\rangle_{\mathbb{H}} \quad \text{for all } x_1, x_2, y \in X. \tag{10}$$

Further, for $x, y \in X$

$$\begin{aligned} \langle ix, y\rangle_{\mathbb{H}} &:= \langle ix, y\rangle + i\langle ix, iy\rangle + j\langle ix, jy\rangle + k\langle ix, ky\rangle \\ &= -\langle x, iy\rangle - i\langle x, i^2 y\rangle - j\langle x, ijy\rangle - k\langle x, iky\rangle \quad \text{(by (5))}, \\ &= -\langle x, iy\rangle + i\langle x, y\rangle - j\langle x, ky\rangle + k\langle x, jy\rangle = i\langle x, y\rangle_{\mathbb{H}}. \end{aligned} \tag{11}$$

Similarly,

$$\langle jx, y\rangle_{\mathbb{H}} = j\langle x, y\rangle_{\mathbb{H}} \quad \text{and} \quad \langle kx, y\rangle_{\mathbb{H}} = k\langle x, y\rangle_{\mathbb{H}} \qquad \text{for all } x, y \in X. \tag{12}$$

Hence for any $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, it follows from (11) and (12) that

$$\langle qx, y\rangle_{\mathbb{H}} = q\langle x, y\rangle_{\mathbb{H}} \quad \text{for all } x, y \in \mathbb{H}. \tag{13}$$

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Equations (8), (9), (10), and (13) imply that $\langle \ , \ \rangle_{\mathbb{H}}$ is a quaternionic inner product on X. Further, (7) shows that the norm induced by the quaternionic inner product $\langle \ , \ \rangle_{\mathbb{H}}$ on X coincides with the norm induced by the original real inner product $\langle \ , \ \rangle$. Since X is complete with respect to this norm, it is a quaternionic Hilbert space. □

Remark 5. Note that in the course of the proof of Theorem 4, we have proved the following: Let A be a real B^*-algebra containing a $*$-subalgebra that is $*$-isomorphic to $\mathbb{H}$. If there exists a real Hilbert space X and a $*$-homomorphism π of A into $BL(X)$, then X is a quaternionic Hilbert space.

Corollary 6. *If, in addition to the hypotheses in Theorem* 4, *every element in A is normal, then A is isometrically $*$-isomorphic to $C(Y, \mathbb{H})$, the algebra of all continuous $\mathbb{H}$-valued functions on a compact Hausdorff space Y.*

Proof. Since every element in A is normal, by Theorem 3 of [4], A is isometrically $*$-isomorphic to a closed $*$-subalgebra $\widehat{A}$ of $C(Y, \mathbb{H})$ for some compact Hausdorff space Y. Since $\widehat{A}$ contains a $*$-subalgebra that is $*$-isomorphic to $\mathbb{H}$, a straightforward application of the real Stone-Weierstrass theorem shows that $\widehat{A} = C(Y, \mathbb{H})$. (See [8] for details.) □

Remark 7. Just as Theorem 4 (which gives a representation of A on a quaternionic Hilbert space) is an analogue of the general (real or complex) Gelfand-Naimark theorems (which give a representation of A on a real or complex Hilbert space), Corollary 6 (which gives a representation of A as an algebra of quaternion-valued functions on a compact Hausdorff space) is an analogue of commutative Gelfand-Naimark Theorems (which give a representation of A as an algebra of complex-valued functions on a compact Hausdorff space).

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