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Scaling and fractal formation in Persistence

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Abstract

The spatial distribution of unvisited/persistent sites in $d = 1$ $A{+}A \to \emptyset$ model is studied numerically. Over length scales smaller than a cut-off $\xi(t) \sim t^z$, the set of unvisited sites is found to be a fractal. The fractal dimension d_f, dynamical exponent z and persistence exponent θ are related through $z(1 - d_f) = \theta$. The observed values of d_f and z are found to be sensitive to the initial density of particles. We argue that this may be due to the existence of two competing length scales, and discuss the possibility of a crossover at late times.

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Persistence properties of spatially extended systems undergoing time evolution has attracted a lot of attention of late. Generically there is a stochastic field $\phi(\mathbf{x},t)$ at each lattice site $\mathbf{x}$, which evolves with time t through interactions with other (usually nearest neighbour) sites. One quantity of interest in the present context is the persistence probability at time t, which is defined as the fraction $P(t)$ of sites in which the stochastic field $\phi(\mathbf{x},t)$ did not change sign in the time interval [0,t]. In a large number of cases, it is found that $P(t) \sim t^{-\theta}$ [1]. The new exponent θ, called persistence exponent, is, in general, unrelated to other known static and dynamic exponents.

The non-trivial nature of θ can be attributed to the interactions between neighbouring sites, which makes the effective stochastic process at any single site non-Markovian. Suppose $\phi(\mathbf{x},t)$ flips sign at time t. This event will increase the chance of neighbouring sites also flipping sign at subsequent times $t' > t$. This leads to the growth of spatial correlations in the system, which die out with increasing separation, on account of the statistical independence of distant flips. The non-Markovian nature of the process is more directly captured in these spatial correlations. Although much effort has been expended for calculation of the persistence exponent θ by exact [2] and approximate [3] methods, little has been done to investigate the associated spatial correlations in the process. In this paper, we undertake such a study in $d = 1$, where the correlations are expected to be most pronounced.

In one dimension, the zeroes of the stochastic field can be viewed as a set of particles, moving about in the lattice, annihilating each other when two of them meet. When a particle moves across a lattice site for the first time, the field there flips sign, and the site becomes non-persistent. If each particle is assumed to perform purely diffusive motion, this reduces to the well-known reaction-diffusion model $A + A \to \emptyset$ [4], with appropriate initial conditions. The simplest case is random initial distribution of particles, with average density n_0, for which $P(t) \sim t^{-\theta}$ with $\theta = 3/8$ [2], independent of n_0 [5]. We investigate spatial correlations in persistence for this simple model. We start with the two-point correlator $C(r,t)$, which is defined as the probability that site $\mathbf{x}+\mathbf{r}$ is persistent, given that site $\mathbf{x}$ is persistent (averaged over $\mathbf{x}$). We define $\rho(\mathbf{x},t)$ as the density of persistent sites: ie., $\rho(\mathbf{x},t) = 1$ if site

$\mathbf{x}$ is persistent at time t, and 0 otherwise. Then, with the previous definition, the expression for $C(r,t)$ is as follows.

$$C(r,t) =< \rho(\mathbf{x},t) >^{-1}< \rho(\mathbf{x},t)\rho(\mathbf{x}+\mathbf{r},t) > \quad (1)$$

where the brackets denote average over the entire lattice and $< \rho(\mathbf{x},t) >= P(t)$.

Our main results are the following. Strong spatial correlations exist in the distribution of persistent sites, with a cut-off length scale $\xi(t)$ separating correlated and uncorrelated regions. At late times t (ie., in the *scaling regime*), this length scale grows as a power of time: $\xi(t) \sim t^z$, where z is the dynamical exponent in this context [6]. In the correlated region $r \ll \xi(t)$, the correlator shows a *power law decay with distance*: $C(r,t) \sim r^{-\alpha}$. The scale-invariant behaviour, indicative of strong correlations, shows that the set of persistent sites is a self-similar fractal with dimension $d_f = 1 - \alpha$. By consistency, the exponents are related as $z\alpha = \theta$. We have analyzed the fractal structure by box-counting method also. Careful measurements of the exponents over several decades of MC time show that the observed values of α and z change with initial density n_0 while satisfying the above scaling relation.

We did our numerical simulation on a 1-d lattice of size $N = 10^5$, with periodic boundary conditions. Particles are initially distributed at random on the lattice with average density n_0, and their positions are sequentially updated— each particle was made to move one step in either direction with equal probability ($D = 1/2$). Whenever such a move resulted in two particles occupying the same position, both are removed from the lattice before moving to the next particle. The starting density of persistent sites is $P(0) = 1 - n_0$, and a persistent site becomes non-persistent when it is occupied by a particle for the first time. The time evolution is done up to 10^5 Monte-Carlo steps (1 MC step is counted after all the particles in the lattice were touched once). These time and lattice scales are the largest possible within our computational resources. We repeated our simulations for a few values of starting density n_0. The results were averaged over 50 different initial realisations.

For distances $r \gg 1$ and late times t, we find that $C(r,t) \sim r^{-\alpha}$ for $r \ll \xi(t)$. In the other extreme of large separations, ie., $r \gg \xi(t)$, the sites are uncorrelated so that $C(r,t) = P(t) \sim t^{-\theta}$, independent of r. Thus $\xi(t)$ is the correlation length for persistence, and consistency demands $\xi(t)^{-\alpha} \sim t^{-\theta}$. This implies a power-law divergence: $\xi(t) \sim t^z$ with a dynamical exponent z related to α and θ through the scaling relation

$$z\alpha = \theta \tag{2}$$

The observed behaviour of $C(r,t)$ can be summarised in the following dynamic scaling form.

$$C(r,t) = P(t) f(r/\xi(t)) \tag{3}$$

with the scaling function $f(x) \sim x^{-\alpha}$ as $x \ll 1$ and $f(x) \simeq 1$ for $x \gg 1$. In Fig.1, the scaling function $f(x) = C(r,t)/P(t)$ is plotted against the scaled distance $x = r/t^z$ for two values of time separated by a decade. The initial density is $n_0 = 0.5$. Excellent data collapse is obtained for $z = 1/2$, and the measured value of the spatial exponent $\alpha \simeq 3/4$ is entirely in accordance with the scaling relation.

The observed power-law decay of $C(r,t)$ with r has a wider significance, apart from showing the strong spatial correlations in the distribution. It implies that, over length scales not too large, the underlying structure is a self-similar fractal. This is most easily seen with the 'box-counting' procedure [7]. We divide the enire lattice into boxes of size l, at time t. After discarding 'empty' boxes, ie., those which contain not even a single persistent site, let $M(l,t)$ be the average number of persistent sites in a box of length l. This quantity is related to $C(r,t)$ through $M(l,t) = \int_0^l C(r,t)dr$. Substituting the scaling form Eq. 3 for $C(r,t)$, one finds

$$M(l,t) \sim l^{1-\alpha} \qquad l \ll \xi(t) \tag{4}$$

$$M(l,t) = lP(t) \qquad l \gg \xi(t) \tag{5}$$

which can be summarised in the scaling form

$$M(l,t) = lP(t)h(l/\xi(t)) \tag{6}$$

with the scaling function $h(x) \sim x^{-\alpha}$ for $x \ll 1$ and $h(x) \simeq 1$ for $x \gg 1$. We see that over small enough length-scales $l \ll \xi(t)$, the set of persistent sites form a self-similar fractal with fractal dimension $d_f = 1 - \alpha$, with a crossover to homogeneous behaviour at larger length scales. This crossover is illustrated in Fig.2, where we have $M(l,t)$ (measured from box-counting) plotted against the box size l for three values of time. The initial density here is $n_0 = 0.5$, and we find $d_f \simeq 0.25$ in agreement with our result from study of the two-point correlation $C(r,t)$.

In Fig.3, we compare the results from box-counting for different starting densities. For $n_0 = 0.2$, we see that the fractal region appears much later compared to higher values. This is presumably due to the large inter-particle separation at $t = 0$, and the consequent delay in reaching the scaling regime. For higher densities, the fractal dimension is seen to decrease continously with n_0, approaching zero in the limit $n_0 \to 1$. We notice that although $\xi(t) \sim 10^3$ in terms of the lattice spacing, it is still much less than the lattice size N, so as to rule out finite-size effects.

In Fig.4, we plot the scaling function $h(\eta) = M(l,t)/lP(t)$ against the scaling variable $\eta = l/t^z$ for two values of time separated by a decade. We have displayed results for $n_0 = 0.8$ and 0.95. For $n_0 = 0.8$ the best data collapse is obtained with $z \simeq 0.45$, wheras for $n_0 = 0.95$, the corresponding value is $z \simeq 0.39$. The exponent α, measured from the small argument divergence of $h(\eta)$, also shows similar changes.

In Table I, we have summarised our exponent values for four initial densities. All measurements were made using the data for the mass-distribution $M(l,t)$ rather than the correlator $C(r,t)$ on account of lesser statistical fluctuations. For the dynamical exponent z, we chose the value which gave the best collapse of data under dynamic scaling. Although it is difficult to measure the exponent very accurately using this method, we have verified by visual inspection that the error involved is less than the reported changes in the exponent values at least by a factor of two. We have omitted the case $n_0 = 0.2$ because no single value of z was found to give good scaling behaviour in the time range studied.

A more direct way to measure the dynamical exponent z is to compute the average separation $L(t)$ between persistent sites. If the spatial distribution were uniform over all length scales, this quantity would be simply $L(t) \sim P(t)^{-1}$. Since this is not the case, we have to proceed more carefully. We define $n(k,t)$ to be the number of nearest neigbour pairs of persistent sites at time t with separation k. By definition, $\int_k n(k,t) = NP(t)$ and $\int_k kn(k,t) = N$. The average separation $L(t) = N^{-1}\int_k k^2 n(k,t)$ and we expect $L(t) \sim t^z$. We computed $L(t)$ numerically by simulating 100 lattices of size $N = 10^5$ upto 10^5 time steps, for each n_0. In Fig.5, we display the results for the running exponent $z_{eff} = d(\text{log}L)/d(\text{log}t)$. The results are seen to be fully supportive of our earlier conclusions.

Our numerical results are strongly suggestive of non-universal behaviour of exponents α and z. The non-universal exponent values have been observed to be valid over at least three decades of MC time (upto 10^5 time steps). We note that there are *two length scales* at work here. For low n_0, the dynamics is dominated by diffusive motion of isolated particles, 'eating into' clusters of persistent sites. Due to annihilation, their average density decays as $n(t) = (8\pi Dt)^{-1/2}$ [8] and hence the average separation is the diffusive scale $\mathcal{L}_D(t) \sim t^{1/2}$. On the other hand, for $n_0 \to 1$, the initial separation of persistent sites $\sim 1/(1-n_0) \gg 1$. The short time behaviour is now dominated by persistent $\to$ non-persistent conversion of isolated sites, with characteristic length scale $\mathcal{L}_p(t) \sim t^{3/8}$. It is possible that the observed non-universal behaviour results from competition between these two scales. According to this picture, one should see a crossover to diffusion dominated regime at later times, but we are yet to see any signature of that. Further numerical work, at least a few orders of magnitude greater than what is reported here, would be required to establish conclusively the possibility of a temporal crossover.

To conclude, we have discovered strong, power-law correlations in the spatial distribution of persistent sites in one-dimensional $A + A \to \emptyset$ model. The correlation length $\xi(t)$ exhibits an algebraic divergence with time. In the correlated region, the set of persistent sites form a self-similar fractal, while over larger length scales, the distribution is homogeneous. These features are not specific to this model or dimension. We have observed identical features in kinetic Ising model in $d = 1$ and 2 [9], showing that this is a general phenomenon in the context of persistence. The interesting aspect of the present model is that the fractal dimension was found to be sensitive to the starting density of particles.

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TABLES

n_0	α	z
0.50	0.7342(8)	0.50
0.80	0.8294(5)	0.45
0.95	0.9517(3)	0.39

TABLE I. Observed values of exponent α as measured from box-counting method (details in text), for four values of initial density n_0. The quoted value of dynamical exponent z is the one which gave the best data collapse over three decades of time, $t = 10^3, 10^4$ and 10^5. The fractal dimension $d_f = 1 - \alpha$.

FIGURE CAPTIONS

FIG. I: The scaling function for two-point correlation $f(x) = C(r,t)/P(t)$ plotted against the scaling variable $x = r/t^z$ on log-scale for two values of time $t = 10^3$ and 10^4. The starting density of particles is $n_0 = 0.5$. The data for different times are seen to collapse into the same curve if scaling is done with $z = 0.50$. The observed $\alpha \simeq 0.75$ is in agreement with the proposed scaling relation (2).

FIG. II: The average number of persistent sites $M(l,t)$ in a box of size l at time t is plotted against the box size l for $t = 10^3, 10^4$ and 10^5. The initial density of particles is $n_0 = 0.5$. The crossover from fractal (dimension $d_f \simeq 1/4$) to homogeneous ($d_f = d = 1$) distribution is clear from the figure.

FIG. III: Same as Fig.2, for four starting densities $n_0 = 0.2, 0.5, 0.8$ and 0.95. All plots correspond to $t = 10^4$. For $n_0 = 0.2$, the fractal region is reached late, but the asymptotic value is seen to be the same as that for $n_0 = 0.5$. For higher n_0, d_f decreases continously, approaching zero in the limit $n_0 \to 1$.

FIG. IV: The scaling function for the mass-distribution $h(\eta) = M(l,t)/lP(t)$ plotted against the scaling variable $\eta = r/t^z$ for two values of time $t = 10^4, 10^5$ and two starting densities $n_0 = 0.8$ and 0.95. The observed data collapse has been obtained with $z = 0.45 (n_0 = 0.8)$ and $z = 0.39 (n_0 = 0.95)$. The corresponding values for α are $\simeq 0.83$ and 0.95. For comparison, a straight line with slope 0.75 is also shown.

FIG. V: The running exponents z_{eff} for four starting densities is plotted against $1/\log t$. These results have been averaged over 100 starting configurations.

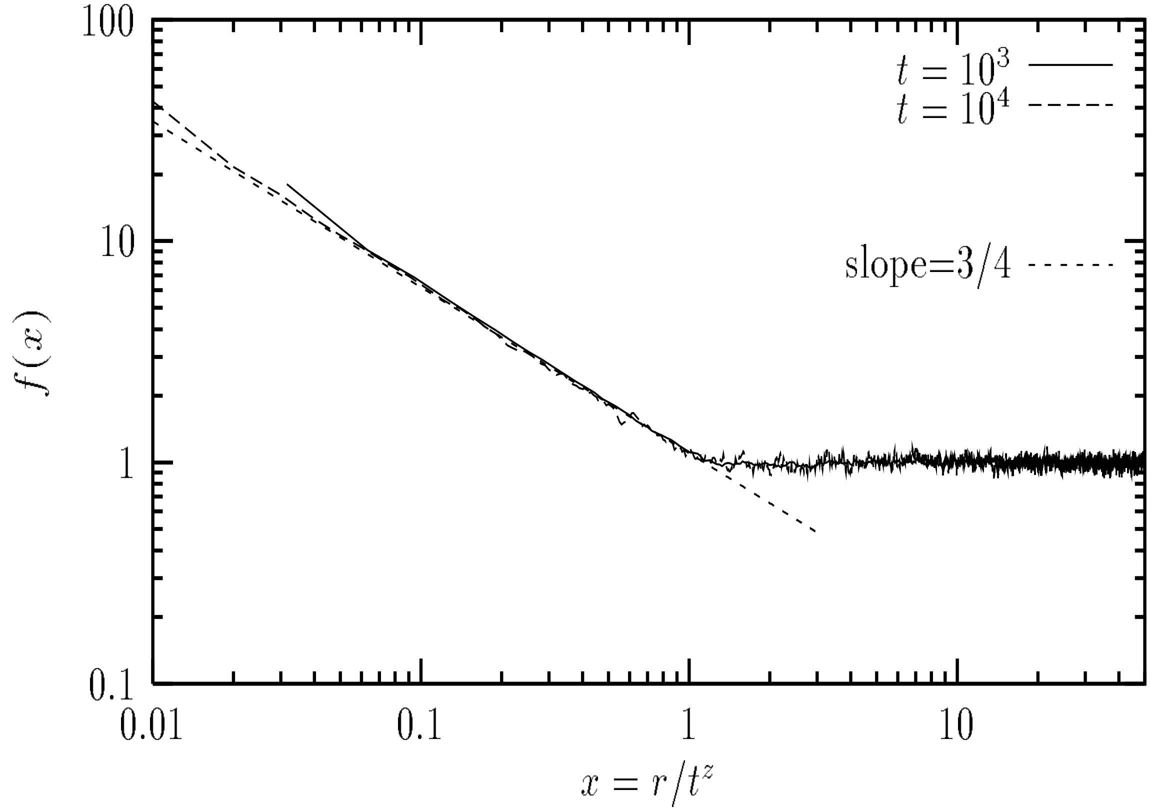
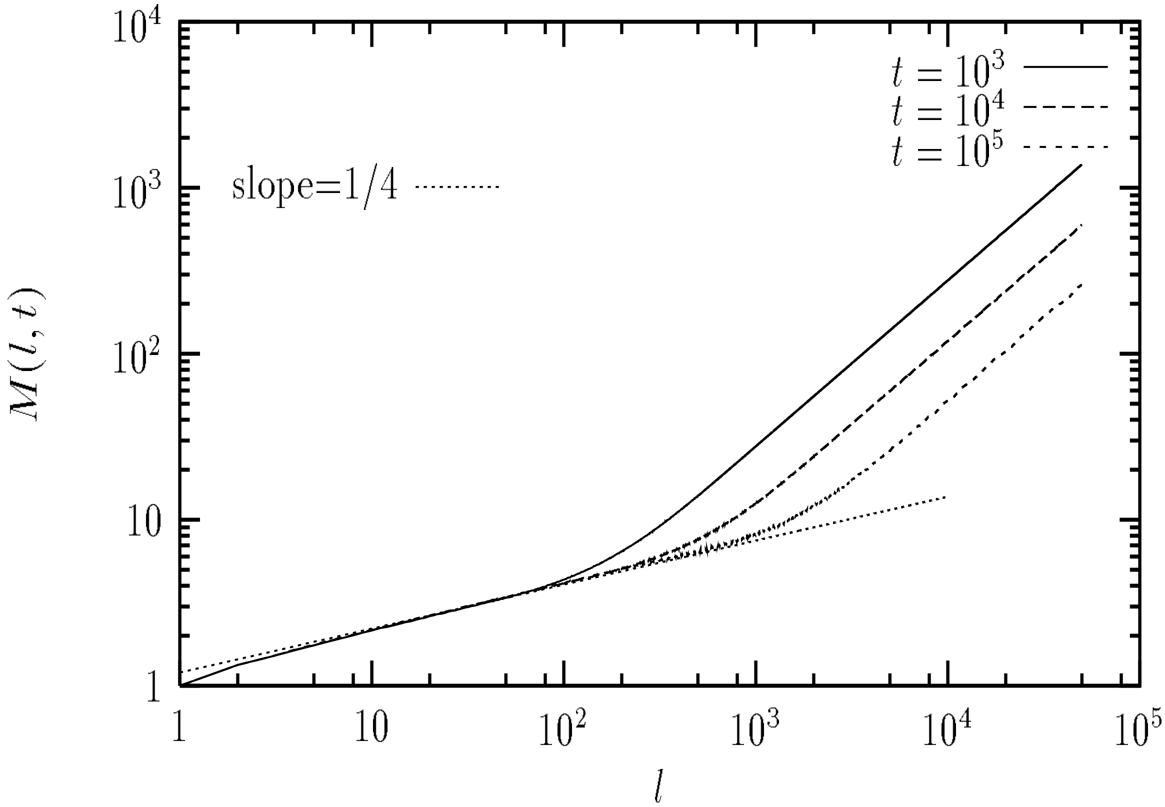
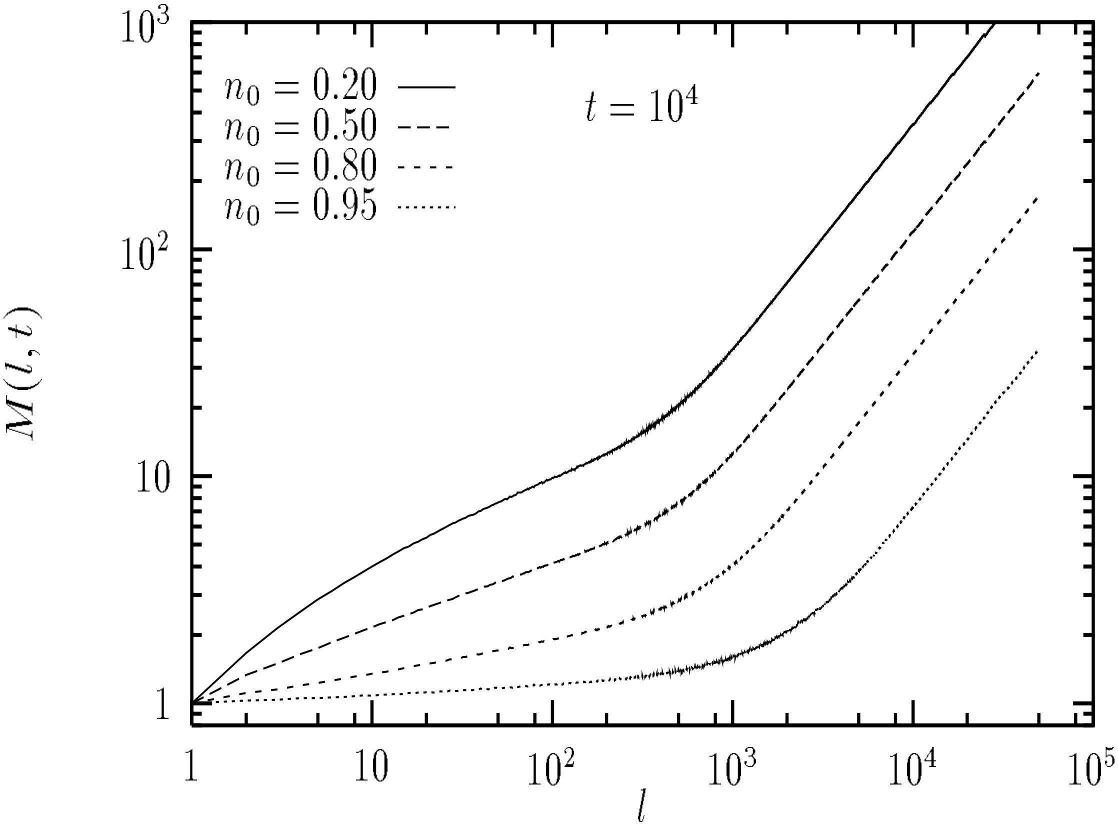


FIG. I



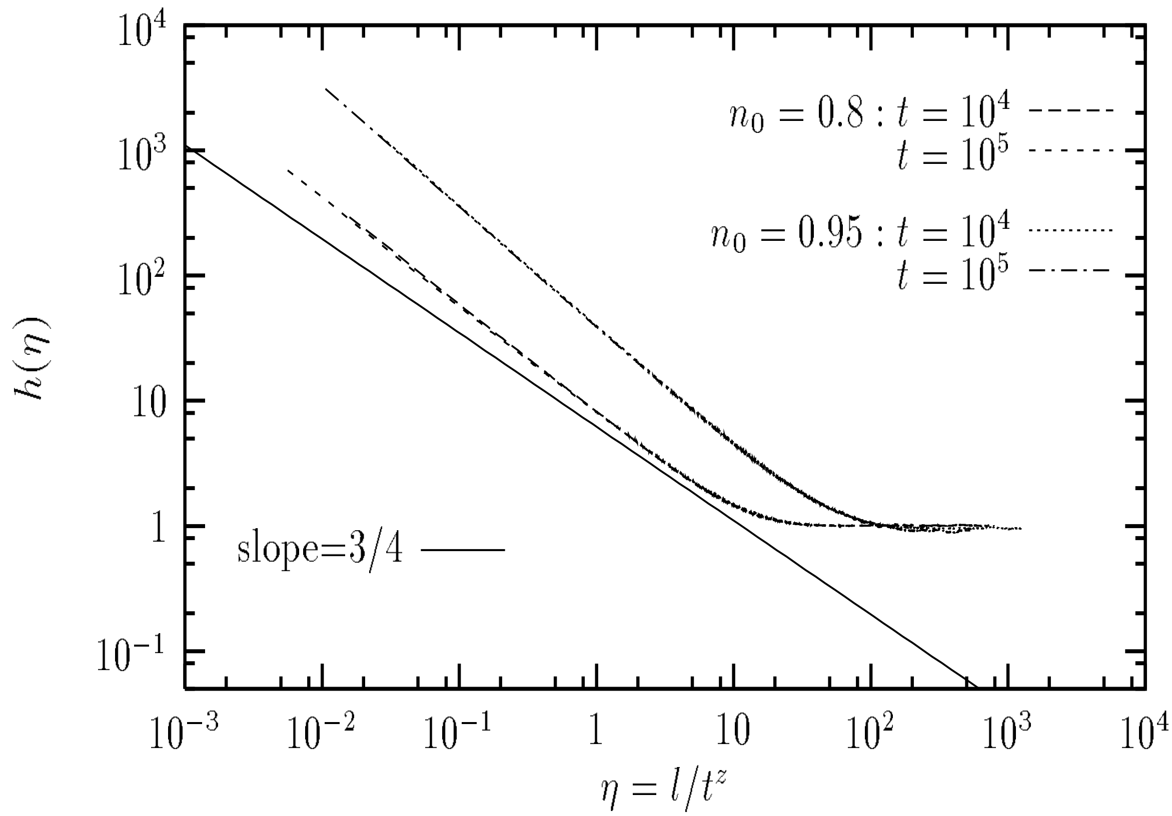
10^4
10^3
10^2
10
1
$M(l,t)$
slope=1/4
$t=10^3$
$t=10^4$
$t=10^5$
1
10
10^2
10^3
10^4
10^5
l

FIG. II



$n_0 = 0.20$
$n_0 = 0.50$
$n_0 = 0.80$
$n_0 = 0.95$
$t = 10^4$
$M(l,t)$
10^3
10^2
10
1
1
10
10^2
10^3
10^4
10^5
l

FIG. III



$n_0 = 0.8 : t = 10^4$
$t = 10^5$
$n_0 = 0.95 : t = 10^4$
$t = 10^5$
slope=3/4
$h(\eta)$
$\eta = l/t^z$
10^4
10^3
10^2
10
1
10^{-1}
10^{-3}
10^{-2}
10^{-1}
1
10
10^2
10^3
10^4

FIG. IV

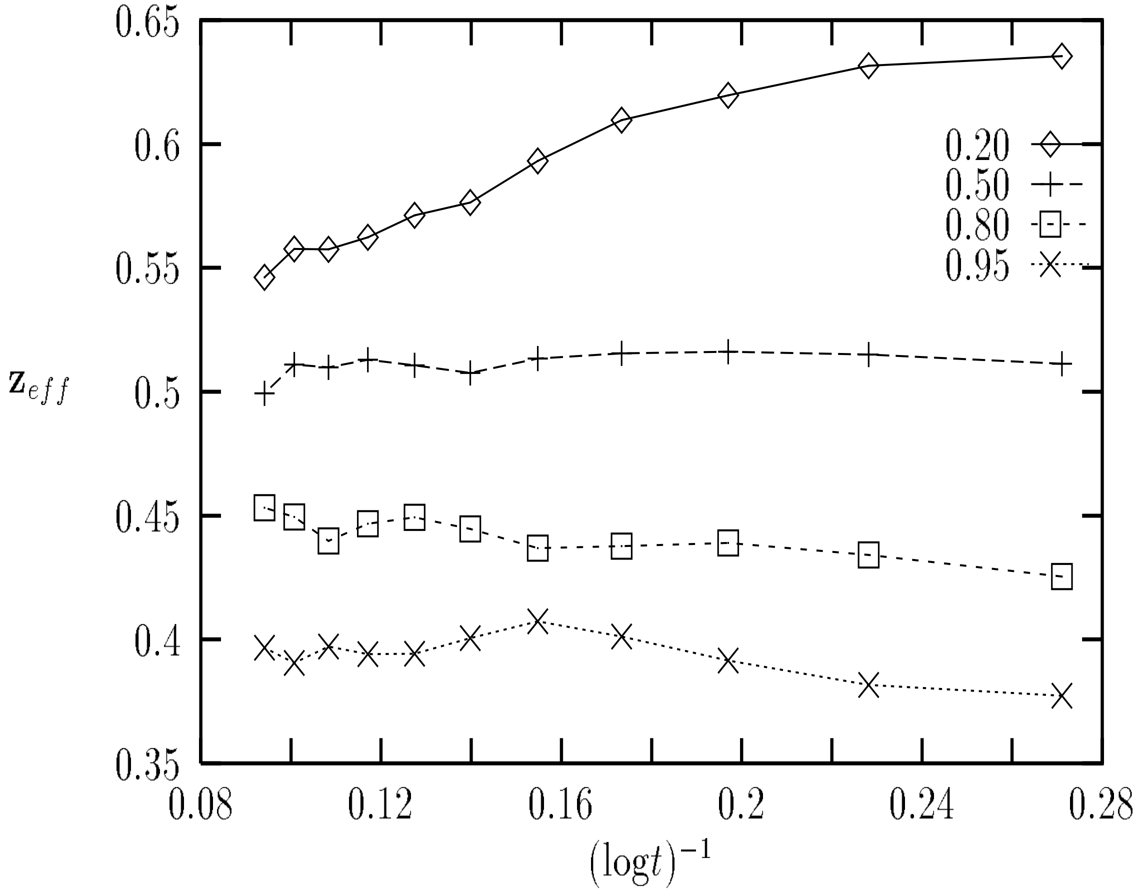

0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.08
0.12
0.16
0.2
0.24
0.28
z_{eff}
$(\log t)^{-1}$
0.20
0.50
0.80
0.95

FIG. V