Spontaneous Layer Polarization and Conducting Domain Walls in the Quantum Hall Regime of Bilayer Graphene

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Bilayer graphene subjected to perpendicular magnetic and electric fields displays a subtle competition between different symmetry broken phases, resulting from an interplay between the internal spin and valley degrees of freedom. The transition between different phases is often identified by an enhancement of the conductance. Here, we propose that the enhanced conductance at the transition is due to the appearance of robust conducting edge states at domain walls between the two phases. We formulate a criterion for the existence of such conducting edge states at the domain walls. For example, for a spontaneously layer polarized state at filling factor $\nu = 2$, domains walls between regions of opposite polarization carry conducting edge modes. A microscopic analysis shows that lattice-scale interactions can favor such a layer polarized state.

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I. INTRODUCTION

Bilayer graphene (BLG) is a rich playground to explore many-body physics[1]. At zero magnetic field, the energy bands exhibits a quadratic touching that can lead to a variety of many-body instabilities[2–10]. Signatures of a symmetry-broken ground state at $B = 0$ have been observed experimentally[11–15], although the precise nature of this state is still debated. When a magnetic field is applied perpendicular to the system, the Landau levels are highly degenerate, including spin, valley, and (for the zeroth Landau level) also an orbital degree of freedom[16]. This degeneracy can be lifted by exchange interactions[12,15,17–19], leading to different kinds of broken symmetry states[20–22]. Which state is favored depends on the nature of the lattice-scale interactions between electrons, which break the approximate SU(4) symmetry in spin and valley space.

In BLG, an electric field perpendicular to the plane couples to the valley degree of freedom of the Landau levels. Upon tuning the strength of the magnetic and electric perpendicular fields (B and E, respectively) at a fixed filling fraction, transitions between different ordered states can be induced[12,19,23,24]. These transitions are identified by peaks in the conductance along lines in the (E, B) plane. The mechanism for this enhanced conductance at the transitions remains unexplained. These transitions are expected to be of first order in the clean limit; they are described as a level crossing of different ground states, without closing of the energy gap above these two states, and hence there is no obvious reason for an enhancement of the conductance.

In this paper, we propose that the enhanced conductance at the transitions can result from the appearance of robust one-dimensional conducting modes at domain walls between different phases. The possibility of the appearance of such (non-chiral) modes, either at the edge of the sample or at domain walls, has been proposed in Refs.[8,25–36]. These edge states are partially protected against backscattering by the approximate conservation of either the spin or pseudospin (valley) quantum numbers, and are robust in the presence of electron-electron interactions. Evidence for such edge state have been observed under a high in-plane magnetic field[37]. We formulate a simple criterion for robust edge states at domain walls between two quantum Hall ferromagnetic phases with the same filling fraction, based on their symmetry properties and their quantum numbers.

As an example, we analyze the case of $\nu = 2$ at $E = 0$. In this case, to leading approximation, the partially filled Landau levels have an SU(2) valley (pseudospin) degree of freedom[38,39]. This symmetry is broken either by an applied electric field or by lattice scale interactions. We argue that the experimental findings of Wietz *et. al.*[12] are consistent with a spontaneously layer polarized phase [an easy axis ferromagnet, in terms of the SU(2) pseudospin]. The domain walls between regions of opposite polarization support conducting edge modes[26,30]. At $E = 0$, the domain walls percolate, leading to an enhanced conductance. We present a Hartree-Fock analysis of a microscopic model, and demonstrate how such a layer polarized phase can be favored over other possible broken-symmetry states in the physically relevant regime of parameters. Interestingly, to obtain such a phase, it is essential to treat all the Landau levels explicitly, rather than projecting to the partially filled zero-energy Landau levels.

II. SETUP

We consider a BLG sheet in the Bernal stacking. The low-energy single particle effective Hamiltonian is written as[16]

$$H_0 = -\frac{1}{2m}\begin{pmatrix} 0 & (\pi_x - i\tau^z\pi_y)^2 \\ (\pi_x + i\tau^z\pi_y)^2 & 0 \end{pmatrix}. \quad (1)$$

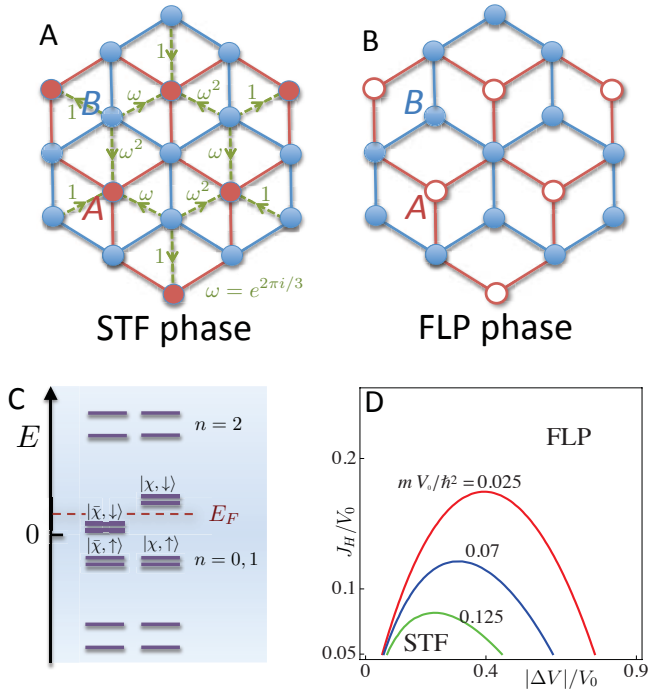

FIG. 1: (A) The staggered flux (STF) phase. In this phase, translational symmetry is spontaneously broken, and there is a spontaneous staggered flux between the two graphene sheets. The order parameter can be described as a complex hopping amplitude between the A sublattice of the bottom layer and the B sublattice of the top layer. (B) The fully layer polarized (FLP) phase, which breaks inversion symmetry spontaneously. The electrons in the $n = 0, 1$ Landau levels with spin antiparallel to the external magnetic field occupy a single valley (and a single layer). (C) Energies of the Landau levels at filling factor $\nu = 2$. Ignoring the Zeeman splitting and electron-electron interactions, there are eight degenerate zero-energy Landau levels, corresponding to spin, valley, and orbital ($n = 0, 1$) indices. The exchange interactions favour a ferromagnet in spin and valley manifold with equal occupancy of $n = 0, 1$ orbitals. The Zeeman coupling picks a direction for spin and splits the spin degeneracy. Finally, exchange interactions spontaneously split the remaining valley degeneracy, favoring a spinor $|\bar{\chi}\rangle$ in valley space over the orthogonal spinor $|\chi\rangle$. (D) Ground state mean-field phase diagram for $\nu = 2$ as a function of J_H and ΔV (see Eq. 3). The other coupling constant were fixed as follows: $U = V_1 = V_0$ and $V_2 = V_3$. The phase boundaries between the FLP and the STF phases are shown for different dimensionless coupling strengths: $mV_0/\hbar^2 = 0.025, 0.07, 0.125$.

Here, m is the effective mass of the bands near zero energy, and $\pi_i = -i\partial_i - eA_i$ $(i = x, y)$ where $\vec{A}$ is the vector potential. $\vec{\tau}$ are Pauli matrices acting on the valley index, such that $\tau^z = \pm 1$ corresponds to the $\pm K$ point in momentum space, where $K = (-4\pi/3\sqrt{3}a_0, 0)$, and a_0 is the inter-atomic spacing within each layer. We set the units such that $\hbar = c = 1$. The 2×2 Hamiltonian acts on the spinor (ψ_A, ψ_B), where $\psi_{A(B)}$ annihilates an electron on sublattice A (B) in the bottom (top) layer, respectively. We define the 8-component spinor Ψ, that contains annihilation operators in layer $\mu^z = \pm 1$, valley $\tau^z = \pm 1$, and spin $s^z = \pm 1$.

In the presence of a uniform orbital magnetic field, the single particle spectrum consists of a series of Landau levels whose energies are $E_n = \pm\omega_c\sqrt{n(n-1)}$, where $\omega_c \equiv eB/m$ and $n = 0, 1, \ldots$ [16]. Each Landau level is four-fold degenerate, with two possible valley labels, $\tau^z = \pm 1$, and two spin labels, $s^z = \pm 1$ (neglecting the Zeeman splitting). In addition, the $n = 0, 1$ Landau levels are degenerate.

We write the full Hamiltonian as

$$\hat{H} = \hat{H}_0 + \hat{H}_Z + \hat{H}_C. \tag{2}$$

Here, $\hat{H}_0 = \int d^2r \Psi^\dagger(r) H_0 \Psi(r)$, $\hat{H}_Z = -g\mu_B B \int d^2r \Psi^\dagger s^z \Psi$ is the Zeeman coupling, and $\hat{H}_C$ is the Coulomb interaction (to be discussed below).

When some of the zero-energy Landau levels are empty, the system tends to form a quantum Hall ferromagnetic state which breaks the symmetry in spin and valley space, in order to gain Coulomb exchange energy. At the lowest Landau level, where there is an additional orbital ($n = 0, 1$) degeneracy, maximum exchange is gained by filling the $n = 0, 1$ orbitals together with the same state in τ, s space[38,39]. We will assume that this form of "Hund's rule" is obeyed below, although it is not essential for the general criterion for conducting edge states between different phases.

The filling fraction ν is defined as the number of electrons per flux quantum, with respect to the charge neutral state. The $\nu = 0$ quantum Hall state is determined by two orthogonal spinors in spin/valley space, χ_1 and χ_2, such that of the eight degenerate zero-energy Landau levels, four are occupied: $|\chi_i, n\rangle$ with $i = 1, 2$ and $n = 0, 1$[21]. Similarly, the $\nu = 2$ state is determined by a single spin/valley spinor χ, such that of the $E = 0$ Landau level states, only the states $|\chi, n\rangle$ $(n = 0, 1)$ are *empty* (see Fig. 1C). At a given filling fraction, there is a manifold of possible states; this degeneracy is lifted by the Zeeman field, and applied electric field perpendicular to the sheet (which breaks the degeneracy between the layers), and by the short-range exchange interactions.

III. CONDITION FOR CONDUCTING MODES AT DOMAIN WALLS

As external parameters are varied at a fixed filling fraction, the system can undergo phase transitions between different ordered states in spin/valley space. If disorder effects are ignored, these transitions are generically of first order. At the transition point, we expect a phase mixture of two phases.

Consider a domain wall between two such phases. Here, we discuss a sufficient condition for the appearance of conducting edge states at the domain wall. Our condition is formulated as follows: suppose that the two

phases on either side of the domain wall are invariant under a common $U(1)$ symmetry generated by an operator $\hat{G}$ in spin and/or valley space. (The symmetry could be generated by s^z, τ^z, or some combination of the two.) Define the weighted filling fraction $\tilde{\nu} = \sum_{j\in\text{filled}} q_j$, where j runs over the filled states, and q_j is the charge of the state j under $\hat{G}$. *If $\tilde{\nu}$ of the two phases is different, there is necessarily a gapless edge state at the domain wall between them.* This edge state is robust in the presence of arbitrary interactions and disorder, as long as they preserve $\hat{G}$. For instance, if $\hat{G} = \tau^z$, the edge states will be protected as long as we neglect lattice-scale disorder that causes inter-valley scattering, and the domain wall itself is sufficiently smooth. The conductance of the edge state is given by $\Delta\tilde{\nu}e^2/2h$, where $\Delta\tilde{\nu}$ is the difference of $\tilde{\nu}$ between the two phases.

A simple way to understand the existence of a gapless edge state is to define the Hall conductance related to the conserved $U(1)$ charge under $\hat{G}$. We introduce a vector potential $\vec{A}_G$ that couples to the charge under $\hat{G}$, by substituting $-i\vec{\partial} \to -i\vec{\partial} - \hat{G}\vec{A}_G$ in $\hat{H}_0$. The response of the current $\vec{j}_G = \partial\hat{H}/\partial\vec{A}_G$ to an applied electric field in the plane, described by the "$\hat{G}$ Hall conductance" σ_G^H, is quantized (this is analogous to the spin Hall conductance in a quantum spin Hall state with conserved spin[40,41]). Phases with different $\tilde{\nu}$ correspond to different σ_G^H, and must have gapless edge states between them. This argument is expected to hold in the presence of arbitrary interactions and disorder, as long as the symmetry $\hat{G}$ is preserved and the gap in the bulk of both phases is maintained.

For example, consider the state with filling factor $\nu = 2$, specified by the four-component spinor χ defined above. The Zeeman coupling favors a particular spin component, say $s^z = 1$. To specify the state, the spinor χ in valley space remains to be determined. Lattice scale exchange interactions will select either a spontaneously layer-polarized state with $\langle\vec{\tau}\rangle \parallel \hat{z}$, or an in-plane polarized phase with $\langle\vec{\tau}\rangle \perp \hat{z}$. If the layer-polarized state is favored, then domain walls between the $\tau^z = \pm 1$ phases carry non-chiral edge states. These edge states are robust as long as we can neglect inter-valley scattering (which requires the domain wall to be smooth on the lattice scale; see next section for a discussion of the characteristic length scale of the domain walls), and remain so for arbitrary interactions. The edge states have conductance of $2e^2/h$ (where the factor of 2 is due to the orbital degeneracy). The in-plane state breaks a continuous symmetry; in this phase, there are no sharp domain walls. If weak inter-valley scattering disorder is present, the valley polarization is locally pinned by the disorder, and twists gradually in space.

Physically, the in-plane polarized state breaks translational symmetry spontaneously, while the valley polarized state breaks inversion symmetry. On the lattice scale, the former is described as a "staggered flux" (STF) state, and the latter is a "fully layer polarized" (FLP) state (see Fig. 1A,B).

IV. HARTREE-FOCK ANALYSIS FOR $\nu = 2$

The long-range part of the Coulomb interactions is symmetric in spin and valley space, and therefore it does not lift the degeneracy between the layer-polarized and the in-plane polarized states. The degeneracy is lifted by short-range (lattice scale) exchange interactions. In order to analyze the competition between these phases, we use the following form for the short-range exchange Hamiltonian:

$$\hat{H}_{\text{ex}} = \int d^2r \left\{ V_0 n^2 + \sum_\mu \left[\sum_\tau U n_{\mu\tau\uparrow} n_{\mu\tau\downarrow} + V_1 n_{\mu,K} n_{\mu,K'} - J_H \vec{S}_{\mu,K} \cdot \vec{S}_{\mu,K'} \right] + \sum_\tau V_2 n_{1,\tau} n_{-1,\tau} + V_3 \sum_\mu n_{\mu,K} n_{-\mu,K'} \right\}. \tag{3}$$

Here, $n_{\mu\tau s} = \psi^\dagger_{\mu\tau s}\psi_{\mu\tau s}$ ($\mu = \pm 1$, $\tau = K, K'$, $s = \uparrow, \downarrow$ are layer, valley, and spin indices, respectively, and we have suppressed the spatial argument $\vec{r}$ for brevity), $n_{\mu\tau} = \sum_s n_{\mu\tau s}$, $n = \sum_{\tau,\mu} n_{\tau\mu}$, and $[\vec{S}_{\mu\tau}]_{s,s'} = \psi^\dagger_{\mu\tau s}\vec{\sigma}_{ss'}\psi_{\mu\tau s'}$ where $\vec{\sigma}$ are Pauli matrices. V_0 is the part of the Coulomb interactions which are isotropic in layer, valley, and spin space; U is the strength of the interaction between two electrons in the same layer and valley; V_1 is an inter-valley, same-layer coupling constant; J_H is an inter-valley Hund's rule coupling constant; and the V_2 (V_3) terms describe inter-layer, intra- (inter-) valley interactions, respectively. For simplicity, we have assumed that the spatial overlap between wavefunctions of electrons in different layers is small, so we can neglect inter-layer exchange interactions. On general grounds, we expect the following relations between the interactions: $V_0 \gtrsim U \geq V_1 > J_H > 0$, $U > V_2 \geq V_3 > 0$. (For more discussion of the microscopics of the exchange interactions, see Appendix A.)

We now proceed to perform a mean-field analysis on the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_Z + \hat{H}_{\text{ex}}$. Interestingly, upon projection to the partially filled Landau level, the exchange Hamiltonian (3) does not lift the degeneracy

between the layer-polarized and the in-plane polarized states at $\nu = 2$. This is because the partially filled Landau level is fully polarized in spin and valley space; therefore, $\langle \hat{H}_{\rm ex} \rangle = 0$ independent of the direction of polarization. In the following, we avoid projecting to the lowest Landau level, and treat the entire Landau spectrum. The contribution of the $n > 1$ Landau levels to the susceptibility in the particle-hole channel is divergent in the limit $B \to 0$, due to the quadratic band touching in the underlying dispersion. This implies that the contribution of the higher (occupied and unoccupied) states to the energetics is significant.

To estimate the ground state energies of the different states, we use a variational Hamiltonian of the form

$$\hat{H}_{\rm MF} = \hat{H}_0 + \hat{H}_Z - \int d^2r \sum_{\alpha,\beta,\zeta} \lambda^{(\zeta)}_{\alpha\beta} \Psi^\dagger \mu^\alpha \tau^\beta \frac{1+\zeta s^z}{2} \Psi, \quad (4)$$

where $\alpha, \beta = 0, x, y, z$ and $\zeta = \pm 1$. The parameters $\lambda^{(\zeta)}_{\alpha\beta}$ are chosen to minimize $\langle \hat{H} \rangle_{\rm MF}$ (see aApendixes B and C for details).

Symmetry can be used to reduce the number of variational parameters. We assume that none of the candidate states break spin rotational symmetry around the z axis. The fully layer polarized (FLP) state breaks lattice inversion symmetry, represented by $\hat{I} = \mu^x \tau^x \times (\vec{r} \to -\vec{r})$, but preserves translational symmetry and three-fold rotational symmetry, $\hat{R}_{2\pi/3} = \exp(2\pi i \mu^z \tau^z/3) \times (\vec{r} \to \mathcal{R}\vec{r})$, where $\mathcal{R}$ is a 2×2 rotation matrix (Appendix B 1). The only mean- field terms that are consistent with these symmetries are: $\{\mu^z, \tau^z, \mu^z\tau^z\}_{\rm FLP}$. The staggered flux (STF) phase breaks translational symmetry, but preserves $\hat{R}_{2\pi/3}$ around a certain three-fold axis and $\hat{I}$. The allowed mean-field terms in this phase are $\{\mu^x\tau^x, \mu^y\tau^y, \mu^z\tau^z\}_{\rm STF}$. Interestingly, the set of mean-field terms in the FLP phase are mapped onto those of the STF phase under a unitary transformation given by $\hat{U} = \exp(i\frac{\pi}{4}\mu^x\tau^y)$, which also leaves $\hat{H}_0$ invariant. Therefore, interactions that are invariant under $\hat{U}$ [e.g. the V_0 term in (3), or long-range Coulomb interactions] do not lift the degeneracy between the FLP and STF phases.

The explicit evaluation of $\langle \hat{H} \rangle_{\rm MF}$ and the minimization over $\lambda^{(\zeta)}_{\alpha\beta}$ is tedious but straightforward, and will be deferred to the appendixes B,C, and D. We will quote some of the result here. The instability towards either an FLP or STF phase occurs for any non-zero strength of the interactions, as a result of the flatness of the Landau levels. The difference in ground state energy between the FLP and STF states can be found analytically in the limit of weak interactions. In this limit, the energy difference per unit area $\Delta E \equiv E_{\rm STF} - E_{\rm FLP}$ is given by

$$\Delta E = \frac{\chi_0}{\ell_B^4}\left[\left(\frac{V_0}{2}+\frac{U}{4}\right)J_H - \left(\frac{V_0}{2}+\frac{2V_3-U}{4}\right)\Delta V + \frac{1}{8}[2J_H^2 + (U-V_3)^2 + 2(\Delta V)^2]\right], \quad (5)$$

where $\Delta V = V_1 - V_2$, $\ell_B = \sqrt{\hbar/eB}$, and $\chi_0 = \frac{m}{\pi}\sum_{n=2}^{N}\frac{\omega_c}{E_n}$ ($N = E_c/\omega_c$, where E_c is a high energy cutoff of the theory). The ground state is the FLP state when $\Delta E > 0$, which occurs when J_H is larger than a critical value of the order of ΔV.

Figure 1D shows the phase diagram as a function of J_H and ΔV. We have fixed the ratios between all the other coupling constants, and present results for different values of the dimensionless interaction parameter mV_0. For weak interactions [where Eq. (5) applies], the FLP phase is favored for large J_H, whereas the STF phase is the ground state for small J_H and an intermediate range of ΔV. As the interaction strength increases, the region of the FLP phase expands (a naive estimate of the realistic interaction strength gives $mV_0 \sim 1$). These findings do not depend sensitively on the precise values of the other coupling constants.

In the above analysis, we have disregarded the long-range part of the Coulomb interaction, and treated only contact (exchange) interactions (whose range is of the order of the short distance cutoff, $a \equiv 1/\sqrt{mE_c}$). As already noted, the $1/r$ part of the Coulomb interaction is symmetric in spin and valley space, and does not distinguish the FLP and STF phases. The dipole-dipole term, which falls as $1/r^3$, favours the STF phase (since it opposes layer polarization). However, a simple estimate shows that the dipolar energy per unit area, $E_{\rm d} \sim e^2d^2/\ell_B^5$ (where d is the inter-layer spacing), is suppressed by a factor of $\sim d^2/a\ell_B \ll 1$ compared to the exchange energy difference between the two phases, Eq. (5) (taking e^2a as the typical magnitude of the exchange couplings). Therefore, the long-range dipolar interaction is typically negligible.

Finally, we comment on the structure of the domain walls between two oppositely polarized regions in the FLP phase. The "easy axis" anisotropy energy is of the order of e^2a per unit area, whereas the stiffness of the valley pseudospin $\vec{\tau}$ is $\sim e^2/\ell_B$. Dimensional analysis gives that the domain wall has a characteristic thickness of $\ell_{\rm DW} \sim \ell_B\sqrt{\ell_B/a} \gg a$. Therefore, we expect that intervalley scattering induced by the domain wall is small.

V. DISCUSSION AND RELATION TO EXPERIMENTS

Experimentally, an enhanced conductance was found along a line in the (E, B) plane for filling factor $\nu = 0$[12], and at $E = 0$ for filling factor $\nu = 2$[12,19]. We interpret this enhancement of the conductance as arising from gapless edge modes at domain walls between two phases at

the transition point. If these phases satisfy the criterion described above, then they are topologically distinct as long as the common $U(1)$ symmetry G is preserved; therefore, there is a sharp phase transition between the two phases, even in the presence of interactions and disorder. The transport near the transition is then described in terms of percolation of the domain walls between the two phases[42–45].

For $\nu = 2$, as we have shown here, the enhanced conductance is readily explained if the $E = 0$ ground state is spontaneously layer polarized. Our microscopic analysis shows how such a state can arise from short-range exchange interactions. Such spontaneous layer polarization can be detected directly by capacitance measurements[46].

At $\nu = 0$, the existence of an enhanced conductance at the finite E transition allows us to put constraints on the nature of the states on either side of the transition. We assume that the $E = 0$ ground state is a canted antiferromagnet[21]. Then, according to our criterion, a domain wall with a partially layer polarized state[21], in which a coherent superposition of the two valley states is occupied, does not carry a protected edge mode, because the two phases do not have a common $U(1)$ symmetry. An edge state with a *fully layer polarized* state, however, does have a conducting edge state, since the two phases have a common valley symmetry, and valley Hall conductance jumps across the domain wall.

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Appendix A: Local Interaction Hamiltonian

To write the interaction Hamiltonian, we start from a tight binding model with the lattice structure of bilayer graphene. We are interested in the local part of the low-energy effective interactions, which are anisotropic in valley and layer space. We will assume weak coupling (small e^2/v), for which we can simply project the microscopic (Coulomb) interactions

$$H_{int} = \sum_{s,s'} \int d^3\vec{r}\, d^3\vec{r'}\, V(\vec{r}-\vec{r'})\varphi_s^\dagger(\vec{r})\varphi_s(\vec{r})\varphi_{s'}^\dagger(\vec{r'})\varphi_{s'}(\vec{r'}), \tag{A1}$$

onto the low-energy subspace. Here, $\varphi_s(\vec{r})$ annihilates an electron at position $\vec{r}$ with spin s, and $V(\vec{r}-\vec{r'}) = e^2/|\vec{r}-\vec{r'}|$. We choose a basis of states whose support in momentum space is in the regions $\left|\vec{k}-\vec{K}\right| < \Lambda$ and $\left|\vec{k}-\vec{K'}\right| < \Lambda$, where Λ is a momentum cutoff. In real space, these wavefunctions are localized within a region of size $a \sim 2\pi/\Lambda$. We assume a to be of the order of a few lattice constants.

Let us denote the basis functions by $\Phi_{\mu\tau}(\vec{r}-\vec{R})$, where $\vec{R}$ is the center of mass of this orbital, $\vec{r}$ is a continuous space variable, and μ, τ are the layer and valley indices. $\psi_{\mu\tau s}$ is an operator that annihilates an electron in orbital $\Phi_{\mu\tau}(\vec{r}-\vec{R})$ with spin s.

The low-energy part of the field operators is given by

$$\varphi_s(r) = \sum_{R,\mu,\tau} \Phi_{\mu\tau}(\vec{r}-\vec{R})\psi_{\mu\tau s}(\vec{R}). \tag{A2}$$

The interaction Hamiltonian takes the form

$$H_{int} = \sum_{s,s',1,2,3,4} \Big[\int dr dr' V(r-r')\Phi_1^*(\vec{r}-\vec{R}_1)\Phi_2(\vec{r}-\vec{R}_2) \times \Phi_3^*(\vec{r}-\vec{R}_3)\Phi_4(\vec{r}-\vec{R}_4)\Big]\, \psi_{1s}^\dagger(\vec{R}_1)\psi_{2s}(\vec{R}_2)\psi_{3s'}^\dagger(\vec{R}_3)\psi_{4s'}(\vec{R}_4). \tag{A3}$$

Here, we have used the short hand notation 1 for $\{\mu_1, \tau_1, R_1\}$, etc. The object within the square bracket is the coupling constant of the particular operator $\psi_{1s}^\dagger(\vec{R}_1)\psi_{2s}(\vec{R}_2)\psi_{3s'}^\dagger(\vec{R}_3)\psi_{4s'}(\vec{R}_4)$.

Calculating the values of the microscopic coupling constants from first principles is very difficult, because these coupling constants are strongly renormalized with respect to their bare values[47]. We will mostly treat them as phenomenological parameters. Below, we make some physically-motivated simplifying assumptions, in order to reduce the number of independent parameters.

1. Simplifying assumptions and explicit form of H_{int}

1. We assume that $\vec{R}_1 = \cdots = \vec{R}_4$.This assumption is justified if the orbitals $\Phi(\vec{r}-\vec{R})$ are sufficiently localized around $\vec{R}$.

2. We consider two types of terms: intra-layer and inter-layer. We assume that the overlap between orbitals localized in the two layers is negligible; therefore, we will not consider terms that hop a pair from one layer to the other. Moreover, the inter-layer interaction terms are spin independent.

Within these assumptions, we get the following form for $H_{\rm int}$:

$$H_{\rm int} = H_{\rm intra} + H_{\rm inter} + V_0(\psi^\dagger\psi)^2 \tag{A4}$$

$$H_{\rm intra} = \sum_\mu \big(U(n_{\mu K\uparrow}n_{\mu K\downarrow} + n_{\mu K'\uparrow}n_{\mu K'\downarrow}) + V_1 n_{\mu K}n_{\mu K'} - J_H \vec{S}_{\mu K}\cdot\vec{S}_{\mu K'}\big) \tag{A5}$$

$$H_{\rm inter} = V_2(n_{1,K}n_{-1,K}+n_{1,K'}n_{-1,K'})+\sum_\mu V_3 n_{\mu,K}n_{-\mu,K'} \tag{A6}$$

with six parameters, $V_0, U, V_1, J_H, V_2, V_3$. On general grounds, we expect the following inequalities to hold:

$$\begin{aligned} &V_0 \gtrsim U \geq V_1 > J_H, \\ &V_1 > V_2 \geq V_3. \end{aligned} \tag{A7}$$

In addition to these local interactions, there are also long-range Coulomb (monopole-monopole and dipole-dipole) interactions.

Appendix B: Mean Field Theory with Local Interactions: general formulation

Consider the following Hamiltonian:

$$H = H_0 + H_{\rm int} - \mu(\psi^\dagger\psi - n_0), \tag{B1}$$

where H_0 is the single particle Hamiltonian of BLG, $H_{\rm int}$ is a local exchange interaction (Eq. A4). The chemical potential μ is chosen such that the density is $\langle\psi^\dagger\psi\rangle = n_0$.

We use a variational mean-field Hamiltonian:

$$H_{\rm MF} = H_0 - \sum_{a=1}^{15}\lambda_{a,s}\psi_s^\dagger O_a\psi_s - \mu_{0s}\psi_s^\dagger\psi_s. \tag{B2}$$

O_a are the following matrices in valley and layer space:

$$\begin{aligned} O_{a=0,..,15} = \{&1, \mu^z\tau^z, \mu^x, \mu^y\tau^z,\ \mu^z, \mu^y\tau^y, \mu^y\tau^x, \tau^z, \\ &\mu^x\tau^x, \mu^x\tau^y, \mu^x\tau^z, \mu^y, \tau^x, \tau^y, \mu^z\tau^x, \mu^z\tau^y\}. \end{aligned} \tag{B3}$$

These form a complete basis of hermitian matrices in the layer and valley space. They satisfy

$${\rm Tr}O_aO_{a'} = \delta_{a,a'}. \tag{B4}$$

The mean-field energy is

$$\begin{aligned} E(\{\lambda_a\},\mu_0) &= \langle H\rangle \\ &= \langle H_{MF}\rangle + \sum_{a=1}^{15}\lambda_{as}\langle\psi_s^\dagger O_a\psi_s\rangle + \mu_{0s}\langle\psi_s^\dagger\psi_s\rangle \\ &+ \langle H_{\rm int}\rangle - \mu(\langle\psi^\dagger\psi\rangle - n_0). \end{aligned} \tag{B5}$$

A general spin diagonal quartic interaction term in $\langle H_{\rm int}\rangle$ can be written as

$$\begin{aligned} &\langle(\psi^\dagger O_a\psi)(\psi^\dagger O_b\psi)\rangle \\ &= \langle\psi^\dagger O_a\psi\rangle\langle\psi^\dagger O_b\psi\rangle - \sum_s\langle\psi^\dagger_{\alpha s}\psi_{\beta' s}\rangle\langle\psi^\dagger_{\beta s}\psi_{\alpha' s}\rangle O_a^{\alpha\alpha'}O_b^{\beta\beta'} \end{aligned} \tag{B6}$$

Summation over repeated indices is implied. Let us write $G_s^{\alpha\beta} \equiv \langle\psi^\dagger_{\alpha s}\psi_{\beta s}\rangle = \sum_a \phi_{as}O_a^{\alpha\beta}$, where $\phi_{as} = \frac{1}{4}{\rm Tr}O_aG_s = \langle\psi_s^\dagger O_a\psi_s\rangle$. Then

$$\begin{aligned} &\langle\psi^\dagger_{\alpha s}\psi_{\beta' s}\rangle\langle\psi^\dagger_{\beta s}\psi_{\alpha' s}\rangle O_a^{\alpha\alpha'}O_b^{\beta\beta'} \\ &= \frac{1}{16}\sum_{c,d}\phi_{cs}\phi_{ds}O_a^{\alpha\alpha'}O_b^{\beta\beta'}O_c^{\alpha\beta'}O_d^{\beta\alpha'} \\ &= \frac{1}{16}\sum_{c,d}\phi_{cs}\phi_{ds}{\rm Tr}[O_aO_c^*O_bO_d^*] \end{aligned} \tag{B7}$$

For the general local H_{int} we will have $a = b$ which gives $c = d$. Then ${\rm Tr}[O_aO_c^*O_aO_c^*] = \pm 4$ and

$$\langle(\psi^\dagger O_a\psi)^2\rangle = \langle\psi^\dagger O_a\psi\rangle^2 - \frac{1}{4}\sum_{c,s}\phi_{c,s}^2{\rm Sgn}[O_aO_c] \tag{B8}$$

Here ${\rm Sgn}[O_aO_c] = +1$ if O_a, O_c commute and -1 if they anti-commute. We can then collect and write all the terms in a compact way as

$$\langle H_{\rm int}\rangle = -\frac{1}{2}\sum_a \phi_a^T M_a\phi_a \tag{B9}$$

where M_a's are 2×2 matrices in spin label for each ϕ_a. The energy functional then becomes

$$\begin{aligned} E(\{\lambda_a\},\mu_{0s}) &= \langle H_{MF}\rangle + \sum_{a=1}^{15}\lambda_a\langle\psi^\dagger O_a\psi\rangle + \mu_{0s}\langle\psi_s^\dagger\psi_s\rangle \\ &- \frac{1}{2}\sum_{a=0}^{15}\phi_a^T M_a\phi_a - \mu(\langle\psi^\dagger\psi\rangle - n_0) \\ &= \langle H_{MF}\rangle + \sum_{a=1}^{15}\lambda_a\phi_a + \mu_{0s}\langle\psi_s^\dagger\psi_s\rangle \\ &- \frac{1}{2}\sum_{a=0}^{15}\phi_a^T M_a\phi_a - \mu(\langle\psi^\dagger\psi\rangle - n_0) \end{aligned} \tag{B10}$$

The saddle point equations are

$$\frac{\partial E(\{\lambda_a\},\mu_{0s})}{\partial\lambda_{a's'}} = \sum_{a=1}^{15}\left(\lambda_a^T - \phi_a^T M_a\right)\frac{\partial\phi_a}{\partial\lambda_{a's'}} - \phi_0^T M_0\frac{\partial\phi_0}{\partial\lambda_{a's'}} + (\mu_{0s}-\mu)\frac{\partial\langle\psi_s^\dagger\psi_s\rangle}{\partial\lambda_{a's'}} = 0 \tag{B11}$$

$$\frac{\partial E(\{\lambda_a\},\mu_{0s})}{\partial \mu_{0s'}} = \sum_{a=1}^{15} \left(\lambda_a^T - \phi_a^T M_a\right) \frac{\partial \phi_a}{\partial \mu_{0s'}} - \phi_0^T M_0 \frac{\partial \phi_0}{\partial \mu_{0s'}} + (\mu_{0s} - \mu)\frac{\partial \langle \psi_s^\dagger \psi_s\rangle}{\partial \mu_{0s'}} = 0 \quad \text{(B12)}$$

$$\frac{\partial E(\{\lambda_a\},\mu_0)}{\partial \mu} = \langle \psi^\dagger \psi\rangle - n_0 = 0 \quad \text{(B13)}$$

This gives the following mean-field equations:

$$\lambda_a = M_a \phi_a$$
$$n_0 = \langle \psi^\dagger \psi\rangle \quad \text{(B14)}$$
$$\mu_{0s} = \mu + M_0 \phi_0 \quad \text{(B15)}$$

Substituting $\phi_a = M_a^{-1}\lambda_a$ and $\langle \psi^\dagger\psi\rangle = n_0$ back into the expression for the energy, we get

$$\tilde{E}(\{\lambda_a\},\mu_0) = \langle H_{MF}\rangle + \frac{1}{2}\lambda_a^T M_a^{-1}\lambda_a + \mu n_0 + \frac{1}{2}(\mu_{0s}-\mu)(M_0^{-1})_{s,s'}(\mu_{0s'}-\mu) \quad \text{(B16)}$$

Note that this energy functional coincides with the original one at the saddle point, and its variation with respect to λ_a and μ_0 gives the correct mean-field equations.

1. Symmetries of the BLG Hamiltonian

The non-interacting Hamiltonian of BLG in absence of external magnetic field has the following symmetries. Below we describe the behaviour of our wavefunctions $|\psi_{\tau_z=K/K',\mu_z=\pm1,\sigma_z=A/B}(\vec{q})\rangle$ under the symmetry operations. (τ, μ, σ represent Pauli matrices acting in valley, layer, and sub lattice space, respectively, and $\vec{K} = \left(-\frac{4\pi}{3\sqrt{3}a_0}, 0\right)$) where a_0 is the inter-atomic spacing in each layer. Other definitions as shown in the figure are $\vec{R}_1 = (0, -a_0)$, $\vec{R}_2 = a_0\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\vec{R}_3 = a_0\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$; $\vec{G}_1 = \frac{2\pi}{a_0}\left(\frac{-1}{\sqrt{3}}, \frac{1}{3}\right)$, $\vec{G}_2 = \frac{2\pi}{a_0}\left(\frac{1}{\sqrt{3}}, \frac{1}{3}\right)$.

1. Time reversal symmetry (TRS): $(\vec{q} \to -\vec{q}) i s_y \tau_x \mathcal{K}$, ($\mathcal{K}$ represents complex conjugation).

2. Rotation by $2\pi/3$ around the A-sublattice of top layer (the stacking point):

$$J_{2\pi/3}\exp\left[\frac{2\pi i}{3}\tau_z\left(\frac{1+\mu_z}{2}\frac{1+\sigma_z}{2}\right) - \frac{2\pi i}{3}\tau_z\left(\frac{1-\mu_z}{2}\frac{1-\sigma_z}{2}\right)\right] = J_{2\pi/3}\exp[\frac{2\pi i}{3}\tau_z\left(\frac{\sigma_z+\mu_z}{2}\right)].$$

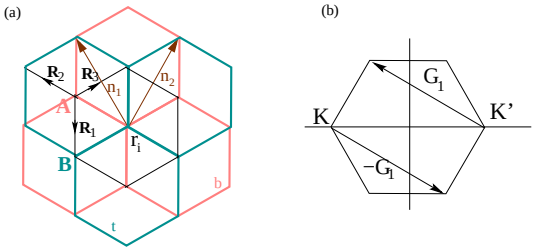

FIG. 2: (a) Schematic of bilayer graphene with top/bottom layer labelled by t/b (with colours blue/pink). Thin black hexagon defines the effective hexagonal lattice. (b) Brilluion zone for the effective hopping Hamiltonian

(Under $J_{2\pi/3}$, $q_x \to q_x\cos(2\pi/3) - q_y\sin(2\pi/3)$, $q_y \to q_x\sin(2\pi/3) + q_y\cos(2\pi/3)$)

3. Translation by $\vec{R} = l_1\vec{n}_1 + l_2\vec{n}_2$: $\exp[i\tau_z\frac{4\pi}{3\sqrt{3}a_0}R_x]\exp[-i\vec{q}.\vec{R}]$,. $\vec{n}_1$ and $\vec{n}_2$ are primitive vectors of the hexagonal lattice (see Fig. 2).

4. Inversion Symmetry: $(\vec{q} \to -\vec{q})\tau_x\mu_x\sigma_x$, with inversion center at mid point of the stacking bond between the two layers.

5. Mirror $x \to -x$: $(q_x \to -q_x)\tau_x$.

Details of the symmetry transformations:
Rotation by $2\pi/3$ for $|\psi_{K,A}(\vec{q})\rangle$:

$$\begin{aligned}
\langle \vec{r}_i|\psi_{K,A}(\vec{q})\rangle &\sim \exp[i(\vec{K}+\vec{q}).\vec{r}_{i,A}] \\
\langle \vec{r}_i|J_{\frac{2\pi}{3}}|\psi_{K,A}(\vec{q})\rangle &\sim \exp[iJ_{\frac{2\pi}{3}}(\vec{K}+\vec{q}).\vec{r}_{i,A}] \\
&= \exp[i(\vec{K}-\vec{G}_1 + J_{\frac{2\pi}{3}}\vec{q}).(\vec{r}_i+\vec{R}_2)] \\
&= \exp[-i\vec{G}_1.\vec{R}_2]\langle \vec{r}_i|\psi_{K,A}(J_{\frac{2\pi}{3}}\vec{q})\rangle \\
&= \exp(i2\pi/3)\,\langle \vec{r}_i|\psi_{K,A}(J_{\frac{2\pi}{3}}\vec{q})\rangle
\end{aligned}$$

Similarly at B sublattice,

$$\begin{aligned}
\langle \vec{r}_i|J_{\frac{2\pi}{3}}|\psi_{K,B}(\vec{q})\rangle &\sim \exp[iJ_{\frac{2\pi}{3}}(\vec{K}+\vec{q}).\vec{r}_{i,B}] \\
&= \exp[i\vec{K}-\vec{G}_1 + J_{\frac{2\pi}{3}}\vec{q}).(\vec{r}_i-\vec{R}_3)] \\
&= \exp[i\vec{G}_1.\vec{R}_3]\langle \vec{r}_i|\psi_{K,A}(J_{\frac{2\pi}{3}}\vec{q})\rangle \\
&= \exp(-i2\pi/3)\,\langle \vec{r}_i|\psi_{K,A}(J_{\frac{2\pi}{3}}\vec{q})\rangle.
\end{aligned}$$

At K', we get $\vec{K}' \to \vec{K}' + \vec{G}_1$ and the signs are reversed for A and B sublattices. Thus we get the above expression for the symmetry operation.

Translation by a lattice translation $\vec{R}$:

$$\begin{aligned}
\langle \vec{r}_i|T_{\vec{R}}\,\psi_{K,A/B}(\vec{q})\rangle &= \langle T_{-\vec{R}}\vec{r}_i|\,\psi_{K,A/B}(\vec{q})\rangle \\
&\sim \exp[i(\vec{K}+\vec{q}).(\vec{r}_{i,A/B}-\vec{R})] \\
&= \exp[-i(\vec{K}+\vec{q}).\vec{R}]\langle \vec{r}_i|\psi_{K,A/B}(\vec{q})\rangle \\
&= \exp[i\frac{4\pi}{3\sqrt{3}a_0}R_x]\exp[-i\vec{q}.\vec{R}]\langle \vec{r}_i|\psi_{K,A/B}(\vec{q})\rangle.
\end{aligned}$$

Similarly at $\vec{K}'(=-\vec{K})$; $\langle\vec{r}_i|T_R\,\psi_{K',A/B}(\vec{q})\rangle = \exp[-i\frac{4\pi}{3\sqrt{3}a_0}R_x]\exp[-i\vec{q}.\vec{R}]\langle\vec{r}_i|\psi_{K',A/B}(\vec{q})\rangle$.

2. Most general interaction Hamiltonian

For BLG, we can write all possible symmetry allowed local interactions making an analysis similar to that of Vafek *et. al.*[5]. There are 9 symmetry allowed spin diagonal terms. We write them in following particular form.

$$\begin{aligned}
I_1 =&(\psi^\dagger\psi)^2\\
I_2 =&(\psi^\dagger\mu^z\tau^z\psi)^2\\
I_3 =&(\psi^\dagger\mu^x\psi)^2+(\psi^\dagger\mu^y\tau^z\psi)^2\\
I_4 =&(\psi^\dagger\mu^z\psi)^2+(\psi^\dagger\mu^y\tau^y\psi)^2+(\psi^\dagger\mu^y\tau^x\psi)^2\\
I_5 =&(\psi^\dagger\tau^z\psi)^2+(\psi^\dagger\mu^x\tau^x\psi)^2+(\psi^\dagger\mu^x\tau^y\psi)^2\\
I_6 =&(\psi^\dagger\mu^x\tau^z\psi)^2+(\psi^\dagger\mu^y\psi)^2+(\psi^\dagger\tau^x\psi)^2+(\psi^\dagger\tau^y\psi)^2\\
&+(\psi^\dagger\mu^z\tau^x\psi)^2+(\psi^\dagger\mu^z\tau^y\psi)^2\\
I_7 =&(\psi^\dagger\mu^z\psi)^2-(\psi^\dagger\mu^y\tau^y\psi)^2-(\psi^\dagger\mu^y\tau^x\psi)^2\\
I_8 =&(\psi^\dagger\tau^z\psi)^2-(\psi^\dagger\mu^x\tau^x\psi)^2-(\psi^\dagger\mu^x\tau^y\psi)^2\\
I_9 =&(\psi^\dagger\mu^x\tau^z\psi)^2+(\psi^\dagger\mu^y\psi)^2-(\psi^\dagger\tau^x\psi)^2-(\psi^\dagger\tau^y\psi)^2\\
&-(\psi^\dagger\mu^z\tau^x\psi)^2-(\psi^\dagger\mu^z\tau^y\psi)^2
\end{aligned}\tag{B17}$$

These combinations of different terms are made such that the first six interaction terms above commute with the unitary the transformation $\hat{U}=\exp(i\frac{\pi}{4}\mu^x\tau^y)$, which maps FLP state order parameter to STF state. Thus, having only first 6 terms will not lift the degeneracy between FLP and STF ground states.

In addition, for a spin symmetric interaction, the allowed spin dependent terms are $I_{1s}=(\psi^\dagger\vec{s}\psi)^2$,$I_{2s}=(\psi^\dagger\mu^z\tau^z\vec{s}\psi)^2,...I_{9s}$.

We can use Fierz identities to find the number of independent interaction terms out of the above 18 terms. These identities can be written in terms of the 32 individual quartic terms appearing in $I_1,...I_9$ and $I_{1s},...I_{9s}$ as:

$$(\psi^\dagger O_a\psi)^2=-\frac{1}{8}\sum_b \mathrm{Sgn}[O_aO_b]\,((\psi^\dagger O_b\psi)^2+(\psi^\dagger O_b\vec{s}\psi)^2)$$

$$(\psi^\dagger O_a\vec{s}\psi)^2=-\frac{1}{8}\sum_b \mathrm{Sgn}[O_aO_b]\,(3(\psi^\dagger O_b\psi)^2-(\psi^\dagger O_b\vec{s}\psi)^2)$$

$$\begin{aligned}
O_a =&\{1,\mu^z\tau^z,\mu^x,\mu^y\tau^z,\,\mu^z,\mu^y\tau^y,\mu^y\tau^x,\tau^z,\mu^x\tau^x,\mu^x\tau^y,\\
&\mu^x\tau^z,\mu^y,\tau^x,\tau^y,\mu^z\tau^x,\mu^z\tau^y\}
\end{aligned}$$

Out of these 18 equations, only 9 are independent and therefore we can express $(\psi^\dagger O_a\vec{s}\psi)^2$ terms in terms of $(\psi^\dagger O_a\psi)^2$. Then $I_{1s},I_{2s}..I_{9s}$ can be rewritten in terms of $I_1,I_2..I_9$ as $I_{is}=\sum\Gamma_{ij}I_j$, where

$$\Gamma=\begin{pmatrix}
-\frac{3}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&0&0&0\\
-\frac{1}{2}&-\frac{3}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&0&0&0\\
-1&1&-1&1&-1&0&0&0&0\\
-\frac{3}{2}&-\frac{3}{2}&\frac{3}{2}&-\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&0&0&0\\
-\frac{3}{2}&-\frac{3}{2}&-\frac{3}{2}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&0&0&0\\
-3&3&0&-1&1&-1&0&0&0\\
\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&-2&-1&1\\
\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-1&-2&-1\\
1&-1&0&1&-1&0&2&-2&-1
\end{pmatrix}.\tag{B18}$$

The most general local interaction Hamiltonian for BLG can now be written as

$$H_{\rm int}=\sum_{i=1}^{9}g_iI_i\tag{B19}$$

Appendix C: Mean field analysis for $\nu=2$ BLG

For filling fraction $\nu=2$, we consider two symmetry broken states and compare their ground state energies to find the favored state at zero electric field. Below we discuss the the mean field analysis for both of them.

1. Layer polarized state

We first consider a state in which inversion symmetry is broken, but translational and rotational symmetries are preserved. In such state, the following mean-field terms are allowed:

$$O_{a=\{1,2,3\}}=\{\mu^z,\tau^z,\mu^z\tau^z\}.\tag{C1}$$

Note that, since we have assumed a uniform state and these are local terms, they must be diagonal in the LL index n. We therefore write the mean-field Hamiltonian as follows:

$$H_{MF}=H_0-\sum_{\zeta=\pm1,a=1,2,3}\lambda_{a\zeta}\psi^\dagger\frac{1+\zeta s^z}{2}O_a\psi-h\psi^\dagger s^z\psi-\mu_0\psi^\dagger\psi,\tag{C2}$$

where h is a Zeeman field. The Hamiltonian can be diagonalized and the spectrum is

$$E_{n,s,\tau}=\begin{cases}-hs-\mu_0-(\lambda_{1s}+\lambda_{2s})\tau-\lambda_{3s} & ,\ n\le1\\ -hs-\mu_0-\lambda_{2s}\tau\pm\sqrt{(\lambda_{1s}+\tau\lambda_{3s})^2+E_n^2} & ,\ n>1\end{cases}\tag{C3}$$

Filling all the Landau levels up to $\nu=2$, the total energy is:

$$E_{MF} = \sum_{\tau=\pm1} (E_{0,\uparrow,\tau} + E_{1,\uparrow,\tau}) + E_{0,\downarrow,\tau=1} + E_{1,\downarrow,\tau=1} + \sum_{n=2}^{\infty} \sum_{\tau,s=\pm1} E_{n,s,\tau}$$
$$= 2\left[-h - 2\lambda_{3\uparrow} - \lambda_{1\downarrow} - \lambda_{2\downarrow} - \lambda_{3\downarrow}\right] - \sum_{n=2}^{\infty} \sum_{\tau,s=\pm1} \sqrt{(\lambda_{1s} + \tau\lambda_{3s})^2 + E_n^2} \quad \text{(C4)}$$

We first consider the weak interaction limit such that $\lambda \ll \hbar\omega_c$ and we can expand to second order in the λ's. We get (ignoring the chemical potential terms)

$$E_{MF} \approx 2\left[-h - 2\lambda_{3\uparrow} - \lambda_{1\downarrow} - \lambda_{2\downarrow} - \lambda_{3\downarrow}\right] - \frac{\chi_0 \ell_B^2}{2} \sum_{s=\pm1} \left(\lambda_{1s}^2 + \lambda_{3s}^2\right) - E_0 \quad \text{(C5)}$$

where $\chi_0 \equiv \sum_{n=2}^{\infty} \frac{2}{E_n \ell_B^2} = \frac{m}{\pi} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ and $E_0 = -4\sum_{n=2}^{\infty} E_n$ (Note that these sums actually diverge, and a cutoff needs to be introduced). E_{MF} above is the energy per ℓ_B^2 area of the system while $\langle H_{int}\rangle$ terms are energy per unit area. Thus, collecting the different terms in Eq. (B16) and matching the dimensions, we get (dropping constants)

$$\tilde{E}(\lambda_\gamma, \mu_0) = \frac{1}{\ell_B^2}(-4\lambda_{3\uparrow} - 2\lambda_{1\downarrow} - 2\lambda_{2\downarrow} - 2\lambda_{3\downarrow}) + \frac{1}{2}\lambda_1^T \left(M_1^{-1} - \chi_0\right)\lambda_1 + \frac{1}{2}\lambda_2^T M_2^{-1}\lambda_2 + \frac{1}{2}\lambda_3^T \left(M_3^{-1} - \chi_0\right)\lambda_3 \quad \text{(C6)}$$

This is conveniently written as

$$\tilde{E}(\lambda_\gamma, \mu_0) = -\frac{1}{\ell_B^2} Q^T \lambda + \frac{1}{2}\lambda^T \tilde{M}^{-1} \lambda, \quad \text{(C7)}$$

where $\lambda^T = (\lambda_{1\uparrow}, \lambda_{1\downarrow}, \ldots, \lambda_{3\downarrow})$. Minimizing the energy over λ gives

$$E_{min} = -\frac{1}{2\ell_B^4} Q^T \tilde{M} Q \quad \text{(C8)}$$

We can now calculate $\langle H_{int}\rangle_{FLP}$, using the general interaction Hamiltonian in Eq. (B19). In FLP, three ϕ's corresponding to the above O_a's are nonzero. Calculating $\langle H_{int}\rangle_{FLP}$, we get the following M_i matrices in Eq. (C6) for FLP phase:

$$M_1 = \begin{pmatrix} \frac{K_1}{2} - 2g_4 - 2g_7 & -2g_4 - 2g_7 \\ -2g_4 - 2g_7 & \frac{K_1}{2} - 2g_4 - 2g_7 \end{pmatrix}, \quad \text{(C9)}$$

$$M_2 = \begin{pmatrix} \frac{K_2}{2} - 2g_5 - 2g_8 & -2g_5 - 2g_8 \\ -2g_5 - 2g_8 & \frac{K_2}{2} - 2g_5 - 2g_8 \end{pmatrix}, \quad \text{(C10)}$$

$$M_3 = \begin{pmatrix} \frac{K_3}{2} - 2g_2 & -2g_2 \\ -2g_2 & \frac{K_3}{2} - 2g_2 \end{pmatrix}, \quad \text{(C11)}$$

where
$K_1 = g_1 + g_2 - 2g_3 - g_4 - g_5 + 2g_6 + 3g_7 + 3g_8 - 6g_9$
$K_2 = g_1 + g_2 + 2g_3 - g_4 - g_5 - 2g_6 + 3g_7 + 3g_8 + 6g_9$
$K_3 = g_1 + g_2 - 2g_3 + 3g_4 + 3g_5 - 6g_6 - g_7 - g_8 + 2g_9$

2. Staggered flux state

The second natural possibility for an ordered state (that lifts the degeneracy of the lowest Landau level) is a staggered flux (STF) state with a wavevector that connects K to K'. This state preserves the three fold rotational symmetry $\hat{R}_{2\pi/3} = \exp(2\pi i \mu^z \tau^z/3)$ around the aligned sites and the inversion symmetry, but breaks translational symmetry. The allowed mean fields in this state are

$$O_{a=\{1,2,3\}} = \{\mu^x\tau^x, \mu^y\tau^y, \mu^z\tau^z\}. \quad \text{(C12)}$$

Now,

$$H_{MF} = H_0 - \sum_{\zeta=\pm1, a=1,2,3} \lambda_{a\zeta} \psi^\dagger \frac{1+\zeta s^z}{2} O_a \psi - h\psi^\dagger s^z \psi - \mu_0 \psi^\dagger \psi \quad \text{(C13)}$$

which has the spectrum

$$E_{n,s,\tau} = \begin{cases} -hs - \mu_0 + (\lambda_{1s} - \lambda_{2s})\tau - \lambda_{3s} & , \ n \le 1 \\ -hs - \mu_0 + \lambda_{1s}\tau \pm \sqrt{(\lambda_{2s} + \tau\lambda_{3s})^2 + E_n^2} & , \ n > 1. \end{cases} \quad \text{(C14)}$$

Filling all the Landau levels up to $\nu = 2$, the total energy is:

$$E_{MF} = \sum_{\tau=\pm1} (E_{0,\uparrow,\tau} + E_{1,\uparrow,\tau}) + E_{0,\downarrow,\tau=1} + E_{1,\downarrow,\tau=1} + \sum_{n=2}^{\infty} \sum_{\tau,s=\pm1} E_{n,s,\tau}$$
$$= 2\left[-h - |\lambda_{1\downarrow} - \lambda_{2\downarrow}| - 2\lambda_{3\uparrow} - \lambda_{3\downarrow}\right] - \sum_{n=2}^{\infty} \sum_{\tau,s=\pm1} \sqrt{(\lambda_{2s} + \tau\lambda_{3s})^2 + E_n^2} \quad \text{(C15)}$$

Again considering the weak interaction limit and expanding to second order in the λ's, we get

$$E_{MF} \approx 2\left[-h - |\lambda_{1\downarrow} - \lambda_{2\downarrow}| - 2\lambda_{3\uparrow} - \lambda_{3\downarrow}\right] - \frac{\chi_0 \ell_B^2}{2} \sum_{s=\pm 1} \left(\lambda_{2s}^2 + \lambda_{3s}^2\right) - E_0 \quad \text{(C16)}$$

Taking $\langle H_{int}\rangle_{STF}$, we get the following M_i matrices corresponding to the STF order parameters above

$$M_{1S} = \begin{pmatrix} \frac{K_{1S}}{2} - 2g_5 + 2g_8 & -2g_5 + 2g_8 \\ -2g_5 + 2g_8 & \frac{K_{1S}}{2} - 2g_5 + 2g_8 \end{pmatrix}, \quad \text{(C17)}$$

$$M_{2S} = \begin{pmatrix} \frac{K_{2S}}{2} - 2g_4 + 2g_7 & -2g_4 + 2g_7 \\ -2g_4 + 2g_7 & \frac{K_{2S}}{2} - 2g_4 + 2g_7 \end{pmatrix}, \quad \text{(C18)}$$

$$M_{3S} = \begin{pmatrix} \frac{K_{3S}}{2} - 2g_2 & -2g_2 \\ -2g_2 & \frac{K_{3S}}{2} - 2g_2 \end{pmatrix}, \quad \text{(C19)}$$

where

$$\begin{aligned} K_{1S} =& g_1 + g_2 + 2g_3 - g_4 - g_5 - 2g_6 - g_7 - g_8 - 2g_9 \\ K_{2S} =& g_1 + g_2 - 2g_3 - g_4 - g_5 + 2g_6 - g_7 - g_8 + 2g_9 \\ K_{3S} =& g_1 + g_2 - 2g_3 + 3g_4 + 3g_5 - 6g_6 - g_7 - g_8 + 2g_9 \end{aligned}$$

$$\begin{aligned} \tilde{E}(\lambda_\gamma, \mu_0) &= \frac{2}{\ell_B^2}\left[-h - (\lambda_{1\downarrow} - \lambda_{2\downarrow}) - 2\lambda_{3\uparrow} - \lambda_{3\downarrow}\right] + \frac{1}{2}\lambda_1^T M_{1S}^{-1}\lambda_1 \\ &+ \frac{1}{2}\lambda_2^T\left(M_{2S}^{-1} - \chi_0\right)\lambda_2 + \frac{1}{2}\lambda_3^T\left(M_{3S}^{-1} - \chi_0\right)\lambda_3 \\ &= -\frac{1}{\ell_B^2} Q_S^T \lambda + \frac{1}{2}\lambda^T \tilde{M_S}^{-1}\lambda, \end{aligned} \quad \text{(C20)}$$

And after minimizing over λ

$$E_{min} = -\frac{1}{2\ell_B^4} Q_S^T \tilde{M_S} Q_S. \quad \text{(C21)}$$

Using equations (C8) and (C21), we can calculate the ground state energies of the two phases ($E_{\rm FLP}$ and $E_{\rm STF}$) and compare which phase has lower energy. We get

$$\begin{aligned} &\Delta E = E_{\rm FLP} - E_{\rm STF} \\ &= \frac{4\chi_0}{\ell_B^4}\left(g_1 g_7 + g_2 g_7 - 2g_3 g_7 - 13 g_4 g_7 - g_5 g_7 + 2g_6 g_7 + g_7^2\right. \\ &- g_1 g_8 - g_2 g_8 + 2g_3 g_8 + 5g_4 g_8 + g_5 g_8 - 2g_6 g_8 - g_8^2 + 2g_1 g_9 \\ &\left. + 2g_2 g_9 - 4g_3 g_9 - 10 g_4 g_9 - 2g_5 g_9 + 4g_6 g_9 + 4g_8 g_9 - 4g_9^2\right) \end{aligned} \quad \text{(C22)}$$

3. Large Interactions

For the case when the interactions are large and above approximation doesn't work, we numerically minimize the energy functionals $E(\{\lambda_a\}, \mu_0)$ w.r.t. λ_a. In Eq. (B10), we use the MF equation $\phi_{as} = -\partial E_{MF}/\partial\lambda_{as}$ and find the minimum of the resulting energy functional.

Appendix D: Change of variables from $V_0, U, V_{1,2,3}$ and J_H to $g_1 ... g_9$ in the Coulomb Hamiltonian

We have found the ground state energies for the most general interactions in the weak coupling limit. We can now use our interaction Hamiltonian in Eq. (A4) and read off the interaction parameters $g_1, ..g_9$. Then we can obtain the difference in ground state energies of FLP and STF phases from Eq. (C22) to determine which state is favored.

To write the U term, we use the identity

$$(n_\uparrow - \frac{1}{2})(n_\downarrow - \frac{1}{2}) = \frac{1}{4} - \frac{1}{6}\left(\psi^\dagger \vec{s}\psi\right)^2 \quad \text{(D1)}$$

that holds for a single orbital.

Up to a chemical potential term this gives,

$$\begin{aligned} &\sum_\mu n_{\mu K\uparrow} n_{\mu K\downarrow} + n_{\mu K'\uparrow} n_{\mu K'\downarrow} \\ &= -\frac{1}{6}\left[\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\vec{s}\psi\right)\cdot\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\vec{s}\psi\right)\right. \\ &\left. + \left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\vec{s}\psi\right)\cdot\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\vec{s}\psi\right)\right] \\ &+ (\mu \to -\mu) \\ &= -\frac{1}{6}\left[\frac{1}{4}\left(\psi^\dagger\vec{s}\psi\right)\cdot\left(\psi^\dagger\vec{s}\psi\right) + \frac{1}{4}\left(\psi^\dagger\mu^z\vec{s}\psi\right)\cdot\left(\psi^\dagger\mu^z\vec{s}\psi\right)\right. \\ &\left. + \frac{1}{4}\left(\psi^\dagger\tau^z\vec{s}\psi\right)\cdot\left(\psi^\dagger\tau^z\vec{s}\psi\right) + \frac{1}{4}\left(\psi^\dagger\mu^z\tau^z\vec{s}\psi\right)\cdot\left(\psi^\dagger\mu^z\tau^z\vec{s}\psi\right)\right] \\ &= -\frac{1}{24}\left(I_{1s} + I_{2s} + \frac{I_{4s} + I_{7s}}{2} + \frac{I_{5s} + I_{8s}}{2}\right) \end{aligned} \quad \text{(D2)}$$

Using the the relations between I_{is} and I_s in Eq. (B18), we get

$$\begin{aligned} &n_{\mu K\uparrow} n_{\mu K\downarrow} + n_{\mu K'\uparrow} n_{\mu K'\downarrow} \\ &= \frac{1}{8}\left(I_1 + I_2 + \frac{I_4 + I_7}{2} + \frac{I_5 + I_8}{2}\right). \end{aligned} \quad \text{(D3)}$$

V_1 term:

$$
\begin{aligned}
&V_1 \sum_{\mu} n_{\mu K} n_{\mu K'} \\
&= V_1 \left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\psi\right)\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\psi\right) + (\mu \to -\mu) \\
&= \frac{V_1}{8}\left[\left(\psi^\dagger\psi\right)\left(\psi^\dagger\psi\right) + \left(\psi^\dagger\mu^z\psi\right)\left(\psi^\dagger\mu^z\psi\right)\right. \\
&\qquad \left. - \left(\psi^\dagger\tau^z\psi\right)\left(\psi^\dagger\tau^z\psi\right) - \left(\psi^\dagger\mu^z\tau^z\psi\right)\left(\psi^\dagger\mu^z\tau^z\psi\right)\right] \\
&= \frac{V_1}{8}\left(I_1 - I_2 + \frac{I_4+I_7}{2} - \frac{I_5+I_8}{2}\right). \qquad \text{(D4)}
\end{aligned}
$$

V_2 term:

$$
\begin{aligned}
&V_2(n_{-1K}n_{1K} + n_{-1K'}n_{1K'}) \\
&= V_2\left[\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\psi\right)\left(\psi^\dagger \frac{1-\mu^z}{2}\frac{1+\tau^z}{2}\psi\right)\right. \\
&\qquad \left. + \left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\psi\right)\left(\psi^\dagger \frac{1-\mu^z}{2}\frac{1-\tau^z}{2}\psi\right)\right] \\
&= \frac{V_2}{8}\left[\left(\psi^\dagger\psi\right)\left(\psi^\dagger\psi\right) - \left(\psi^\dagger\mu^z\psi\right)\left(\psi^\dagger\mu^z\psi\right)\right. \\
&\qquad \left. + \left(\psi^\dagger\tau^z\psi\right)\left(\psi^\dagger\tau^z\psi\right) - \left(\psi^\dagger\mu^z\tau^z\psi\right)\left(\psi^\dagger\mu^z\tau^z\psi\right)\right] \\
&= \frac{V_2}{8}\left(I_1 - I_2 - \frac{I_4+I_7}{2} + \frac{I_5+I_8}{2}\right). \qquad \text{(D5)}
\end{aligned}
$$

V_3 term:

$$
\begin{aligned}
&V_3(n_{-1K}n_{1K'} + n_{-1K'}n_{1K}) \\
&= V_3\left[\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\psi\right)\cdot\left(\psi^\dagger \frac{1-\mu^z}{2}\frac{1-\tau^z}{2}\psi\right)\right. \\
&\qquad \left. + \left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\psi\right)\cdot\left(\psi^\dagger \frac{1-\mu^z}{2}\frac{1+\tau^z}{2}\psi\right)\right] \\
&= \frac{V_3}{8}\left[\left(\psi^\dagger\psi\right)\left(\psi^\dagger\psi\right) - \left(\psi^\dagger\mu^z\psi\right)\left(\psi^\dagger\mu^z\psi\right)\right. \\
&\qquad \left. - \left(\psi^\dagger\tau^z\psi\right)\left(\psi^\dagger\tau^z\psi\right) + \left(\psi^\dagger\mu^z\tau^z\psi\right)\left(\psi^\dagger\mu^z\tau^z\psi\right)\right] \\
&= \frac{V_3}{8}\left(I_1 + I_2 - \frac{I_4+I_7}{2} - \frac{I_5+I_8}{2}\right). \qquad \text{(D6)}
\end{aligned}
$$

J_H term:

$$
\begin{aligned}
&-J_H \sum_{\mu} \vec{s}_{\mu K}\cdot\vec{s}_{\mu K'} \\
&= -J_H\left[\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1+\tau^z}{2}\vec{s}\psi\right)\cdot\left(\psi^\dagger \frac{1+\mu^z}{2}\frac{1-\tau^z}{2}\vec{s}\psi\right)\right. \\
&\qquad \left. +(\mu\to-\mu)\right] \\
&= -\frac{J_H}{8}\left[\left(\psi^\dagger\vec{s}\psi\right)^2 + \left(\psi^\dagger\mu^z\vec{s}\psi\right)^2 - \left(\psi^\dagger\tau^z\vec{s}\psi\right)^2 - \left(\psi^\dagger\mu^z\tau^z\vec{s}\psi\right)^2\right] \\
&= -\frac{J_H}{8}\left(I_{1s} - I_{2s} + \frac{I_{4s}+I_{7s}}{2} - \frac{I_{5s}+I_{8s}}{2}\right) \\
&= -\frac{J_H}{8}\left(-I_1 + I_2 - \frac{I_4+I_7}{2} + \frac{I_5+I_8}{2} - I_6 + I_9\right) \qquad \text{(D7)}
\end{aligned}
$$

These give the following values of the interaction parameters.

$$
\begin{aligned}
g_1 &= V_0 + \frac{1}{8}\left(U + V_1 + V_2 + V_3 + J_H\right) \\
g_2 &= \frac{1}{8}\left(U - V_1 - V_2 + V_3 - J_H\right) \\
g_3 &= 0 \\
g_4 &= \frac{1}{16}\left(U + V_1 - V_2 - V_3 + J_H\right) \\
g_5 &= \frac{1}{16}\left(U - V_1 + V_2 - V_3 - J_H\right) \\
g_6 &= \frac{J_H}{8} \\
g_7 &= \frac{1}{16}\left(U + V_1 - V_2 - V_3 + J_H\right) \\
g_8 &= \frac{1}{16}\left(U - V_1 + V_2 - V_3 - J_H\right) \\
g_9 &= -\frac{1}{8}J_H \qquad \text{(D8)}
\end{aligned}
$$

Plugging these in Eq. (C22), we get the difference in mean field ground state energies of FLP and STF phases for the weak interaction limit where we can expand the Mean Field energies to 2^{nd} order in λ_a.

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