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Existence results for classes of infinite semipositone problems

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Abstract

We consider the problem

$$\begin{cases} -\Delta_p u = \frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $a > 0$, $b > 0$, $c \geq 0$, $\gamma > p$ and $\alpha \in (0, 1)$. Given a, b, γ and α, we establish the existence of a positive solution for small values of c. These results are also extended to corresponding exterior domain problems. Also, a bifurcation result for the case $c = 0$ is presented.

1 Introduction

Consider the nonsingular boundary value problem:

$$\begin{cases} -\Delta u = au - bu^2 - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $a > 0$, $b > 0$, $c \geq 0$, $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u and $h : \bar{\Omega} \to R$ is a $C^1(\bar{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, $\max_{x\in\bar{\Omega}} h(x) = 1$ and $h(x) = 0$ for $x \in \partial\Omega$. Existence of positive solutions of problem (1) was studied in [1]. In particular, it was proved that given an $a > \lambda_1$ and $b > 0$ there exists a $c^*(a, b, \Omega) > 0$ such that for $c < c^*$ (1) has positive solutions. Here, λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Nonexistence of a positive solution was also proved when $a \leq \lambda_1$. Later in [2], these results were extended to the case of the p-Laplacian operator, Δ_p, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. Boundary value problems of the form (1) are known as semipositone problems since the nonlinearity $f(s, x) = as - bs^2 - ch(x)$ satisfies $f(0, x) < 0$ for some $x \in \Omega$. See [3–9] for some existence results for semipositone problems.

In this paper, we study positive solutions to the singular boundary value problem:

$$\begin{cases} -\Delta_p u = \frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $a > 0$, $b > 0$, $c \geq 0$, $\alpha \in (0,1)$, $p > 1$, and $\gamma > p$. In the literature, problems of the form (2) are referred to as infinite semipositone problems as the nonlinearity $f(s) = \frac{as^{p-1}-bs^{\gamma-1}-c}{s^\alpha}$ satisfies $\lim_{s\to 0^+} f(s) = -\infty$. One can refer to [10–14], and [15–17] for some recent existence results of infinite semipositone problems. We establish the following theorem.

Theorem 1.1 *Given $a, b > 0$, $\gamma > p$, and $\alpha \in (0,1)$, there exists a constant $c_1 = c_1(a, b, \alpha, p, \gamma, \Omega) > 0$ such that for $c < c_1$, (2) has a positive solution.*

Remark 1.1 In the nonsingular case ($\alpha = 0$), positive solutions exist only when $a > \lambda_1$ (the principal eigenvalue) (see [1, 2]). But in the singular case, we establish the existence of a positive solution for any given $a > 0$.

Next, we study positive radial solutions to the problem:

$$\begin{cases} -\Delta_p u = K(|x|)(\frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha}), & x \in \Omega, \\ u = 0, & \text{if } |x| = r_0, \\ u \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{3}$$

where $\Omega = \{x \in \mathbb{R}^n \mid |x| > r_0\}$ is an exterior domain, $n > p$, $a > 0$, $b > 0$, $c \geq 0$, $\alpha \in (0,1)$, $p > 1$, $\gamma > p$ and $K : [r_0, \infty) \to (0, \infty)$ belongs to a class of continuous functions such that $\lim_{r\to\infty} K(r) = 0$. By using the transformation: $r = |x|$ and $s = (\frac{r}{r_0})^{\frac{-n+p}{p-1}}$, we reduce (3) to the following boundary value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = h(s)(\frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha}), & 0 < s < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{4}$$

where $h(s) = (\frac{p-1}{n-p})^p r_0^p s^{\frac{-p(n-1)}{n-p}} K(r_0 s^{\frac{-(p-1)}{n-p}})$. We assume:

(H_1) $K \in C([r_0, \infty), (0, \infty))$ and satisfies $K(r) < \frac{1}{r^{n+\theta}}$ for $r \gg 1$, and for some θ such that $(\frac{n-p}{p-1})\alpha < \theta < \frac{n-p}{p-1}$.

With the condition (H_1), h satisfies:

$$\text{there exists } \epsilon_1 > 0 \text{ such that } h(s) \leq \frac{1}{s^\rho} \text{ for all } s \in (0, \epsilon_1),$$
$$\text{where } \rho = \frac{n-p-\theta(p-1)}{n-p}. \tag{5}$$

We note that if $\theta \geq \frac{n-p}{p-1}$ then $h(s)$ is nonsingular at 0 and $h \in C([0,1], (0,\infty))$. In this case, problem (4) can be studied using ideas in the proof of Theorem 1.1. Hence, our focus is on the case when $\theta < \frac{n-p}{p-1}$ in which, h may be singular at 0. Note that in this case $\hat{h} = \inf_{s\in(0,1)} h(s) > 0$.

Remark 1.2 Note that $\rho + \alpha < 1$ since $\theta > (\frac{n-p}{p-1})\alpha$.

We then establish the following theorem.

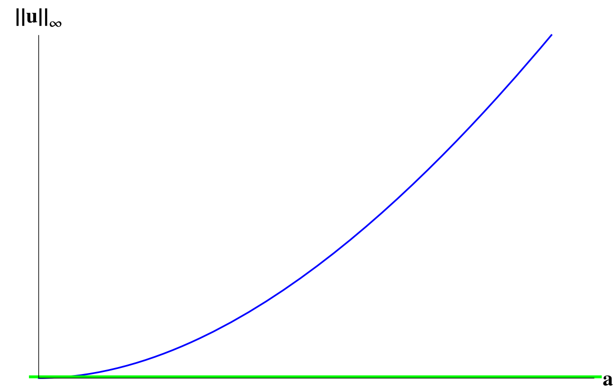

Figure 1 Bifurcation diagram, *a* vs. $\|u\|_\infty$ for (6).

Theorem 1.2 *Given $a, b > 0$, $\gamma > p$, $\alpha \in (0,1)$, and assume (H_1) holds. Then there exists a constant $c_2 = c_2(a, b, \alpha, p, \gamma) > 0$ such that for $c < c_2$, (3) has a positive radial solution.*

Finally, we prove a bifurcation result for the problem

$$\begin{cases} -\Delta_p u = \frac{au^{p-1} - bu^{\gamma-1}}{u^\alpha}, & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, a is a positive parameter, $b, \alpha > 0$, $p > 1 + \alpha$ and $\gamma > p$. We prove the following.

Theorem 1.3 *The boundary value problem* (6) *has a branch of positive solutions bifurcating from the trivial branch of solutions* $(a, 0)$ *at* $(0, 0)$ (*as shown in Figure* 1).

Our results are obtained *via* the method of sub-super solutions. By a subsolution of (2), we mean a function $\psi \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ that satisfies

$$\begin{cases} \int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \, dx \leq \int_\Omega \frac{a\psi^{p-1} - b\psi^{\gamma-1} - c}{\psi^\alpha} w \, dx, & \text{for every } w \in W, \\ \psi > 0, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases}$$

and by a supersolution we mean a function $Z \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} \int_\Omega |\nabla Z|^{p-2}\nabla Z \cdot \nabla w \, dx \geq \int_\Omega \frac{aZ^{p-1} - bZ^{\gamma-1} - c}{Z^\alpha} w \, dx, & \text{for every } w \in W, \\ Z > 0, & \text{in } \Omega, \\ Z = 0, & \text{on } \partial\Omega, \end{cases}$$

where $W = \{\xi \in C_0^\infty(\Omega) | \xi \geq 0 \text{ in } \Omega\}$. The following lemma was established in [13].

Lemma 1.4 (see [13, 18]) *Let ψ be a subsolution of* (2) *and Z be a supersolution of* (2) *such that $\psi \leq Z$ in Ω. Then* (2) *has a solution u such that $\psi \leq u \leq Z$ in Ω.*

Finding a positive subsolution, ψ, for such infinite semipositone problems is quite challenging since we need to construct ψ in such a way that $\lim_{x\to\partial\Omega} -\Delta_p\psi = -\infty$ and $-\Delta_p\psi > 0$ in a large part of the interior. In this paper, we achieve this by constructing subsolutions of the form $\psi = k\phi_1^\beta$, where k is an appropriate positive constant, $\beta \in (1, \frac{p}{p-1})$ and ϕ_1 is the eigenfunction corresponding to the first eigenvalue of $-\Delta_p\phi = \lambda|\phi|^{p-2}\phi$ in Ω, $\phi = 0$ on $\partial\Omega$.

In Sections 2, 3, and 4, we provide proofs of our results. Section 5 is concerned with providing some exact bifurcation diagrams of positive solutions of (2) when $\Omega = (0,1)$ and $p = 2$.

2 Proof of Theorem 1.1

We first construct a subsolution. Consider the eigenvalue problem $-\Delta_p\phi = \lambda|\phi|^{p-2}\phi$ in Ω, $\phi = 0$ on $\partial\Omega$. Let ϕ_1 be an eigenfunction corresponding to the first eigenvalue λ_1 such that $\phi_1 > 0$ and $\|\phi_1\|_\infty = 1$. Also, let $\delta, m, \mu > 0$ be such that $|\nabla\phi_1| \geq m$ in Ω_δ and $\phi_1 \geq \mu$ in $\Omega - \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$. Let $\beta \in (1, \frac{p}{p-1+\alpha})$ be fixed. Here, note that since $\alpha \in (0,1)$, $\frac{p}{p-1+\alpha} > 1$. Choose a $k > 0$ such that $2bk^{\gamma-p} + \beta^{p-1}\lambda_1 k^\alpha \leq a$. Define $c_1 = \min\{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p, \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a - \beta^{p-1}\lambda_1 k^\alpha)\}$. Note that $c_1 > 0$ by the choice of k and β. Let $\psi = k\phi_1^\beta$. Then

$$-\Delta_p\psi = k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove ψ is a subsolution, we need to establish:

$$\begin{aligned} &k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}} \\ &\quad\leq ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} - \frac{c}{k^\alpha\phi_1^{\alpha\beta}} \end{aligned} \tag{7}$$

in Ω if $c < c_1$. To achieve this, we split the term $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}$ into three, namely,

$$\begin{aligned} k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} &= ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right) \\ &\quad - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right). \end{aligned}$$

Now to prove (7) holds in Ω, it is enough to show the following three inequalities:

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right) \leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}, \quad \text{in } \Omega, \tag{8}$$

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right) \leq -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \quad \text{in } \Omega - \Omega_\delta, \tag{9}$$

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}} \leq -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \quad \text{in } \Omega_\delta. \tag{10}$$

From the choice of k, $-(a - \beta^{p-1}\lambda_1 k^\alpha) \leq -2bk^{\gamma-p}$, hence,

$$\begin{aligned} -\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right) &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)} \\ \leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}. \end{aligned} \tag{11}$$

Using $\phi_1 \geq \mu$ in $\Omega - \Omega_\delta$ and $c < \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a - \beta^{p-1}\lambda_1 k^\alpha)$

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\bigl(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\bigr) \leq \frac{-k^{p-1}\phi_1^{\beta(p-1)}(a - k^\alpha\lambda_1\beta^{p-1})}{2k^\alpha\phi_1^{\alpha\beta}} \leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}}. \tag{12}$$

Finally, since $|\nabla\phi_1| \geq m$, in Ω_δ, and $c < k^{p-1+\alpha}\beta^{p-1}(\beta - 1)(p - 1)m^p$,

$$-k^{p-1}\beta^{p-1}(\beta - 1)(p - 1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}} \leq \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta - 1)(p - 1)m^p}{k^\alpha\phi_1^{\alpha\beta}\phi_1^{p-\beta(p-1)-\alpha\beta}} \leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}\phi_1^{p-\beta(p-1+\alpha)}}.$$

Since $p - \beta(p - 1 + \alpha) > 0$,

$$-k^{p-1}\beta^{p-1}(\beta - 1)(p - 1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}} \leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}}. \tag{13}$$

From (11), (12) and (13) we see that equation (7) holds in Ω, if $c < c_1$. Next, we construct a supersolution. Let e be the solution of $-\Delta_p e = 1$ in $\Omega, e = 0$ on $\partial\Omega$. Choose $\bar{M} > 0$ such that $\frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha} \leq \bar{M}^{p-1}\ \forall u > 0$ and $\bar{M}e \geq \psi$. Define $Z = \bar{M}e$. Then Z is a supersolution of (2). Thus, Theorem 1.1 is proven.

3 Proof of Theorem 1.2

We begin the proof by constructing a subsolution. Consider

$$\begin{aligned} &-\bigl(|\phi'|^{p-2}\phi'\bigr)' = \lambda|\phi|^{p-2}\phi, \quad t \in (0,1), \\ &\phi(0) = \phi(1) = 0. \end{aligned} \tag{14}$$

Let ϕ_1 be an eigenfunction corresponding to the first eigenvalue of (14) such that $\phi_1 > 0$ and $\|\phi_1\|_\infty = 1$. Then there exist $d_1 > 0$ such that $0 < \phi_1(t) \leq d_1 t(1 - t)$ for $t \in (0,1)$. Also, let $\epsilon < \epsilon_1$ and $m, \mu > 0$ be such that $|\phi_1'| \geq m$ in $(0,\epsilon] \cup [1 - \epsilon, 1)$ and $\phi_1 \geq \mu$ in $(\epsilon, 1 - \epsilon)$. Let $\beta \in (1, \frac{p-\rho}{p-1+\alpha})$ be fixed and choose $k > 0$ such that $2bk^{\gamma-p} + \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}} \leq a$. Define $c_2 = \min\{\frac{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{d_1^\rho}, \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}})\}$. Then $c_2 > 0$ by the choice of k and β. Let $\psi = k\phi_1^\beta$. This implies that:

$$-\bigl(|\psi'|^{p-2}\psi'\bigr)' = k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta - 1)(p - 1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove ψ is a subsolution, we need to establish:

$$\begin{aligned} &k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta - 1)(p - 1)\frac{\phi_1'^p}{\phi_1^{p-\beta(p-1)}} \\ &\quad\leq h(t)\left(ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} - \frac{c}{k^\alpha\phi_1^{\alpha\beta}}\right). \end{aligned} \tag{15}$$

Here, we note that the term $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} = \frac{\hat{h}k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}}{\hat{h}} \leq h(t)(ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}) - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}))$, where $\hat{h} = \inf_{s\in(0,1)} h(s)$. Now to prove (15) holds in $(0,1)$, it is enough to show the following three inequalities:

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\right) \leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}, \quad \text{in } (0,1), \tag{16}$$

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\right) \leq -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \quad \text{in } (\epsilon, 1-\epsilon), \tag{17}$$

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} \leq -\frac{ch(t)}{k^\alpha\phi_1^{\alpha\beta}}, \quad \text{in } (0,\epsilon] \cup [1-\epsilon, 1). \tag{18}$$

From the choice of k, $-(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}}) \leq -2bk^{\gamma-p}$, hence,

$$\begin{aligned} -\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\right) &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)} \\ &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}. \end{aligned} \tag{19}$$

Using $\phi_1 \geq \mu$ in $(\epsilon, 1-\epsilon)$ and $c < \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}})$

$$\begin{aligned} -\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\right) &\leq \frac{-k^{p-1}\phi_1^{\beta(p-1)}(a - \frac{k^\alpha\lambda_1\beta^{p-1}}{\hat{h}})}{2k^\alpha\phi_1^{\alpha\beta}} \\ &\leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}}. \end{aligned} \tag{20}$$

Next, we prove (18) holds in $(0,\epsilon]$. Since $|\phi_1'| \geq m$, and $p - \beta(p-1) > \alpha\beta + \rho$

$$\begin{aligned} -k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} &\leq \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi_1^{\alpha\beta}\phi_1^\rho} \\ &\leq \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi_1^{\alpha\beta}d_1^\rho t^\rho}. \end{aligned}$$

Since $h(t) \leq \frac{1}{t^\rho}$ in $(0,\epsilon]$, and $c < \frac{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{d_1^\rho}$,

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} \leq \frac{-ch(t)}{k^\alpha\phi_1^{\alpha\beta}}. \tag{21}$$

Proving (18) holds in $[1-\epsilon, 1)$ is straightforward since h is not singular at $t = 1$. Thus, from equations (19), (20) and (21), we see that (15) holds in $(0,1)$. Hence, ψ is a subsolution. Let $Z = \bar{M}e$ where e satisfies $-(|e'|^{p-2}e')' = h(t)$ in $(0,1)$, $e(0) = e(1) = 0$ and $\bar{M}$ is such that $\frac{au^{p-1} - bu^{\gamma-1} - c}{u^\alpha} \leq \bar{M}^{p-1}$ $\forall u > 0$ and $\bar{M}e \geq \psi$. Then Z is a supersolution of (4) and there exists a solution u of (4) such that $u \in [\psi, Z]$. Thus, Theorem 1.2 is proven.

4 Proof of Theorem 1.3

We first prove (6) has a positive solution for every $a > 0$. We begin by constructing a subsolution. Let ϕ_1 be as in the proof of Theorem 1.1 (see Section 2). Let $\beta \in (1, \frac{p}{p-1})$, and choose a $k > 0$ such that $bk^{\gamma-p} + \beta^{p-1}\lambda_1 k^\alpha \le a$. Let $\psi = k\phi_1^\beta$. Then

$$-\Delta_p \psi = k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove ψ is a subsolution, we will establish:

$$k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} \le ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} \tag{22}$$

in Ω. To achieve this, we rewrite the term $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}$ as $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} = ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1)$. Now to prove (22) holds in Ω, it is enough to show $-k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1) \le -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}$. From the choice of k, $-(a - \beta^{p-1}\lambda_1 k^\alpha) \le -bk^{\gamma-p}$, hence,

$$\begin{aligned}-k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right) &\le -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\\ &\le -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}.\end{aligned}$$

Thus, ψ is a subsolution. It is easy to see that $Z = (\frac{a}{b})^{\frac{1}{\gamma-p}}$ is a supersolution of (6). Since k, can be chosen small enough, $\psi \le Z$. Thus, (6) has a positive solution for every $a > 0$. Also, all positive solutions are bounded above by Z. Hence, when a is close to 0, every positive solution of (6) approaches 0. Also, $u \equiv 0$ is a solution for every a. This implies we have a branch of positive solutions bifurcating from the trivial branch of solutions $(a, 0)$ at $(0, 0)$.

5 Numerical results

Consider the boundary value problem

$$\begin{cases} -u''(x) = \frac{au-bu^2-c}{u^\alpha}, & x \in (0,1),\\ u(0) = 0 = u(1), \end{cases} \tag{23}$$

where $a, b > 0$, $c \ge 0$ and $\alpha \in (0,1)$. Using the quadrature method (see [19]), the bifurcation diagram of positive solutions of (23) is given by

$$G(\rho, c) = \int_0^\rho \frac{ds}{\sqrt{[2(F(\rho) - F(s))]}} = \frac{1}{2}, \tag{24}$$

where $F(s) := \int_0^s f(t)\,dt$ where $f(t) = \frac{at-bt^2-c}{t^\alpha}$ and $\rho = u(\frac{1}{2}) = \|u\|_\infty$. We plot the exact bifurcation diagram of positive solutions of (23) using Mathematica. Figure 2 shows bifurcation diagrams of positive solutions of (23) when $a = 8$ ($< \lambda_1$) and $b = 1$ for different values of α.

Bifurcation diagrams of positive solutions of (23) when $a = 15$ ($> \lambda_1$) and $b = 1$ for different values of α is shown in Figure 3.

Finally, we provide the exact bifurcation diagram for (6) when $p = 2$, and $\Omega = (0, 1)$. Consider

$$\begin{cases} -u''(x) = \frac{au-bu^2}{u^\alpha}, & x \in (0,1),\\ u(0) = 0 = u(1), \end{cases} \tag{25}$$

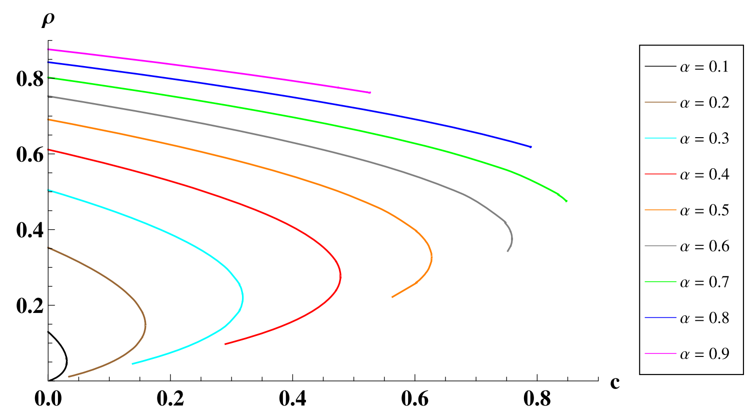

Figure 2 Bifurcation diagrams, *c* vs. ρ for (23) with $a = 8$, $b = 1$.

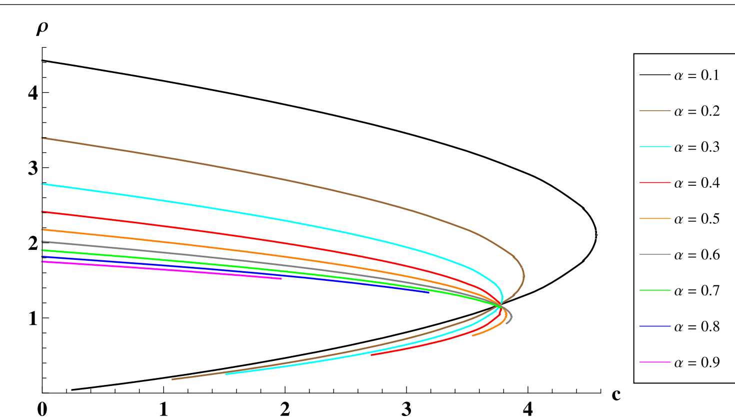

Figure 3 Bifurcation diagrams, *c* vs. ρ for (23) with $a = 15$, $b = 1$.

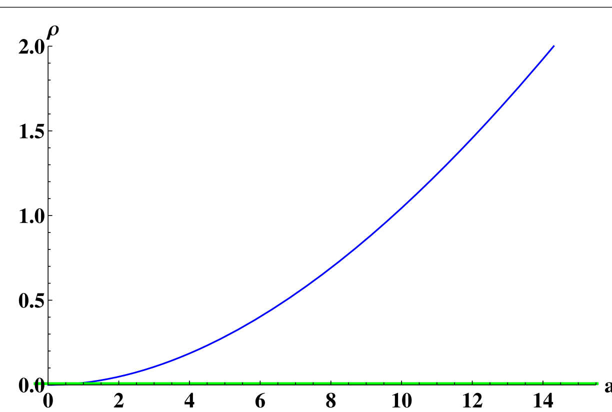

Figure 4 Bifurcation diagram, *a* vs. ρ for (25) with $\alpha = 0.5$, $b = 1$.

where $a, b, \alpha > 0$. The bifurcation diagram of positive solutions of (25) is given by

$$\tilde{G}(\rho, a) = \int_0^\rho \frac{ds}{\sqrt{[2(\tilde{F}(\rho) - \tilde{F}(s))]}} = \frac{1}{2}, \tag{26}$$

where $\tilde{F}(s) := \int_0^s \tilde{f}(t)\,dt$ where $\tilde{f}(t) = \frac{at-bt^2}{t^\alpha}$ and $\rho = u(\frac{1}{2}) = \|u\|_\infty$. The bifurcation diagram of positive solutions of (25) as well as the trivial solution branch are shown in Figure 4 when $\alpha = 0.5$ and $b = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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