On monopole operators in supersymmetric Chern-Simons-matter theories

Ofer Aharony, Prithvi Narayan and Tarun Sharma

Department of Particle Physics and Astrophysics, Weizmann Institute of Science,
Rehovot, 7610001 Israel

E-mail: Ofer.Aharony@weizmann.ac.il, prithvi@weizmann.ac.il,
Tarun.Sharma@weizmann.ac.il

ABSTRACT: We discuss monopole operators in $U(N_c)$ Chern-Simons-matter theories in three space-time dimensions. We mention an apparent problem in the matching of such operators in dualities between non-supersymmetric theories, and suggest a possible resolution. A similar apparent problem exists in the mapping of chiral monopole operators in theories with $\mathcal{N} = 2$ supersymmetry. We show that in many theories the lowest naive chiral monopole operator is actually not chiral, and we find the lowest monopole operator that is actually chiral in these theories. It turns out that there are several different forms of this operator, depending on the number of colors, the number of flavours, and the Chern-Simons level. Since we use the supersymmetric index to find the lowest chiral monopoles, our results for these monopoles are guaranteed to be invariant under the dualities in supersymmetric theories. The theories we discuss are believed to be dual in the 't Hooft large N_c limit to classical high-spin gravity theories. We argue that these theories (supersymmetric or not) should not have classical solutions charged under the U(1) gauge field in the high-spin multiplet.

Contents

1 Introduction

In the last twenty years, many examples of dualities between different quantum field theories in three and four space-time dimensions have been discovered. In particular, following [1], many examples of pairs of theories that are the same at low energies have been found, both in three and in four space-time dimensions.

A particular class of interesting gauge theories in three space-time dimensions is $\mathrm{U}(N_c)$ gauge theories with matter fields in the fundamental representation and with a Chern-Simons (CS) coupling for the gauge field.[1] These theories can either be defined as the low-energy limit of gauge theories which have both the Yang-Mills kinetic term and the Chern-Simons term (these theories can flow to non-trivial conformal field theories at low energies if all relevant couplings are tuned to zero), or directly (without a Yang-Mills term) as conformal field theories in which all beta functions vanish. In either case at low energies the gauge field is not dynamical, but the matter fields are dynamical and their couplings are affected by the Chern-Simons gauge fields.

For theories of this type with $\mathcal{N} = 2$ supersymmetry, dualities were discovered in [9] for the case with N_f chiral superfields in the fundamental representation, and $N_a = N_f$ chiral superfields in the anti-fundamental representation of $\mathrm{U}(N_c)$ (this duality can be derived by adding real mass terms to the duality without Chern-Simons coupling that was discovered in [10]). This was later generalized in [11] to the case with $N_a \neq N_f$.

Theories of this type without supersymmetry were studied in [5, 6], and this led to a conjecture that they also satisfy a duality between $\mathrm{U}(N_c)_k$ theories ($k > 0$) with N_f scalar matter fields and $\mathrm{U}(k - N_c)_{-k + \frac{N_f}{2}}$ theories with N_f fermion matter fields; this duality was presented explicitly in [12].[2] In the non-supersymmetric theories it is only known how to perform explicit computations at weak coupling or in the large N_c 't Hooft limit, so the evidence for the non-supersymmetric dualities at finite N_c is much weaker. It was shown in [13] that one can flow (at least for large enough N_c) from the $\mathcal{N} = 2$ dualities to the non-supersymmetric dualities, providing evidence for the validity of the latter at finite N_c.

The statement of the duality is that in the low-energy conformal field theory (CFT), all operators should match (including their scaling dimensions), and all their correlation functions as well. In the supersymmetric case, it is possible to check that all chiral operators agree between the two theories by computing their "superconformal index" [14–16] which is a sum over all chiral operators. This index, proportional to the partition function on $S^2 \times S^1$ with appropriate background fields, can be computed [17, 18] using localization in terms of the high-energy degrees of freedom, and indeed in all cases that have been checked the index agrees between pairs of dual theories [19–22]. It is not known how to compare non-chiral operators (or any operators in the non-supersymmetric case at finite N_c), since it is not known how to compute their dimension except at weak coupling.

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[1]One reason that these theories are interesting is that in the 't Hooft large N_c limit with a finite number of matter fields, they are believed [2–7] to be dual to classical high-spin gravity theories on AdS_4 [8].

[2]Here we use the convention for k that is natural from the low-energy point of view, as in [12]. In the non-supersymmetric theories this differs by a shift of k by $N_c \operatorname{sign}(k)$ from the high-energy value of k in Yang-Mills-Chern-Simons theories, so that we always have $|k| > N_c$.

In this paper we discuss "monopole operators" in these CFTs. A monopole operator is defined as a point-like defect such that there is some magnetic flux on the S^2 surrounding it (this flux can be chosen to be in the Cartan subalgebra of $U(N_c)$). It is related by the state-operator mapping to a state of the conformal field theory on S^2 which has a non-zero gauge field flux on the S^2. In theories without Chern-Simons couplings, such operators were studied extensively in the literature (see e.g. [23–27]). Their dimensions can be computed at weak coupling, but in non-supersymmetric theories essentially nothing is known about them at higher values of the coupling. In general gauge theories (say, with $SU(N_c)$ gauge group) it is not even clear how to identify monopole operators at strong coupling. However, in $U(N_c)$ theories there is a 'topological' $U(1)_J$ global symmetry whose current is the dual of the diagonal $U(1)$ field strength, and monopoles (and only monopoles) are charged under this symmetry. This enables a simple identification of monopole operators even away from weak coupling.

In Chern-Simons theories there is an extra complication. The Chern-Simons term implies that monopole operators carry an electric charge, so that to form a gauge-invariant operator they must be dressed with extra charged fields. For example, the simplest monopoles in the $U(N_c)_k$ theory break $U(N_c) \rightarrow U(1) \times U(N_c - 1)$ and carry $\pm k$ units of charge under the $U(1)$, and this charge must be balanced by extra fields carrying $\mp k$ units of charge.

In a theory that contains scalar fields φ in the fundamental and anti-fundamental representation, one would expect the lightest (lowest dimension) monopole operator to arise from a product of the monopole defect operator X with $|k|$ scalar fields in the fundamental or anti-fundamental representation (depending on the sign of k and on the monopole charge), so that it takes the form $X\varphi^{|k|}$. At weak coupling (large $|k|$) this naively gives an operator with a dimension of order $|k|$. Note that in the theory on S^2 the lowest energy scalar states charged under the $U(1)$ have spin $\frac{1}{2}$ in the monopole background [28], so this product is actually not a scalar but an operator with spin $\frac{|k|}{2}$.

If there are no scalar fields of the appropriate representation (which happens on one-side of the non-supersymmetric duality) one needs to put in $|k|$ fermions ψ, but then because of anti-symmetry one needs to add also $O(k^{\frac{3}{2}})$ derivatives in the large $|k|$ limit (see appendix B) to form a non-vanishing operator of the schematic form $(X\psi\partial\psi\partial^2\psi\cdots)$. At weak coupling this operator seems to have a dimension of order $|k|^{\frac{3}{2}}$. For the non-supersymmetric duality one needs to map monopoles with scalars to monopoles with fermions, but this seems problematic since their classical scaling dimensions are very different from each other, and even scale differently with N_c in the 't Hooft limit (in which one takes large N_c with fixed 't Hooft coupling $\lambda \equiv N_c/k$). Recall that the dualities match the $U(1)_J$ symmetries on the two sides, so monopoles must map to monpoles under the duality. Presumably the monopoles acquire large anomalous dimensions at strong coupling that make this matching work, but the needed anomalous dimensions do not satisfy the usual large N_c scaling, and it would be interesting to understand where they come from.[3]

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[3]The duality actually maps the free scalar theory coupled to Chern-Simons to a Gross-Neveu model coupled to Chern-Simons, but we do not expect going from the free theory to the critical one to affect the large N_c scaling of the monopole dimensions.

In order to shed more light on this one can look at similar questions in $\mathcal{N} = 2$ supersymmetric theories, where at least for chiral monopole operators we have more control. In supersymmetric theories such a monopole looks like a chiral field of the form (say) $V_+\Phi^{|k|}$, where V_+ is the standard chiral monopole operator with the minimal positive monopole charge (see [23, 24, 29, 30]) and Φ is a chiral superfield in the fundamental representation (if there are several such fields they could all appear). This seems to give a chiral operator of spin $\frac{|k|}{2}$. It is easy to compute the classical dimension of this operator, and for the chiral operator one expects this dimension to be protected. Thus, naively we would expect the dimensions of the $V_+\Phi^{|k|}$ operators to match across supersymmetric dualities. However, it is easy to see that because of the different scaling dimensions of the monopole operators V_+ this is not the case, both in the dualities of [9] and in the more general dualities of [11].

How is this possible, given that the indices of the two theories, and thus all chiral operators, match? A deeper look at the index reveals that in many cases the operators $V_+\Phi^{|k|}$ do not appear in the index (namely, there is no contribution to the index with the corresponding quantum numbers), implying that they are actually not chiral. When a naively chiral operator does not appear in the index, this means that it can join with another operator to form a non-chiral multiplet, and generically we expect this to happen whenever it can. As we discuss in detail below, from the point of view of the index which is computed in the UV theory, there are other operators with the same quantum numbers as $V_+\Phi^{|k|}$ that involve gluinos, which can join together with these operators to form long (non-chiral) multiplets of the superconformal algebra. From the point of view of the low-energy theory that contains no gluinos (and no other naively chiral operators with the same quantum numbers) this non-chirality is more surprising, but this theory is generally strongly coupled.

In this paper we study in detail the spectrum of chiral monopole operators in $\mathrm{U}(N_c)_k$ theories, focusing for simplicity on the two cases $N_a = N_f$ and $N_a = 0$. The latter case is particularly interesting because it is used (for $N_f = 1$) to flow to the non-supersymmetric duality [13]. We will show that in some cases the naive chiral operators are chiral, but in other cases they are not, and for every value of N_c, N_f and k we identify the lightest monopole operator that appears in the index. Our results are based partly on a numerical evaluation of the index for small values of N_c, N_f and k, which we use to conjecture the general result, and partly on analytic arguments that are valid for some ranges of values of N_c, N_f and k. We verify that these lightest operators match across the duality, as implied by the equality of the indices of dual theories. Note that in general the lightest monopole operators carry a non-zero spin.

In some cases we find that the lightest chiral monopole operator has a dimension of order N_c^2 in the 't Hooft large N_c limit. This implies that all the naive chiral monopole operators with dimensions of order N_c are actually not chiral. We then go back to the apparent mismatch in non-supersymmetric theories, and argue that already at weak coupling in the scalar theory, the monopole operators could get large anomalous dimensions that may change their N_c-scaling in the 't Hooft large N_c limit.

We begin in section 2 with a review of background material about monopole operators, superconformal indices, and how to read off the charges and field content of chiral

monopole operators from the index. In section 3 we present a conjecture (based on numerical evaluations of the index) for the dimensions and flavour representations of the lowest monopole operators for the case of $N_a = N_f$, and in section 4 we do the same for $N_a = 0$. In section 5 we discuss the 't Hooft large N_c limit of our results. We use the duality between CS-matter theories in this limit and high-spin gravity theories to argue that the latter theories should not have classical charged solutions. In section 6 we prove our conjecture for the form of the lowest chiral monopole operator in a simple case; other cases are analyzed in appendix A. In section 7 we briefly discuss the duals of non-chiral monopole operators under the supersymmetric dualities. In section 8 we discuss the perturbative corrections to dimensions of monopole operators in non-supersymmetric Chern-Simons-matter theories, and argue that they can be as large as $O(k^{\frac{3}{2}})$. We summarize our results in section 9. Several appendices contain technical details.

2   The superconformal index and BPS monopole operators

The superconformal index I of a 3d $\mathcal{N} = 2$ supersymmetric theory [16–18, 31, 32] is defined as a weighted sum over the Hilbert space of the theory on S^2 as follows:

$$I = \text{Tr}\left[(-1)^F e^{-\beta\{Q,S\}} x^{\epsilon+j_3} \prod_n t_n^{f_n}\right],\tag{2.1}$$

where

- F is a fermion number operator and $(-1)^F$ gives $(+1)$ for bosonic and (-1) for fermionic states.

- Q and S are particular supercharges in the superconformal algebra which satisfy

$$\{Q, S\} = \epsilon - j_3 - R \geq 0,\tag{2.2}$$

  where ϵ is the energy in units of the radius of the S^2, j_3 is the charge under the Cartan subalgebra of the Spin(3) rotation group of the S^2, and R is the R-charge of the $\mathcal{N} = 2$ superconformal algebra. Under radial quantization Q are S are Hermitian conjugates of each other and (2.2) is positive semi definite.

- Only states with $\{Q, S\} = 0$ contribute to I, so it is actually independent of β.

- I is a non-trivial function of x and of all the other fugacities for global symmetries t_n, and f_n are the charges under these symmetries.

- I is invariant under continuous deformations of the theory which preserve the superconformal symmetry of the theory.

Up to an overall factor related to the vacuum energy of the theory on S^2, I is equal to the partition function of the theory on $S^2 \times S^1$ with appropriate background fields. Thus, it can be evaluated by a path integral of the theory on $S^2 \times S^1$, with periodic boundary conditions for both fermions and bosons, and with the relevant chemical potentials turned

on. This is a supersymmetric quantity and can be evaluated via supersymmetric localization on $S^2 \times S^1$. Since the index does not change under renormalization group flow (except possibly for changes in the R-symmetry that sits in the superconformal algebra), it can be defined even for theories which are not conformal, as long as they have an exact $U(1)_R$ symmetry, and in asymptotically free theories it can be easily computed in the UV. In particular, in our case, it can computed in a Yang-Mills-Chern-Simons theory that is weakly coupled at high energies.

In this paper we are interested in $U(N_c)$ CS theories at level k coupled to N_f chiral multiplets Φ_a in the fundamental and N_a chiral multiplets $\tilde{\Phi}_{\tilde{b}}$ in the anti-fundamental representation of $U(N_c)$.[4] The flavour symmetry group of these theories is given by $(U(N_f) \times U(N_a))/U(1)$, with a combination of the two $U(1)$'s being a part of the gauge symmetry. We will take Φ_a ($\tilde{\Phi}_{\tilde{b}}$) to be in the fundamental representation of $U(N_f)$ ($U(N_a)$). It will be convenient to write the fugacities of the global symmetry $SU(N_f) \times SU(N_a) \times U(1)_A$ as $t_a, \tilde{t}_{\tilde{b}}, y$, respectively, satisfying $\prod_{a=1}^{N_f} t_a = \prod_{\tilde{b}=1}^{N_a} \tilde{t}_{\tilde{b}} = 1$. In the special case $N_f = 0$ or $N_a = 0$ there is no $U(1)_A$ symmetry, so one must set $y = 1$. There is also a topological $U(1)_J$ symmetry, whose current includes $\epsilon^{\mu\nu\rho} \text{tr}(F_{\nu\rho})$, and whose fugacity we denote by w. With these definitions, the superconformal index takes the following explicit form:

$$
I = \sum_{\{m_i\} \in \mathbb{Z}} (-1)^{\sum \left(-k m_i - \frac{1}{2}(N_f - N_a)|m_i|\right)} \frac{w^{\sum m_i}}{(sym)} \oint \left(\prod_{i=1}^{N_c} \frac{dz_i}{2\pi i z_i} z_i^{-k m_i}\right) Z_g \left(\prod_{a=1}^{N_f} Z_{\Phi_a}\right) \left(\prod_{\tilde{b}=1}^{N_a} Z_{\tilde{\Phi}_{\tilde{b}}}\right),
$$

$$
Z_g = \prod_{(i \neq j)=1}^{N_c} x^{-|m_i - m_j|/2} \left(1 - \frac{z_i}{z_j} x^{|m_i - m_j|}\right),
$$

$$
Z_{\Phi_a} = \prod_{i=1}^{N_c} \left((x^{1-r} z_i^{-1} t_a^{-1} y^{-1})^{|m_i|/2} \prod_{j=0}^{\infty} \frac{(1 - z_i^{-1} t_a^{-1} y^{-1} x^{|m_i|+2-r+2j})}{(1 - z_i\, t_a\, y\, x^{|m_i|+r+2j})} \right),
$$

$$
Z_{\tilde{\Phi}_{\tilde{b}}} = \prod_{i=1}^{N_c} \left((x^{1-\tilde{r}} z_i \tilde{t}_{\tilde{b}}^{-1} y^{-1})^{|m_i|/2} \prod_{j=0}^{\infty} \frac{(1 - z_i \tilde{t}_{\tilde{b}}^{-1} y^{-1} x^{|m_i|+2-\tilde{r}+2j})}{(1 - z_i^{-1}\tilde{t}_{\tilde{b}}\, y\, x^{|m_i|+\tilde{r}+2j})} \right),
$$

(2.3)

where (sym) is the dimension of the subgroup of the S_{N_c} Weyl group that is unbroken by the monopole background with fluxes $\{m_i\}$ ($i = 1, \cdots, N_c$) on S^2 in the Cartan of $U(N_c)$, r is the R-charge of the Φ_a, and $\tilde{r}$ of the $\tilde{\Phi}_{\tilde{b}}$ (these charges may be modified by mixing them with other global symmetry charges, using the appropriate fugacities[5]). We include the phase factor $(-1)^{\sum(-k m_i - \frac{1}{2}(N_f - N_a)|m_i|)}$ which was pointed out in [37] and which plays a crucial role in the factorization properties of the index studied in [38, 39].

As the Chern-Simons-matter theory we are studying is superconformal [40], there is a one-to-one map between local operators on $\mathbb{R}^3$ and states on $S^2 \times \mathbb{R}$. In the sector with

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[4]For convenience we will use the shorthand notation $U(N_c)_k(N_f, N_a)$ for these theories.

[5]The precise R-symmetry of the superconformal theory in the IR can in principle be found by F-maximization [33–36], but this will not play any role in our analysis. Note that the IR R-symmetry can contain also accidental symmetries that are not captured by the index [36], but the index must still match between IR-dual theories.

fluxes $\{m_i\}$ on S^2, the U(N_c) gauge symmetry of the theory is broken to a subgroup which keeps the flux invariant. The flux state on S^2, which carries gauge charge $\{-km_i\}$ due to the CS coupling, is dual to a local operator on $\mathbb{R}^3$ which is charged under the unbroken gauge symmetry. This is referred to as the '*bare*' monopole operator. This operator can be dressed with charged fields to make gauge-invariant monopole operators.

It is useful to keep track of the basic fields ('letters') which have the correct charges to contribute to the index. When we compute the contribution to the index from a sector with fluxes $\{m_i\}$, we need to take into account how this shifts the quantum numbers of the various fields; we determine this from the states of each field in the monopole background on S^2, by using the state/operator correspondence. Note that here we need to use the supersymmetric monopole background, which involves also an expectation value for the scalar field in the vector multiplet [24]. We denote the scalar and fermion components of Φ_a ($\tilde{\Phi}_{\tilde{b}}$) by ϕ_a and ψ_a ($\tilde{\phi}_{\tilde{b}}$ and $\tilde{\psi}_{\tilde{b}}$), respectively. The quantum numbers $(\epsilon, j_3, R; A)$ of the basic letters with R-charge R and U(1)$_A$ charge A, in the flux background $\{m_i\}$, are given by:

$$\phi_i^a \colon \left(r + \frac{1}{2}|m_i|, \frac{1}{2}|m_i|, r; 1\right),$$

$$\tilde{\phi}_{\tilde{b}}^i \colon \left(\tilde{r} + \frac{1}{2}|m_i|, \frac{1}{2}|m_i|, \tilde{r}; 1\right),$$

$$\bar{\tilde{\psi}}_{+i}^{\tilde{b}} \colon \left(\frac{3}{2} - \tilde{r} + \frac{1}{2}|m_i|, \frac{1}{2}(1 + |m_i|), 1 - \tilde{r}; -1\right),$$

$$\bar{\psi}_{+a}^i \colon \left(\frac{3}{2} - r + \frac{1}{2}|m_i|, \frac{1}{2}(1 + |m_i|), 1 - r; -1\right),$$

$$(\lambda_-)_j^i \text{ with } m_i \neq m_j \colon \left(\frac{1}{2}(|m_i - m_j| + 1), \frac{1}{2}(|m_i - m_j| - 1), 1; 0\right),$$

$$\partial_{++} \colon (1, 1, 0; 0),$$

where $i, j = 1, \cdots, N_c$, and λ_j^i are the gauginos of the high-energy Yang-Mills-Chern-Simons theory, which contribute only for $m_i \neq m_j$. The $\pm$'s denote which component we are considering, according to its charge under the Cartan subalgebra of the Spin(3) rotation group, before taking into account the shift by the monopole background. Here we only wrote down the 'letters' which obey an equality in (2.2), since others do not contribute to the index; whenever we write down a field contributing to an operator from here on, we will mean the specific component of the field which is listed in (2.4). Operators containing (say) $(\partial^j \phi_i^a)^n$ are identified by having a contribution to the index from the appropriate term in the Taylor expansion of the denominator of Z_{Φ_a}, and $\partial^j \bar{\psi}_a^i$'s by contributions from the numerator.[6] Note that the gauginos are not expected to be part of the low-energy CS-matter theory, so the interpretation of their contributions from the point of view of

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[6]When we write down operators with derivatives, they should always be intereperted as gauge-covariant derivatives.

this theory is not clear,[7] but we will still write our operators using these gauginos (since this is how we can identify their form from the index, by contributions from Z_g).

The quantum numbers of the bare chiral monopole operator $V_{\{m_i\}}$ can be computed as in [29], and they can also be read off from the Index. This operator is rotationally symmetric ($j_3 = 0$), invariant under the $\mathrm{SU}(N_f) \times \mathrm{SU}(N_a)$ flavour symmetry, and it carries R-charge and axial charge[8]

$$\epsilon(V_{\{m_i\}}) = R(V_{\{m_i\}}) = -\sum_{(i \neq j)=1}^{N_c} \frac{|m_i - m_j|}{2} + (N_f(1-r) + N_a(1-\tilde{r}))\sum_{i=1}^{N_c} \frac{|m_i|}{2},$$

$$A(V_{\{m_i\}}) = (-1)(N_f + N_a)\sum_{i=1}^{N_c} \frac{|m_i|}{2}.$$

$$(2.5)$$

For simplicity let us first consider $\mathrm{U}(N_c)_k(N_f, N_f)$ theories, in which $\tilde{r} = r$. Naively, for $k > 0$ the lightest BPS monopole operator (namely, the one with the lowest value of $\epsilon + j_3$) with unit topological charge is then[9]

$$\mathcal{M}_{(1,\vec{0})} = V_{(1,\vec{0})}(\phi_1)^k, \text{ contributing a factor } x^{N_f - N_c + 1 + k + r(k - N_f)} \qquad (2.6)$$

for all choices of flavours of the ϕ_1 operators. It turns out that this is not always the case. As we will discuss in later sections, in many cases this operator cancels with other operators in the index, and is thus presumably not chiral. One can also see that this has to be the case from the Giveon-Kutasov (GK) duality in these theories [9]. The GK duality relates the $\mathrm{U}(N_c)_k(N_f, N_f)$ theory with $R_\phi = r$ and $A_\phi = 1$ to a $\mathrm{U}(|k| + N_f - N_c)_{-k}(N_f, N_f)$ theory with $R_\phi = 1 - r$ and $A_\phi = -1$ and with N_f^2 extra gauge-singlet chiral superfields M. For consideration of the lightest monopole operator, M is irrelevant. This implies that if (2.6) is always the lightest monopole operator then its index contribution must match with the GK dual. This is easily seen not to be the case.

The above observation implies that in some theories the naive leading chiral monopole operator must cancel in the index (since if not it would be below the lightest monopole operator in the dual theory), and is thus *not* chiral. This leads to the following questions:

- When does the naive leading monopole operator (2.6) survive?

- What is the leading non-canceling monopole operator in the cases when (2.6) is not chiral?

In the following sections we will answer these questions in detail, and further verify that the leading operators match across the GK duality.

Before proceeding to our computations, we would like to make a remark on different ways of computing the integral in (2.3). The index is represented as a contour integral

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[7]This is possible since this theory has non-zero couplings. The gauginos become auxiliary fields at low energies, that can be written as combinations of the basic fields of the schematic form $\phi\bar{\psi}$. These combinations differ from the naively chiral combinations of ϕ and $\bar{\psi}$ from (2.4) that contribute independently to the index, but apparently (for $m_i \neq m_j$) they are chiral operators in the low-energy CS-matter theory.

[8]The axial U(1) is a symmetry only when both N_f and N_a are non zero.

[9]Monopole operators from other GNO sectors turn out to be heavier.

over the unit circle in the complex plane for the holonomy variables z_i. The integrand has an infinite number of simple poles coming from the contributions Z_{Φ_a}, $Z_{\tilde{\Phi}_{\tilde{b}}}$ of fundamental charged letters and their derivatives. Apart from these poles there are poles at $z_i = 0$ or ∞ as well, depending on the value of N_f, N_a and k. Naively the integral can be evaluated by summing over the residues at these poles, and this has been done for $\mathrm{U}(N_c)_k(N_f, N_a)$ theories in [38]. However, in many cases this procedure does not work (see the revised version of [39]), since it is not clear precisely how to take into account the poles at $z_i = 0$ and $z_i = \infty$. Thus, we will not use this method. Instead we will evaluate the Index by performing a Laurent expansion of the integrand for small chemical potential x.

3 Conjecture for the leading chiral monopoles in the $\sum_i m_i = 1$ sector of $\mathrm{U}(N_c)_k(N_f, N_f)$ theories

In this section we compute the lowest monopole operator (namely, the one with least $\epsilon + j_3$) which survives in the index of "non-chiral" theories (with $N_f = N_a$), in the sector with charge $+1$ under $\mathrm{U}(1)_J$ (namely, $\sum_i m_i = 1$). We do this by expanding the Index (2.3) order by order in x using Mathematica and identifying the lowest power of x which survives. The results of Mathematica suggest that the lowest nontrivial operator occurs (as expected) in the sector with GNO charges $(1, 0, 0, \cdots)$. Note that the global symmetries do not distinguish sectors with different GNO charges $\{m_i\}$ and the same $\mathrm{U}(1)_J$ charge $\sum_i m_i$, and these can mix (even though they appear separately in (2.3)). For simplicity we give the results for $k > 0$, from which the result for $k < 0$ can easily be obtained, as will be explained in section 3.1.

These operators all come with a factor of w^{+1}. The other global symmetries are $\mathrm{SU}(N_f)_l \times \mathrm{SU}(N_f)_r$ (whose fugacities are $(t_1, \ldots t_{N_f})$ and $(\tilde{t}_1, \ldots \tilde{t}_{N_f})$, respectively),[10] along with the axial $\mathrm{U}(1)_A$ symmetry (whose fugacity is y).

Using the results above, the x power, gauge charge and flavour charges of some basic relevant objects are[11]

$$
\begin{aligned}
V_+ &\to z_1^{-k} x^{N_f - N_c + 1 - r N_f} y^{-N_f}, \\
\{\phi_1^a, \ \lambda_1^i \phi_i^a\} &\to z_1 x^{1+r} y \, t_a.
\end{aligned}
\tag{3.1}
$$

One can easily see that the lightest gauge-neutral monopole operator is obtained by dressing V_+ by k ϕ_1's. However, in a somewhat surprising result, we find that this operator does not usually survive in the index, because the operators $(\lambda_1^i \phi_i^a)$ have the same quantum numbers as ϕ_1^a and come with an opposite sign. In fact, we find four regimes of N_c, k, N_f (called Cases 1, 2, 3, 4) where we find different monopole operators giving the leading contribution to the index.

Table 1 gives the results found using a numerical evaluation of the Index in Mathematica (extrapolated from small values of N_c, N_f and k), where $N_c^d \equiv |k| + N_f - N_c$. In some

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[10]With the condition $\prod_{a=1}^{N_f} t_a = 1 = \prod_{a=1}^{N_f} \tilde{t}_a$.

[11]From here on we use the shorthand notation $V_\pm \equiv V_{(\pm 1, \vec{0})}$ for the simplest monopole operator. This monopole breaks $\mathrm{U}(N_c) \to \mathrm{U}(1) \times \mathrm{U}(N_c - 1)$, and from here on sums over i run over the $\mathrm{U}(N_c - 1)$ index, from 2 to N_c.

	Region	Operator: $w^{\sum_i m_i} x^{\epsilon+j_3} y^A$	$\mathrm{SU}(N_f)_l \times \mathrm{SU}(N_f)_r$ flavour rep.
1	$N_f > N_c, k \geq N_c$	$V_+ \phi_1^{k-N_c+1}(\lambda_1^i \phi_i)^{N_c-1}:$ $wx^{N_c^d+1+r(k-N_f)}y^{k-N_f}$	$\left(N_c\left\{ \overbrace{\begin{array}{c}\square\square\cdots\square\\ \square \\ \vdots\end{array}}^{k-N_c+1}, \; 1 \right.\right)$
2	$k \geq N_c \geq N_f$	$V_+ \phi_1^{k-N_c+1}(\lambda_1^i \phi_i)^{N_f-1}(\lambda_1^i \tilde{\bar{\psi}}_i)^{N_c-N_f}:$ $wx^{k-N_f+N_c+1+r(N_c^d-N_c)}y^{N_c^d-N_c}$	$\left(\overbrace{\square\square\cdots\square}^{k-N_c}, \; \overbrace{\square\square\cdots\square}^{N_c-N_f} \right)$
3	$N_c \geq N_f, N_c > k$	$V_+ \phi_1(\lambda_1^i \phi_i)^{N_c^d-1}(\lambda_1^i \tilde{\bar{\psi}}_i)^{N_c-N_f}(\phi_i \tilde{\phi}^i)^{N_c-k}:$ $wx^{k-N_f+N_c+1+r(N_f-k)}y^{N_f-k}$	$\left(1, \; {}_{k+N_f-N_c}\left\{ \overbrace{\begin{array}{c}\square\square\cdots\square\\ \square \\ \vdots\end{array}}^{N_c-N_f+1} \right.\right)$
4	$N_f \geq N_c \geq k$	$V_+(\lambda_1^i \phi_i)^k(\phi_i \tilde{\phi}^i)^{N_c-k}:$ $wx^{N_c^d+1+r(N_c-N_c^d)}y^{N_c-N_c^d}$	$\left(N_c\left\{ \begin{array}{c}\square\\\square\\\vdots\end{array}, \; N_c-k\left\{ \begin{array}{c}\square\\\square\\\vdots\end{array} \right.\right.\right)$

Table 1. The leading $\mathcal{M}_+$ monopole operator in $\mathrm{U}(N_c)_k(N_f, N_f)$ theories in various parameter ranges.

cases we can confirm these results by analytic methods, as described below. The charges of the lowest monopole operators which survive[12] in various regimes are listed in the third column, along with a typical operator (there are generally other operators with the same charges, this is just a representative). The flavour representation of the leading surviving operator is given in the last column. Note that in all cases the results are continuous at the boundaries of the different regimes, $k = N_c$ and $N_f = N_c$.

Some more details are given in appendix A, where we compute the lowest monopole operator in the Index analytically for several cases. The spins and R-charges of these monopoles are listed in appendix C.

3.1 Consistency with duality

Given the result for non-chiral $\mathrm{U}(N_c)_k$ theories with $k > 0$, it is easy to read off the results for $k < 0$. Let us denote the Index contribution in the GNO sector $\{m_i\}$ of the $\mathrm{U}(N_c)_k(N_f, N_f)$ theory with $R_\Phi = R_{\tilde{\Phi}} = r$ as $I_{N_c,k,N_f,N_f}^{\{m_i\}}(r; x, y, w, t_a, \tilde{t}_a)$. From (2.3) it is easy to see that

$$I_{N_c,-k,N_f,N_f}^{\{m_i\}}(r; x, y, w, t_a, \tilde{t}_a) = I_{N_c,k,N_f,N_f}^{\{m_i\}}(r; x, y, w, \tilde{t}_a, t_a). \tag{3.2}$$

Therefore for $\mathrm{U}(N_c)_{-k}$ theories, the same operator survives as in table 1, except that the flavour Young tableaux are interchanged. Note also that the Index of $\mathrm{U}(N_c)_{-k}$ in the $\{m_i\}$ sector is the same as the Index of $\mathrm{U}(N_c)_k$ in the $\{-m_i\}$ sector, except for the power of w (so the full indices are the same up to $w \leftrightarrow w^{-1}$).

---

[12]One can verify that the lowest surviving operator is independent of the choice of $0 < r < 1$.

As a consistency check, we can now confirm that the results we obtained are consistent with the GK duality. Under the duality, Case 2 and Case 4 map into themselves, and Case 1 and Case 3 map into each other. This is consistent with the observation that specific monopole operators like $V_+ \phi_1^k$ do not map to themselves under the duality. The results in table 1 for the lowest surviving monopole operators are consistent with the GK duality, along with $r \to 1 - r$, $y \to y^{-1}$, accompanied by complex-conjugating the flavour representation, as expected. More precisely, the lowest surviving operator is consistent with the duality relation

$$
\begin{aligned}
I_{N_c,k,N_f,N_f}(r; x, y, w, t_a, \tilde{t}_a) &= I_{N_c^d,-k,N_f,N_f}(1 - r; x, y^{-1}, w, t_a^{-1}, \tilde{t}_a^{-1}) \\
&= I_{N_c^d,k,N_f,N_f}(1 - r; x, y^{-1}, w, \tilde{t}_a^{-1}, t_a^{-1})
\end{aligned}
\tag{3.3}
$$

(this is not a precise equality in general because of the extra singlet meson operators that need to be added on the right-hand side).

$k = 0$ is a special case. In this case the bare monopole operators $V_\pm$ are gauge-invariant by themselves and need not be dressed by charged matter fields. Since all other operators in the $\sum m_i = \pm 1$ sectors have larger values of $\epsilon + j_3$, $V_\pm$ itself has the lowest non-canceling contribution in the index. The matching of the lightest monopole across the duality also works differently in this case. The dual theories contain extra singlet chiral superfields $V_\pm$ charged under $U(1)_J$, along with superpotential terms for their monopole operators $\tilde{V}_\pm$

$$
W = \tilde{V}_+ V_- + \tilde{V}_- V_+.
\tag{3.4}
$$

These superpotential terms make the bare monopole operators $\tilde{V}_\pm$ of the dual theory Q-exact and remove them from the chiral spectrum, while the gauge singlets $V_\pm$ map to $V_\pm$ of the original theory [10].

3.2 Other GNO sectors

In previous sections we presented the lowest monopole operator in the sector with GNO charge $(1, 0, \cdots)$. One might be worried whether sectors with different GNO charges but the same topological charge $\sum_i m_i = 1$ (say $(2, -1, 0, \cdots)$) could give rise to a lower monopole operator, or cancel the contributions of the monopoles we presented. Using our numeric code (for low values of N_c, N_f, k and the GNO charges) we explicitly checked that this is not the case. The fact that the duality is consistent with the results of the previous subsection also suggests that this is not the case.

In the simplest case (case 1), when the lowest surviving monopole operator has the same power of x as $V_+ \phi_1^k$, we can explicitly show that this is indeed the lowest monopole operator with $\sum_i m_i = 1$. Consider a monopole operator with GNO charge $\{m_i\}$, where m_i are ordered by $|m_1| \geq |m_2| \geq \cdots$. The naive gauge-invariant monopole operator[13] in this case is

$$
V_{\{m_i\}} \left(\phi_1^{m_1} \cdots \phi_{N_c}^{m_{N_c}} \right)^k : \epsilon + j_3 = -\sum_{i \neq j} \frac{|m_i - m_j|}{2} + N_f(1 - r) \sum_i |m_i| + k \sum_i |m_i|(r + |m_i|),
\tag{3.5}
$$

---

[13]Note that this might not always survive in the index. Also assume $k > 0$ for simplicity.

where for any $m_i < 0$, we should use $(\tilde{\phi}^i)^{km_i}$ instead of $\phi_i^{km_i}$. Using the triangle inequality $|m_i + m_j| \leq |m_i| + |m_j|$, we find that for this operator

$$\epsilon + j_3 \geq \sum_i |m_i|(N_c^d + 1 + (k - N_f)r) + k \sum_i |m_i|(|m_i| - 1). \tag{3.6}$$

It is now obvious that the only way we can minimize this charge keeping $\sum m_i = 1$ for arbitrary r charge is to choose $m_1 = 1$ and $m_i = 0$ for $i > 1$ (in this case there is equality in (3.6)).

4 Conjecture for the leading chiral monopoles in the $\sum_i m_i = \pm 1$ sectors of $U(N_c)_k(N_f, 0)$ theories

Let us now turn to a chiral case, $U(N_c)_k$ with $(N_f, 0)$ matter fields. Note that one has to set $y = 1$ in the general formula (2.3), since the axial symmetry is part of the gauge symmetry in this case. One can again perform a series of computations (similar to the non-chiral case) to find the lowest monopole operators in the sectors $\sum_i m_i = \pm 1$. We find that the lowest operator occurs at GNO charge $(\pm 1, 0, \ldots)$. Note that in this case there is no relation between the operators with $\{m_i\}$ and with $\{-m_i\}$ for the same value of k. In particular, unlike the non-chiral case, the monopole operators with $\sum_i m_i = 1$ are very different from $\sum_i m_i = -1$.

Another important feature of these theories is that the duality works differently depending on the sign of $(k - \frac{1}{2}N_f)$ [11]:

$$k - \frac{1}{2}N_f \geq 0: \qquad U(N_c)_k(N_f, 0) \overset{\text{dual}}{\leftrightarrows} U\left(|k| + \frac{1}{2}N_f - N_c\right)_{-k}(0, N_f),$$

$$k - \frac{1}{2}N_f \leq 0: \qquad U(N_c)_k(N_f, 0) \overset{\text{dual}}{\leftrightarrows} U(N_f - N_c)_{-k}(0, N_f), \tag{4.1}$$

where for $k \neq \frac{N_f}{2}$ no extra singlet operators are needed for the duality. We will call the two cases in (4.1) Case 1 and Case 2, respectively. It will turn out that each of these cases has further subcases, where the lowest monopole operator surviving in the index has a different form.

4.1 Monopole GNO charge $(1, 0, \ldots)$ sector

The charges of the bare monopole V_+ correspond to a contribution to the index of the form

$$V_+ \rightarrow z_1^{-k - \frac{N_f}{2}} x^{-(N_c - 1) + (1 - r)\frac{N_f}{2}}. \tag{4.2}$$

In table 2 we give the results of Mathematica for the lowest monopole operator appearing in the index (again, these results are based on extrapolating numerical evaluations of the index for small values of N_c, N_f and k, and in some cases they can be verified by analytic arguments).

Depending on the sign of $k - \frac{N_f}{2}$ we have Case 1 and Case 2, which are further divided into subcases. Note that the results for Cases 1a and 2b are almost the same as the non-chiral Case 1 analyzed in the previous section, except for some shifts in the monopole

	Region	Operator: $w^{\sum m_i} x^{\epsilon + j_3}$	SU(N_f) flavour rep.
1a	$\left(k > \frac{1}{2}N_f, k + \frac{1}{2}N_f > N_c\right)$; $N_f \geq N_c$	$V_+ \phi_1^{k+\frac{N_f}{2}-N_c+1} \left(\lambda_1^i \phi_i\right)^{N_c-1}$: $wx^{k+N_f-N_c+1+kr}$	
1b	$\left(k > \frac{1}{2}N_f, k + \frac{1}{2}N_f > N_c\right)$; $N_f \leq N_c < 2N_f - 1$	$V_+ \phi_1^{k-\frac{1}{2}N_f+1} \left(\lambda_1^i \phi_i\right)^{N_f-1} \left(\phi_i \bar\psi^i\right)^{N_c-N_f}$: $wx^{k-N_f+N_c+1+kr}$	
1c	$\left(k > \frac{1}{2}N_f, k + \frac{1}{2}N_f > N_c\right)$; $N_c + 1 = (n+1)N_f + m$, with $n \geq 1, 0 \leq m < N_f$	$V_+ \phi_1^{N_c^d + nN_f + m} \left(\lambda_1^i \phi_i\right)^{N_f-1} \left(\phi_i \bar\psi^i\right)^{N_f-1}$ $\left(\partial \phi_i \bar\psi^i\right)^{N_f} \left(\partial^2 \phi_i \bar\psi^i\right)^{N_f} \ldots \left(\partial^{n-1}\phi_i\bar\psi^i\right)^{N_f}$ $\left(\partial^n \phi_i \bar\psi^i\right)^m$: $wx^{k+N_f-N_c-1+N_f n(n+1) + 2m(n+1)+kr}$	
2a	$\left(k \leq \frac{1}{2}N_f, N_f \geq N_c\right)$; $k + \frac{1}{2}N_f < N_c$	$V_+ \phi_1 \left(\lambda_1^i \phi_i\right)^{k+\frac{1}{2}N_f-1} \left(\phi_i \bar\psi^i\right)^{N_c - \left(k+\frac{1}{2}N_f\right)}$: wx^{-k+N_c+1+kr}	
2b	$\left(k \leq \frac{1}{2}N_f, N_f \geq N_c\right)$; $k + \frac{1}{2}N_f \geq N_c$	$V_+ \phi_1^{k+\frac{N_f}{2}-N_c+1} \left(\lambda_1^i \phi_i\right)^{N_c-1}$: $wx^{k+N_f-N_c+1+kr}$	

Table 2. The leading $\mathcal{M}_+$ monopole operator in U(N_c)$_k(N_f, 0)$ theories in various parameter ranges.

charges. The typical lowest operator here consists of a bare monopole operator dressed by operators with the same charges as $(\phi_1^{k+\frac{N_f}{2}})$ so as to cancel the gauge charge. In other cases these operators do not survive, and the lowest one which does survive has more gauge-invariants attached to it (some appropriate number of $\phi_i\bar\psi^i$'s). We analytically analyze the contributions of the simplest monopole operators in section 6 below, while some of the other cases are analyzed in appendix A.

4.2 Monopole GNO charge $(-1, 0, \ldots)$ sector

The charges of the bare monopole V_- now correspond to

$$V_- \to z_1^{k - \frac{N_f}{2}} x^{-(N_c-1)+(1-r)\frac{N_f}{2}}. \tag{4.3}$$

In table 3 we give the results of Mathematica for the lowest monopole operator in each case. Again, depending on the sign of $k - \frac{N_f}{2}$, we have Case 1 and Case 2, which are further divided into subregimes. The typical operator consists of a bare monopole operator

	Region	Operator: $w^{\sum m_i} x^{\epsilon + j_3}$	SU(N_f) flavour rep.
1 a	$k + \frac{N_f}{2} - N_c > 0$ $k > \frac{N_f}{2}$ $N_f \geq N_c^d \equiv k + \frac{N_f}{2} - N_c$	$V_- \bar\psi^1 (\lambda_i^1 \bar\psi^i)^{k - \frac{1}{2}N_f - 1} (\phi_i \bar\psi^i)^{N_c - k + \frac{N_f}{2}}$: $w^{-1} x^{k + N_c + 1 - kr}$	Young tableau with N_c+1 boxes across top, $k + \frac{N_f}{2} - N_c$ rows down
1 b	$k + \frac{N_f}{2} - N_c > 0$ $k > \frac{N_f}{2}$ $2N_f - 1 > N_c^d \geq N_f$	$V_- (\bar\psi^1)^{k - \frac{1}{2}N_f - N_c + 1} (\lambda_i^1 \bar\psi^i)^{N_c - 1}$: $w^{-1} x^{3k + 1 - N_c - N_f - kr}$	Young tableau with N_c boxes across top, $k - \frac{N_f}{2} - N_c + 1$ rows down
1 c	$k + \frac{N_f}{2} - N_c > 0$ $k > \frac{N_f}{2}$ $N_c^d + 1 = (n+1)N_f + m$ where $n \geq 1,\ 0 \leq m < N_f$	$V_- (\lambda_i^1 \bar\psi^i)^{N_c - 1} (\bar\psi^1)^{N_f} (\partial\bar\psi^1)^{N_f} \dots$ $(\partial^{n-1}\bar\psi^1)^{N_f} (\partial^n \bar\psi^1)^m$: $w^{-1} x^{3k - N_c + 1 + N_f(n^2 - n - 1) + 2mn - kr}$	Young tableau $N_c - 1$ boxes across $\times\, m$, with column of boxes
2 a	$\frac{N_f}{2} - k > 0,\, N_f > N_c$ $k + \frac{N_f}{2} < N_f - N_c \equiv N_c^d$	$V_- \phi_1^{\frac{N_f}{2} - k - N_c + 1} (\lambda_1^i \phi_i)^{N_c - 1}$: $w^{-1} x^{N_f - k + 1 - N_c - kr}$	Young tableau with $\frac{N_f}{2} - k - N_c + 1$ boxes across top, N_c rows down
2 b	$\frac{N_f}{2} - k > 0,\, N_f > N_c$ $k + \frac{N_f}{2} \geq N_f - N_c \equiv N_c^d$	$V_- \phi_1 (\lambda_1^i \phi_i)^{\frac{N_f}{2} - k - 1} (\phi_i \bar\psi^i)^{N_c - \left(\frac{N_f}{2} - k\right)}$: $w^{-1} x^{k + N_c + 1 - kr}$	Young tableau with $k - \frac{N_f}{2} + N_c + 1$ boxes across top, $N_f - N_c$ rows down

Table 3. The leading $\mathcal{M}_-$ monopole operator in $\mathrm{U}(N_c)_k(N_f, 0)$ theories in various parameter ranges.

dressed with fermions (bosons) if the sign of $k - \frac{N_f}{2}$ is positive (negative), so as to cancel the gauge charge of the bare monopole. Note that for $k = \frac{N_f}{2}$ the bare monopole V_- is gauge-invariant by itself, and does not need to be dressed. The details of the spins and R-charges of these operators are given in appendix C.

4.3 Consistency with duality

From our results above we can easily derive the result for $\mathrm{U}(N_c)_{-k}(0, N_f)$ theories. Again, denoting the index of $\mathrm{U}(N_c)$ CS theories with N_f fundamental and N_a antifundamental chiral multiplets by $I_{N_c, k, N_f, N_a}(r; x, w, t_a, \tilde{t}_a)$, one can see from (2.3) that

$$I_{N_c, k, N_f, 0}(r; x, w, t_a, *) = I_{N_c, k, 0, N_f}(r; x, w^{-1}, *, t_a) = I_{N_c, -k, N_f, 0}(r; x, w^{-1}, t_a, *). \quad (4.4)$$

Now we can check that the results in the tables for the lowest surviving monopole operators with $\sum m_i = \pm 1$ are consistent with the expected dualities [11] (here these are

exact dualities when $k \neq \frac{N_f}{2}$, not just for the lowest monopole operators):

Case 1: $\qquad k \geq \frac{1}{2} N_f, \qquad N_c^d = k + \frac{1}{2} N_f - N_c$

$$I_{N_c,k,N_f,0}(r; x, w, t_a, *) = I_{N_c^d, -k, 0, N_f} \left(1 - r; x, x^{-\frac{N_f}{2}} w^{-1}, *, t_a^{-1} \right),$$

$$= I_{N_c^d, k, N_f, 0} \left(1 - r; x, x^{-\frac{N_f}{2}} w^{-1}, t_a^{-1}, * \right),$$

Case 2: $\qquad k \leq \frac{1}{2} N_f, \qquad N_c^d = N_f - N_c$

$$I_{N_c,k,N_f,0}(r; x, w, t_a, *) = I_{N_c^d, -k, 0, N_f}(1 - r; x, x^{-k} w^{-1}, *, t_a^{-1}),$$

$$= I_{N_c^d, k, N_f, 0}(1 - r; x, x^{-k} w^{-1}, t_a^{-1}, *). \tag{4.5}$$

The last equality implies that the duality should map the operators of table 2 to the ones of table 3 for the same value of k, up to complex conjugation of the flavour representation and a shift in the power of x. We find that in these theories all the different subcases map to themselves under the duality.

The case $|k| = \frac{1}{2} N_f$ is special as the bare monopole operator V_-,[14] is gauge-invariant in this case and hence survives in the $\sum m_i = -1$ sector as the lightest operator. Further, as in the non-chiral case with $k = 0$, the duality matching works differently for V_- as it maps (using the bottom lines of (4.5)) to an extra singlet chiral superfield V_+ in the dual theory, while the singlet $\tilde{V}_-$ is removed from the chiral spectrum of the dual theory by the superpotential $W = V_+ \tilde{V}_-$.[15] The leading contribution in the $\sum_i m_i = 1$ sector of the original theory is given by Case 1a (or 2b) of table 2 for $k = \frac{1}{2} N_f$. To find its dual one has to take into account the contribution of the gauge-singlet chiral multiplet V_+ in the dual theory. Using the results from a Mathematica computation, we claim that the dual of $\mathcal{M}_+$ of the original theory actually comes from the $\{m_i\} = 0$ sector (recall that the singlet V_+ also carries a $U(1)_J$ charge), and has the same charges as

$$\mathcal{M}_-: \quad \bar{\psi}_{V_+} (\phi_i \bar{\psi}^i)^{N_c^d}. \tag{4.6}$$

Note that even though $\bar{\psi}_{V_+}$ is non-chiral as described above, this is not necessarily true for its descendants or its products with other operators; for instance, descendants by derivatives appear in the index for $\bar{\psi}_{V_+}$ but not for $\tilde{V}_-$, and the latter operator can be separately multiplied by $(\phi_1 \bar{\psi}^1)$ which is a singlet of $U(1) \times U(N_c^d - 1)$, while the former operator in the $\{m_i\} = 0$ sector cannot.

5 The leading chiral monopole operators in the 't Hooft large N_c limit

The $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theories described above are particularly interesting in the 't Hooft large N_c limit (keeping fixed $\lambda = N_c/k$ and N_f); in this limit

---

[14]Recall that we assume $k > 0$. For $k < 0$, V_+ is gauge-invariant.

[15]More precisely, the superpotential means that $\tilde{V}_-$ cancels in the index with a fermionic component $\bar{\psi}_{V_+}$ of $\bar{V}_+$, which sits in the same non-chiral multiplet.

their thermal partition function can be computed exactly [41], and for finite large N_c we can flow from the supersymmetric dualities to non-supersymmetric dualities [13].

For non-chiral theories in the 't Hooft limit, the relevant case is Case 2 in table 1. Note that in this case the leading monopole operator does not take the naively expected form, and includes a large number of fermions. In this case, the scaling dimension of the lowest chiral monopole operator scales as N_c in the 't Hooft limit, as expected.

For chiral theories in the 't Hooft limit, the relevant case is Case 1c of tables 2 and 3. Also in this case the monopole operators do not take the naive form, and include a large number of fermions. Notice that for these theories the scaling dimension of the lowest chiral monopole operators scales as N_c^2 in the 't Hooft limit (since n in the tables scales as N_c), unlike the non-chiral case. This implies that for this case all monopole operators with a dimension scaling as N_c are not actually chiral. The difference between the two cases is that in the non-chiral case we can use the operators $\tilde{\phi}$ to construct chiral operators, but these are not available in our chiral case.

These conclusions can be avoided if we keep $(k - N_c)$ fixed in the large N_c limit (and in particular take $\lambda = 1$). For instance, in the non-chiral case if we take $k < N_c$ but also $N_c - k < N_f$ (as required to preserve supersymmetry), then we are actually in Case 3 of table 1. The dual theory in this case has finite N_c^d so it is not in the 't Hooft limit.

5.1 The mapping to high-spin gravity theories

As we mentioned in the introduction, CS-matter theories are believed to be dual to high-spin gravity theories, such that their 't Hooft large N_c limit corresponds to classical high-spin gravity theories (see [42] for a review). States with high-spin gravity particles correspond to operators with dimensions of order 1 in the large N_c limit, while classical solutions of the high-spin gravity theories correspond to operators with dimensions of order N_c (recall that the coupling constants in these theories are of order $1/N_c$). The same non-supersymmetric high-spin gravity theories are dual to the CS-scalar and CS-fermion theories, and they have a parameter θ_0 that corresponds to the 't Hooft coupling constant of these theories (there is also a choice of boundary conditions that determines whether the dual is a free theory coupled to CS, or a critical one). The supersymmetric versions of these high-spin theories have similar properties, and map to various supersymmetric CS-matter theories (chiral or non-chiral) [7].

The $U(1)_J$ global symmetry that the monopoles are charged under maps on the gravity side to the $U(1)$ gauge field in the high-spin multiplet (this multiplet, in the "non-minimal" high-spin theory, contains gauge fields of all integer spins). This is true both in the supersymmetric and in the non-supersymmetric cases. We thus expect classical solutions that carry this charge to correspond to monopole operators with dimensions of order N_c.

However, our arguments imply that such solutions should not exist in many high-spin gravity theories. In the CS-fermion theories we argued (see appendix B) that there are no monopole operators with dimensions of order N_c at large N_c, so no such solutions should exist in the original non-supersymmetric high-spin theory. In the $\mathcal{N} = 2$ supersymmetric theories such monopole operators may exist (and they certainly exist in the non-chiral theories), but we argued that for the $N_a = 0$ chiral theories all chiral monopole operators

have dimensions at least of order N_c^2. Thus, the corresponding high-spin gravity theories should not have any classical BPS charged solutions. Note that even in the cases where monopoles do exist with dimensions of order N_c, we expect these dimensions at weak coupling to be at least of order $k = N_c/\lambda$, such that they diverge in the $\lambda \to 0$ limit (which corresponds to $\theta_0 = 0, \frac{\pi}{2}$; note that the coupling in the gravity theory goes as $1/N_c$ rather than $1/k$ in this limit). Thus, in any high-spin gravity theory we do not expect to have classical charged solutions in the parity-preserving $\theta_0 = 0, \frac{\pi}{2}$ theories. Note that we cannot say if specific monopole operators correspond to classical gravity solutions or not, but when there is no monopole operator there cannot be a corresponding gravity solution.

Some classical solutions of the non-supersymmetric high-spin gravity theories were found in [43], and were generalized to supersymmetric cases (including some of our chiral and non-chiral theories) in [43, 44]. A linearized analysis suggests that these solutions carry a charge under the U(1) gauge symmetry in the high-spin multiplet, but it is difficult to verify this. Our arguments above imply that these solutions actually cannot carry this charge (assuming that the duality to CS-matter theories is correct), and it would be interesting to verify this directly.

Note that one way to avoid these arguments would be if the gravity theories are actually dual to $\mathrm{SU}(N_c)$ CS-matter theories, rather than to $\mathrm{U}(N_c)$ theories; as far as we know, none of the computations performed up to now can distinguish between these two cases. However, the U(1) global symmetry has a very different interpretation in the $\mathrm{SU}(N_c)$ theories, where it is a baryon number symmetry (and there is no global symmetry carried by monopoles). So, the arguments above do not rule out classical charged solutions if the dual gauge theories are $\mathrm{SU}(N_c)$ theories. Naively, such theories should always have baryons with dimensions of order N_c, which could correspond to classical charged solutions on the gravity side. However, in the CS-scalar theories this is actually not the case, because the baryon operator must be anti-symmetric in the color index. An argument similar to the one in appendix B then implies that when $N_f \ll N_c$ it must have a dimension at least of order $N_c^{\frac{3}{2}}$. Thus, even if the gauge group is $\mathrm{SU}(N_c)$, we still claim that the non-supersymmetric high-spin theories cannot have classical charged solutions (and in particular this still means that the solutions of [43] cannot be charged).

6 Analytic arguments for chirality of $V_+\phi_1^k$ and related operators

The results presented in tables 1, 2 and 3 for the lowest lying monopole operators for non-chiral and chiral theories, respectively, are conjectural and based on extrapolating Mathematica computations done for low values of N_c, k and N_f. In this section we present analytic arguments for the simplest operators of the schematic form $V_+\phi_1^k$.[16]

Let us consider the GNO sector $\{m_i\} = \{+1, \overrightarrow{0}\}$ in the $\mathrm{U}(N_c)_k(N_f, N_f)$ theory. The relevant supersymmetric letters to build gauge-invariant operators with the same $(\epsilon + j_3)$

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[16]Here we assume $k > 0$. For a more detailed discussion, and for similar arguments for some other operators, see appendix A.

and axial charge as $V_+\phi_1^k$ (in this monopole background) have (ϵ, j_3, R) equal to:

$$\phi_1 \rightarrow \left(\frac{1}{2} + r, \frac{1}{2}, r\right), \qquad \phi_i \rightarrow (r, 0, r), \qquad \lambda_1^i \rightarrow (1, 0, 1). \tag{6.1}$$

Using these letters we want to construct $\mathrm{U}(1) \times \mathrm{U}(N_c - 1)$ gauge-invariant operators.

Notice that replacing any of the ϕ_1's with $\lambda_1^i \phi_i$ keeps the x and y charges. Since there are only $(N_c - 1)$ λ_1^i's, and they are anti-commuting, the maximum number of ϕ_1's that one can replace with $\lambda_1^i \phi_i$ is $\min(N_c - 1, k)$. It is easy convince oneself that the operators generated in this way exhaust all the naively chiral operators at this level (this power of x). Furthermore, each such replacement flips the sign of $(-1)^F$ and also changes the $\mathrm{SU}(N_f)_l$ flavour representation, since the ϕ's are symmetric in flavour, while the $(\lambda\phi)$'s are anti-symmetric. Thus there are potential cancellations, and whether or not any contribution survives at this level depends on whether all these flavour representations cancel or not. Whenever there is a cancellation we expect that the corresponding bosonic and fermionic operators (that have the same global charges) join together into a single non-chiral multiplet of the superconformal algebra. In this section we perform this analysis.

The total Index contribution of all the operators at this level can be schematically written as

$$\sum_{n=0}^{\min(N_c-1,k)} (-1)^n (\phi_1^a)^{k-n} (\lambda_1^i \phi_i^a)^n. \tag{6.2}$$

Note that for $n > N_f$ these operators vanish due to anti-symmetry of the last factor in $\mathrm{SU}(N_f)_l$. Since the x and y charges of all these operators are the same, looking only at the $\mathrm{SU}(N_f)_l$ flavour representations (these operators are singlets of $\mathrm{SU}(N_f)_r$), we get

In the first line the $(k - n)$-box symmetric representation comes from $(\phi_1^a)^{k-n}$, while the n-box antisymmetric representation comes from $(\lambda_1^i \phi_i^a)^n$ (taking into account the anticommutation of the λ_1^i's). The second line gives the decomposition into irreducible representations of the tensor product in the first line. The third line uses the fact that representations cancel pairwise between the n'th and $(n+1)$'th terms, and the only (if at all) non-canceling

contribution comes from the last term in the series when $n = N_c - 1$. This is precisely Case 1 of table 1.

For two of the remaining three cases, namely Cases 2 and 4 listed in table 1, we will present similar but slightly more involved analytic arguments in appendix A.

The argument presented above can be straightforwardly applied for similar operators in chiral theories as well. In a $U(N_c)_k(N_f, 0)$ theory the corresponding operators are $V_+\phi_1^{k+\frac{1}{2}N_f}$, as the gauge charge of the bare monopole operator V_+ is $-(k + \frac{1}{2}N_f)$. The only difference here is thus a shift of k by $\frac{1}{2}N_f$. The above argument then implies that a non-vanishing contribution at this level occurs for

$$k + \frac{1}{2}N_f \geq N_c \qquad \text{and} \qquad N_f \geq N_c, \tag{6.4}$$

and the surviving $SU(N_f)$ representation is

$$\tag{6.5}$$

This gives Cases 1a and 2b in table 2. Notice that the conditions in (6.4) imply that the rank of the dual gauge theory is non-negative, which is required for unbroken supersymmetry.

In appendix A.2 we present similar arguments for a subset of the other cases listed in tables 2 and 3.

7 A possible dual of $V_+\phi_1^k$ when it is not chiral

In this section we discuss the dual operator to $V_+\phi_1^k$ in non-chiral $U(N_c)_k(N_f, N_f)$ theories, when this operator is not chiral (which is true for all $N_c > 1$). Since the dualities in Chern-Simons-matter theories are strong-weak dualities, in the case where the operator $V_+\phi_1^k$ is not chiral, the operator dual to it will in general have a very different weak coupling scaling dimension. But it must have the same values of the other global charges, namely spin, axial charge and flavour representation. Moreover, since we expect $V_+\phi_1^k$ to be the lowest operator with the same quantum numbers even when it is not chiral, we expect it to be dual to the lowest operator with these quantum numbers in the dual theory, because there should be no level-crossing of the operators in a fixed representation.[17]

For $N_c > 1$, the operator $V_+\phi_1^k$ sits in the k-box symmetric $SU(N_f)_l$ flavour representation and is not chiral. To find its dual we need to look for operators in the $U(N_c^d)_k$ theory which have

- $j_3 = \frac{1}{2}k$,

---

[17]This argument is not rigorous, because in CS-matter theories the coupling constant that we use to go from weak to strong coupling is discrete, rather than continuous. However, we do expect it to be valid at least in the 't Hooft large N_c limit, where this parameter becomes effectively continuous.

- A conjugate symmetric k-box representation under the $\mathrm{SU}(N_f)$ flavour symmetry acting on the $\tilde{\phi}$'s of the dual theory,

- Axial charge $= k - N_f$,

- R-charge $= N_f - N_c + 1 + r(k - N_f)$,

where the axial charge and R-charge are those of the original theory. Note that when the operator $V_+\phi_1^k$ is not chiral, there is no reason for the dual operator to be constructed out of only the supersymmetric letters that we discussed until now. Allowing for non-supersymmetric letters of the dual theory (taken to have level k as in the second line of (3.3)) the simplest possible operator with the same axial charge, R-charge, j_3 and flavour representation is

$$V_-^\dagger (\tilde{\tilde{\phi}}_1)^k \lambda_1^i \lambda_i^1. \tag{7.1}$$

Since $V_-^\dagger$ and $\lambda_1^i \lambda_i^1$ have $j_3 = 0$ and are flavour singlets, the spin and flavour representations match trivially (if we choose the $\tilde{\tilde{\phi}}_1$'s in the monopole background to have $j_3 = \frac{1}{2}$). Note that the gauge charge of $V_-^\dagger$ is $-k$. The $\lambda_1^i \lambda_i^1$ factor is just to compensate for the R-charge. So, we conjecture that this operator is dual to $V_+\phi_1^k$ in the original theory.

In the chiral $\mathrm{U}(N_c)_{N_f,0}$ case, the above argument goes through except for the minor change that here the original theory has the operator $V_+\phi_1^{k+\frac{N_f}{2}}$. In the dual theory (taken to have level k as in (4.5)), we look for operators with topological charge $\sum m_i = -1$. The only subtlety is that the R-charge of the dual theory is shifted by $\frac{N_f}{2}$ as in (4.5) compared to the original theory. Keeping track of this shift, the obvious candidate for the dual operator is

$$V_+^\dagger (\bar{\phi}_1)^{k+\frac{N_f}{2}} \lambda_1^i \lambda_i^1. \tag{7.2}$$

Note that the gauge charge of $V_+^\dagger$ is $k + \frac{N_f}{2}$. The spin matches if we again choose the $\bar{\phi}_1$'s in the monopole background to have $j_3 = \frac{1}{2}$.

8 Perturbative corrections to $V_+\phi_1^k$ in Chern-Simons-scalar theories

In this section, we return to our original motivation of understanding the mismatch of the N_c scaling of the classical dimensions of monopole operators under the non-supersymmetric Chern-Simons duality in the 't Hooft large N_c limit.

Consider the monopole operator $V_+\phi_1^k$ in a $\mathrm{U}(N_c)_k$ Chern-Simons theory coupled to a single scalar field (the analysis is similar for theories with fermions, except that there already the classical dimension scales as $|k|^{\frac{3}{2}}$ for large $|k|$). Using radial quantization, the scaling dimension of any local operator in the flat space theory is mapped to the energy of the corresponding state on S^2. The operator $V_+\phi_1^k$ corresponds to a state with unit magnetic flux on S^2, with k lowest energy scalar ϕ_1 modes excited to neutralize the charge of the bare flux state.

For operators of this type, whose classical energy scales as N_c in the 't Hooft large N_c limit (in which $\lambda \equiv N_c/k$ is kept fixed), one expects perturbation theory not to be valid, and perturbative corrections to the energy to also be of order N_c (see, for instance, [45]; this is

the case even when classical solutions for these monopoles exist in the Chern-Simons-matter theory, as in [45, 46][18]). The general arguments are very similar to the analysis of baryons in the large N_c limit of QCD [47], and we will discuss this analogy further below. However, at least in some cases one expects such operators to correspond to classical solutions of some 'master field' theory whose coupling constants scale as $1/N_c$ (an example of this is the Skyrme model description of baryons in QCD; in our theories the role of this 'master field' theory is played by the dual high-spin gravity theory). In this context one may expect corrections to the dimensions coming from the classical solutions (which are of order N_c) to be suppressed by powers of $1/N_c$, such that the energy of these configurations would remain of order N_c in the 't Hooft large N_c limit. In our case, as we discussed, such a scaling does not seem to be consistent with duality. In this section we will argue that the perturbative corrections to the anomalous dimensions of monopole operators might violate the naive large N_c counting, even at very weak coupling (this implies that these operators do not correspond to classical solutions of any 'master field' theory).

We will work with a normalization of the action where the gauge propagators come with k^{-1}, while the scalar propagators and interaction vertices have no factors of k or N_c (the 3-gluon vertex scales as k). In the 't Hooft large N_c limit at weak coupling ($\lambda \to 0$) we can restrict to planar diagrams with no loops (each loop comes with a factor of λ and hence is suppressed at weak coupling). Thus, the leading contribution to the ground state energy of a unit flux state comes from the diagrams of the form shown in figure 1a, 1b, 1c with an arbitrary number of horizontal gluon propagators.[19] The k vertical lines here are scalar propagators, which all have the same color index. We will show below that all these diagrams have a contribution to the energy that has the same scaling with k at large k; note that the connections between different scalar lines do not have to be planar (as figure 1c illustrates). Furthermore, the diagrams can be divided into two subclasses: "connected" and "disconnected". The diagrams which are "connected" have the property that one can reach any of the gluons from any other gluon by only moving along the vertical scalar propagators and the horizontal gluon propagators, without having to go through the horizontal lines at the top or bottom. All other diagrams are "disconnected". When computing the evolution of the monopole state for a time T, the "connected" diagrams scale as T (compared to figure 1a), while all other diagrams scale as higher powers of T. Thus, diagrams which are "connected" in the above sense contribute to the energy of the state directly, while those which are "disconnected" result from the expansion of the exponential of "connected" diagrams (they give e^{iHT} in the partition function with a time-difference T). Some leading "connected" diagrams are shown in figures 1a, 1b, 1c while figure 1d is an example of a "disconnected" diagram.

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[18]The appendix of [46] constructs classical BPS monopole solutions for the $\mathcal{N} = 2$ supersymmetric theories we discuss in this paper, that correspond to operators like $V_+ \Phi^k$. As discussed above, in most cases we expect this operator not to be chiral in the full theory, and then the corresponding classical solutions could also acquire large quantum corrections.

[19]The Chern-Simons-scalar theories also have $(|\phi|^2)^3$ couplings whose coefficients scale as $1/k^2$ in the small λ limit [5]. Thus, there are also diagrams with vertices of this type and with no loops, which contribute at the same order as the diagrams with gluons, and do not modify our qualitative arguments.
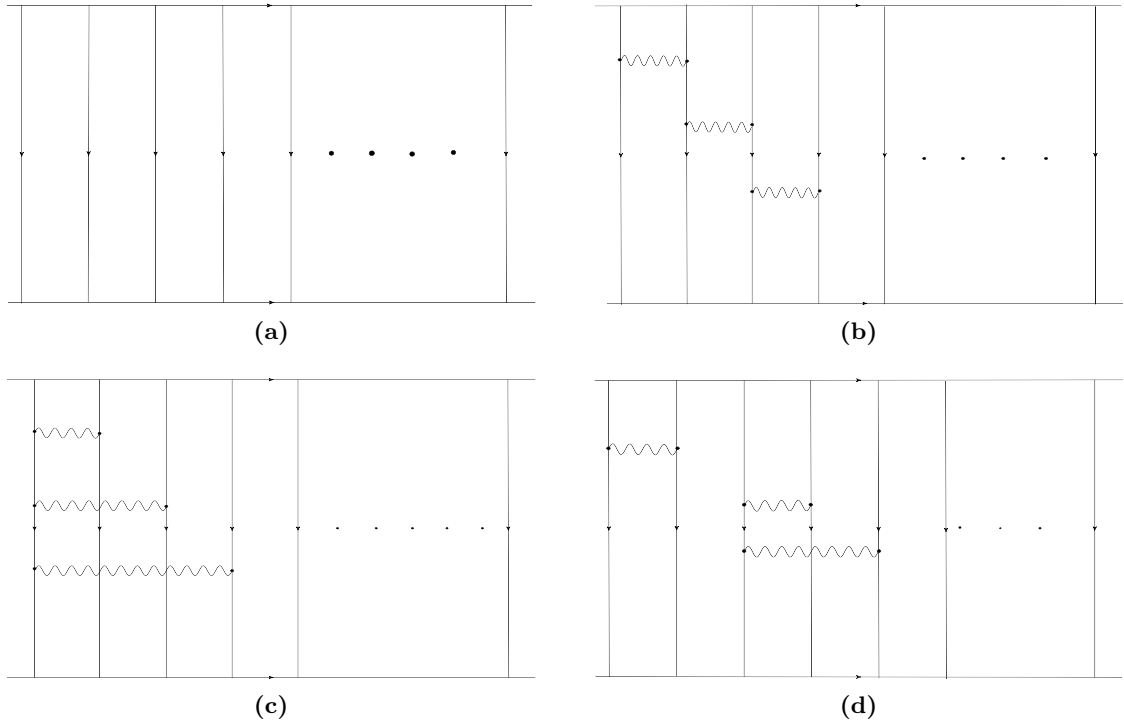
Figure 1. Some of the diagrams contributing to the energy of the lowest flux state, which is related to the dimension of the leading monopole operator.

Even within the restricted class of "connected" diagrams, the number of diagrams with a given number of gluon propagators grows very fast.[20] We will not compute these diagrams explicitly but just perform an estimate of a subset of these diagrams to show that quantum corrections can potentially change the large k scaling of the dimension of $V_+\phi_1^k$ (we assume here that there is no reason for these corrections to all cancel, as would be the case for chiral monopole operators in supersymmetric theories).

Let us normalize the contribution of the leading diagram without gluon propagators, figure 1a, to be $k!$ (this is just the number of ways of contracting k ϕ's with k $\bar\phi$'s). Let us also restrict to the subset of diagrams of the type shown in figure 1b. Such a diagram with n gluons comes with a combinatoric factor of

$$\frac{(k(k-1)\cdots(k-n))^2}{2k^n}(k-n-1)! = k(k!)\prod_{l=1}^{n}\left(1-\frac{l}{k}\right). \tag{8.1}$$

Each factor of $k(k-1)\cdots(k-n)$ comes from the need to choose which scalar (anti-scalar) connects to the first gluon line, which to the second gluon line, and so on (and we get a factor of $\frac{1}{2}$ by inverting the order of the gluon lines). The factor of k^n in the denominator comes from the gluon propagators, and the factor of $(k-n-1)!$ comes from the possible contractions of all the scalars that are not attached to gluon lines. Thus, dividing by the

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[20]One can estimate that for n gluon propagators, the number of relevant diagrams is related to the number of integer partitions of n, which grows exponentially for large n.

diagram of figure 1a, any such diagram with $n \ll \sqrt{k}$ gives a contribution of order k to the energy, which is the expected scaling of the monopole dimension. Note that this has no powers of λ, so these diagrams contribute even at very weak coupling (namely, in the limit of large k with finite N_c).

For the purpose of our estimation we assume that the full contribution from such a diagram differs from the above combinatoric factor by an $\mathcal{O}(1)$ number, since there are no obvious large factors involved. The contribution to the monopole energy from these diagrams is then estimated by summing over the contributions of this subset of "connected" diagrams. We expect that the approximation of the sum over "disconnected" diagrams by the exponential of the "connected" diagrams should be good at least for "connected" diagrams with up to $\sqrt{k}$ gluon propagators. Assuming that all extra factors are equal for all these diagrams, they sum up to

$$\sum_{n=1}^{\sqrt{k}} k(k!) \prod_{l=1}^{n} \left(1 - \frac{l}{k}\right) \sim k^{\frac{3}{2}}(k!). \tag{8.2}$$

It is easy to verify numerically that correcting this sum by a similar contribution from diagrams with a higher number of gluon propagators (by including "connected" diagrams in (8.2) with a number of gluon propagators larger then $\sqrt{k}$) does not affect the leading large k behaviour of the sum.

As described above, this is just an estimate for a very small subset of the leading diagrams at large k. Taking into account all of the other leading diagrams could generate an even larger change in the large k scaling of the monopole ground state energy compared to the "classical" value. On the other hand, clearly there is no reason to expect all these diagrams to be equal (or even to have the same sign) as we assumed. But anyway, this shows that quantum corrections could affect the scaling dimensions of these monopole operators in a very drastic way in the large k limit. In particular, we see that quantum corrections can potentially lead to a change in the k scaling of the dimension of monopole operators, which could resolve the puzzle stated in the introduction regarding the difference in the large k (large N_c) scaling of the naive dimensions of the leading monopole operators across the dual pair of bosonic and fermionic Chern-Simons theories.

Note that the analysis above is very similar to the analysis of the masses of baryons in large N_c QCD [47]. At weak coupling and leading order in large N_c the masses of baryons are $\mathcal{O}(N_c)$. All the diagrams shown above will also contribute to the masses of baryons (note that in the baryon case the quarks that replace the scalars all have different color indices, but because they are contracted with an epsilon symbol, their wave function is eventually symmetric, just like the one of the scalars in our case). Though the same diagrams are suppressed by a factor of λ^n, where λ is the 't Hooft coupling,[21] they are all comparable for a coupling of $\mathcal{O}(1)$, and the above argument would suggest that the masses of baryons could change from $\mathcal{O}(N_c)$ to some higher power of N_c (at least when their spin is of order N_c). It is widely believed that this is not the case for baryons in the large N_c

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[21]In large N_c QCD the gluon propagator goes as g_{YM}^2 rather than as $1/N_c$, while the combinatorics is the same as above.

limit [47], and there is substantial evidence for this. However, there are various differences between our case and that of baryons that could lead to a difference in the scaling of the two cases.

9 Summary

In this paper we discussed monopole operators in Chern-Simons-matter theories. We discussed in detail the chiral monopole operators in such theories with $\mathcal{N} = 2$ supersymmetry, and showed that in many cases the lowest monopole operator is rather different from the simplest naively-chiral monopole operator. In the non-supersymmetric case we noted that duality implies large corrections to the dimensions of monopole operators, which are naively inconsistent with the 't Hooft large N_c limit. We argued that these operators could have large perturbative corrections to their dimensions, that may invalidate the usual large N_c counting. It would be interesting to make these arguments more precise, and perhaps even to compute the monopole dimensions for large N_c and to verify that they are consistent with non-supersymmetric dualities.

There are many possible generalizations of our analysis. We analyzed only theories with $N_a = N_f$ or $N_a = 0$, and the generalization to arbitrary values of N_a and N_f should be straightforward. We also discussed only the simplest monopole operators with $\mathrm{U}(1)_J$ charge $\sum m_i = \pm 1$, and it would be interesting to generalize our analysis to higher charges. It would be interesting to study the chiral rings in these theories including the monopole operators, as done for some theories in [20].[22] It is not clear if all chiral monopoles with $\sum m_i = 1$ are products of the leading chiral monopoles we found with operators in the $\sum m_i = 0$ sector, and if all chiral monopoles with $\sum m_i > 1$ can be written as products of chiral monopoles with $\sum m_i = 1$. One can also use similar methods to study theories with product gauge groups. Theories of this type with a larger amount of supersymmetry were analyzed in detail in the literature, but most of the discussion in the literature (except [46]) is about monopoles that have rather different properties from the monopoles we discuss here.

For gauge groups that do not involve $\mathrm{U}(N_c)$, it is not obvious how to identify the monopole operators, since there is no $\mathrm{U}(1)_J$ symmetry. Nevertheless, the index in these theories is still written as a sum over monopole sectors with different GNO charges, and it would be interesting to try to possibly identify and match different monopole states also for such other gauge groups. In particular it would be interesting to do this for $\mathrm{SU}(N_c)$ gauge theories, noting that their analysis is completely different from the $\mathrm{U}(N_c)$ analysis we

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[22]Naively, one may expect to find a moduli space whenever there is a scalar chiral operator (appearing in the chiral ring). In the case of monopole operators, the corresponding branch of the moduli space would include a vacuum expectation value for the scalar in the vector multiplet, but we know that in the theories we discuss with $k \neq 0$ there is no such moduli space. Hence, we expect that even if some of our theories have scalar chiral monopole operators, the chiral ring would not be freely generated by these operators, but rather they have to vanish when raised to some power. In our analysis of the lightest monopole operators, we only found a scalar operator in Case 4, and in this case it is easy to believe that this operator raised to some power would vanish because fermions are explicilty involved in its construction.

presented here. In these theories there is no $U(1)_J$ symmetry, but there is a $U(1)_B$ baryon-number symmetry whose gauging leads to the $U(N_c)$ theory, and it would be interesting to use the index to understand which baryon operators are chiral.

Acknowledgments

A   Analytic arguments for chirality of monopole operators

In this section we present analytic arguments for the leading chiral monopole operators in the $(\pm 1, \overrightarrow{0})$ sector, for the cases when the leading operator does not involve derivatives or gauge-invariants of the form $(\phi_i \bar{\psi}^i)$ attached to gauge-invariant monopole operators.

The monopole operators with the background flux $(1, \overrightarrow{0})$ break the $U(N_c)$ gauge symmetry to $U(1) \times U(N_c - 1)$. In this background we have (in the weakly coupled high-energy Yang-Mills-Chern-Simons theory which is used to compute the Index) $2(N_c - 1)$ lightest gaugino states coming from λ_1^i, λ_i^1, with $j_3 = 0$, transforming under the unbroken $U(1) \otimes U(N_c - 1)$ as

$$\lambda_1^i: (-1, N_c - 1), \qquad \lambda_i^1: (+1, \overline{N_c - 1}). \tag{A.1}$$

The contribution of these operators corresponds to the following factor in the Index:

$$\prod_{i=2}^{N_c} \left(1 - \frac{z_1}{z_i} x\right) \left(1 - \frac{z_i}{z_1} x\right). \tag{A.2}$$

The other similar factors from the product over $i \neq j > 1$ constitute the Haar measure for the unbroken $U(N_c - 1)$. Thus, in the $(1, \overrightarrow{0})$ sector, the gauge-invariant operators are constructed as $U(1) \otimes U(N_c - 1)$-invariants with λ_1^i and λ_i^1 as additional supersymmetric letters (compared to the zero flux sector).

A.1 $U(N_c)_k(N_f, N_f)$ theories

Our strategy to construct gauge-invariant chiral monopole operators will be exactly the same as the usual construction of local gauge-invariant chiral non-monopole operators in perturbative gauge theories, i.e. to first identify the basic supersymmetric letters that

can contribute to the index,[23] and then to form combinations of these letters that give $\mathrm{U}(1) \times \mathrm{U}(N_c - 1)$-invariant operators.

To proceed, let us first identify the basic supersymmetric letters in the $(1, \overrightarrow{0})$ sector, satisfying $\epsilon = j_3 + R$. Using (2.4), their (ϵ, j_3, R) values are:

$$
\begin{aligned}
\phi_1, \tilde{\phi}^1 &\to \left(\frac{1}{2} + r, \frac{1}{2}, r\right), \\
\phi_i, \tilde{\phi}^i &\to (r, 0, r), \\
\bar{\psi}_+^1, \tilde{\bar{\psi}}_+^1 &\to (2 - r, 1, 1 - r), \\
\bar{\psi}_+^i, \tilde{\bar{\psi}}_+^i &\to \left(\frac{3}{2} - r, \frac{1}{2}, 1 - r\right), \\
(\lambda_-)_1^i, (\lambda_-)_i^1 &\to (1, 0, 1), \\
\partial_{++} &\to (1, 1, 0).
\end{aligned}
\tag{A.3}
$$

Since the bare monopole operator is charged only under the $\mathrm{U}(1)$ and is invariant under $\mathrm{U}(N_c - 1)$, combinations of basic letters which are $\mathrm{U}(N_c - 1)$ invariant but carry $\mathrm{U}(1)$ charges are relevant for our counting purpose. Apart from these we also have combinations of letters invariant under the full $\mathrm{U}(1) \times \mathrm{U}(N_c - 1)$ gauge group which we need to use. All these combinations and their charges and flavour representations relevant for the Index are listed in table 4.

In this subsection we concentrate on monopole operators with $\sum_i m_i = +1$. This is due to the invariance of the Index of these non-chiral theories under

$$
\{m_i\} \to \{-m_i\}, \qquad \{z_i\} \to \{z_i^{-1}\}, \qquad \{w \to w^{-1}\}, \qquad \{t_a \leftrightarrow \tilde{t}_a\}.
\tag{A.4}
$$

This just says that for each operator in a given $\{m_i\}$ sector one can obtain a monopole operator with $\{-m_i\}$ by interchanging

$$
(\phi, \bar{\psi}) \leftrightarrow (\tilde{\phi}, \tilde{\bar{\psi}}).
\tag{A.5}
$$

Another useful property of the Index is its invariance under a second set of transformations,

$$
k \to -k, \qquad \{m_i\} \to \{-m_i\}, \qquad \{w \to w^{-1}\}.
\tag{A.6}
$$

Using this property we can restrict to $k > 0$.

Yet another useful property of the Index of these theories is the fact that the R-charge in these theories can be shifted by mixing it with the $\mathrm{U}(1)_A$. Specifically, $y \to yx^{r_0}$ shifts the R-charge by $r \to r + r_0$. This freedom can be used to set the R-charge r of $\phi, \tilde{\phi}$ to any convenient value, but we will not use this here.

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[23]Including the "bare" monopole operator, transforming under some gauge group representation.

Operator	U(1)	$\epsilon + j_3$	A	$(\mathrm{SU}(N_f)_l, \mathrm{SU}(N_f)_r)$
$V_{1,\vec{0}}$	$-k$	$(1-r)N_f - N_c + 1$	$-N_f$	$(\mathcal{I}, \mathcal{I})$
$\phi_1, \lambda_1^i \phi_i$	$+1$	$1+r$	1	$(N_f, \mathcal{I})$
$\tilde{\phi}^1, \lambda_i^1 \tilde{\phi}^i$	-1	$1+r$	1	$(\mathcal{I}, N_f)$
$\bar{\psi}^1, \lambda_i^1 \bar{\psi}^i$	-1	$3-r$	-1	$(\bar{N}_f, \mathcal{I})$
$\tilde{\bar{\psi}}^1, \lambda_i^1 \tilde{\bar{\psi}}^i$	$+1$	$3-r$	-1	$(\mathcal{I}, \bar{N}_f)$
∂_{++}	0	2	0	$(\mathcal{I}, \mathcal{I})$
$\phi_i \bar{\psi}^i$	0	2	0	$(\mathcal{I} \oplus adj, \mathcal{I})$
$\tilde{\phi}^i \tilde{\bar{\psi}}_i$	0	2	0	$(\mathcal{I}, \mathcal{I} \oplus adj)$

Table 4. Some of the basic $\mathrm{U}(N_c - 1)$-invariant combinations of supersymmetric letters and their gauge and global charges. Summations over the index i run from 2 to N_c.

A.1.1 Case 1: $k \geq N_c$ and $N_f \geq N_c$

This case has already been discussed in section 6. The surviving $\mathrm{SU}(N_c)_l$ flavour representation in this case is

$$(-1)^{N_c-1} {}_{N_c}\left\{ \overbrace{\begin{array}{|c|c|c|c|} \hline & & \cdots & \\ \hline \end{array}}^{k-N_c+1} \right. . \tag{A.7}$$

This representation survives only when

$$k \geq N_c \qquad \text{and} \qquad N_f \geq N_c. \tag{A.8}$$

If any of these two conditions is violated, cancellation at this level is complete and we have to look for other lightest operators.

In the next two subsubsections we give arguments for the lightest monopole operators in the non-chiral theories for Cases 4 and 2, respectively.

A.1.2 Case 4: $m \equiv N_c - k > 0$ and $N_f \geq N_c$

In this case we will show that the leading monopole operator appears at the level of $V_+ \phi_1^k (\tilde{\phi}^i \phi_i)^m$. The Index contribution of all the operators of this general form can be schematically arranged as the following series:

$$\sum_n (\tilde{\phi}^i \phi_i)^n \left(\sum_{l=0}^{k} (-1)^l \phi_1^{k-l} (\lambda_1^i \phi_i)^l \right). \tag{A.9}$$

Naively from the above series it looks like that the Index contribution at this level also vanishes, as the series in the bracket vanishes for $k < N_c$, using the arguments given earlier for Case 1 in section 6. But notice that to actually evaluate the Index contribution of this series we have to take the tensor product of the flavour representations of each term in the series with that of the ϕ_i^n, taking into account that we only have $(N_c - 1)$ different ϕ_i's, and hence more than this number cannot be antisymmetrized in constructing the flavour representations. This makes a difference only when the total number of ϕ_i's in the operator exceeds $N_c - 1$. This shows that all operators at the level of $V_+\phi_1^k(\tilde{\phi}^i\phi_i)^n$ cancel in the index for $n < m = N_c - k$.

For $n = m$, the first special case arises. This happens because in the tensor product of flavour representations of ϕ_i^m from $(\tilde{\phi}^i\phi_i)^{N_c-k}$, and of ϕ_i^k from the $(\lambda_1^i\phi_i)^k$ in the last term in the sum over l, the totally antisymmetric representation vanishes. This leads to a non-canceling contribution from the penultimate term in the series, namely the totally antisymmetric representation of $\phi_1\phi_i^{N_c-1}$. All other representations cancel as for $n < m$. Thus, the lightest surviving operator in this case has the $\mathrm{SU}(N_f)_l \times \mathrm{SU}(N_f)_r$ flavour representation

$$\left(N_c\left\{ \begin{array}{c}\square\\\square\\\vdots\end{array}\right. , N_c-k\left\{\begin{array}{c}\square\\\square\\\vdots\end{array}\right. \right), \tag{A.10}$$

since the $\tilde{\phi}_i$ must also be multiplied anti-symmetrically.

For completeness we now show that the operators at the level of $\phi_1^{k-n}\bar{\tilde{\psi}}_1^n$ with $n > 0$, which could be *lighter* than the operators considered above, actually vanish. The index contribution at this level, for a fixed value of n, is given schematically by the sum

$$\sum_{l=0}^{n} \bar{\tilde{\psi}}_1^{n-l}(\lambda_1^i\bar{\tilde{\psi}}_i)^l \left(\sum_{p=0}^{k-n} (-1)^{n-l+p}\phi_1^{k-n-p}(\lambda_1^i\phi_i)^p \right). \tag{A.11}$$

Notice that the summation over p, for a fixed value of l in the outer summation, is exactly the Index contribution of operators at the level of operators $\phi_1^{k'}$ in a $\mathrm{U}(N_c')_{k'}$ theory with $N_c' = N_c - l, k' = k - n$. Since $N_c' - k' = (N_c - k) + (n - l) > 0$, the contribution of this series vanishes for each allowed value of $n \geq l \geq 0$, and hence the whole series vanishes including the sum over l.

A.1.3 Case 2: $k \geq N_c$ and $m \equiv N_c - N_f > 0$

In this case we show that the operators at the leading order occur at the level of $V_+\bar{\tilde{\psi}}_1^n\phi_1^{k-n}$ for $n = m$, while for $n < m$ they all cancel. The Index contribution at this level for fixed $n > 0$ is given by the following series

$$\sum_{l=\max(0,n-N_f)}^{n} \bar{\tilde{\psi}}_1^{n-l}(\lambda_1^i\bar{\tilde{\psi}}_i)^l \left(\sum_{p=0}^{\min(k-n,N_c-1-l,N_f)} (-1)^{n-l+p}\phi_1^{k-n-p}(\lambda_1^i\phi_i)^p \right). \tag{A.12}$$

Now for $n < m = N_c - N_f$ and $k \geq N_c$ we have

$$\min(k - n, N_c - 1 - l, N_f) = N_f, \tag{A.13}$$

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which results in the vanishing of the series in brackets for each value of l in the outer sum, using the arguments above.

For $n = l = m = N_c - N_f$ though, we have

$$\min(k - m, N_c - 1 - l, N_f) = N_f - 1. \tag{A.14}$$

In this case as well the series inside the brackets in (A.12) vanishes for all terms in the outer sum except the last one, $l = m$, for which the flavour representation is easily obtained from (A.7). Including the $\mathrm{SU}(N_f)_r$ representation of the $\bar{\tilde{\psi}}$'s we get

$$\left(\overbrace{\boxed{|\cdots|}}^{k-N_c}, \overbrace{\boxed{||\cdots|}}^{N_c-N_f} \right). \tag{A.15}$$

Again for completeness we now argue that the operators at the level of $V_+ \phi_1^k (\tilde{\phi}^i \phi_i)^n$ do not contribute for all $n > 0$. The Index contribution at this level is given by the series:

$$(\tilde{\phi}^i \phi_i)^n \left(\sum_{l=0}^{N_f} (-1)^l \phi_1^{k-l} (\lambda_1^i \phi_i)^l \right). \tag{A.16}$$

Here, since $N_c > N_f$, all possible representations in the flavour tensoring of ϕ_i^n with $(\lambda_1^i \phi_i)^l$ are present, and hence the sum cancels identically due to the arguments above for the vanishing of the series inside the brackets.

A.2 $\mathrm{U}(N_c)_k(N_f, 0)$ theories

In these "*chiral*" theories, since we only have chiral multiplets in the fundamental representation of the gauge group, we only have as supersymmetric letters positively charged ϕ's and negatively charged $\bar{\psi}$'s under the Cartan of the gauge group. Thus, in contrast to $\mathrm{U}(N_c)_k(N_f, N_f)$ theories, the positively charged bare monopole operators have to be dressed with ϕ's, while negatively charged bare monopole operators have to be dressed with $\bar{\psi}$'s and are thus very different.

Another important feature of these theories is that the duality for these theories works differently depending on the sign of $(k - \frac{1}{2} N_f)$:

$$
\begin{aligned}
k - \frac{1}{2} N_f \geq 0 : &\qquad \mathrm{U}(N_c)_k(N_f, 0) \leftrightarrows \mathrm{U}\left(|k| + \frac{1}{2} N_f - N_c \right)_{-k} (0, N_f), \\
k - \frac{1}{2} N_f \leq 0 : &\qquad \mathrm{U}(N_c)_k(N_f, 0) \leftrightarrows \mathrm{U}(N_f - N_c)_{-k} (0, N_f).
\end{aligned}
\tag{A.17}
$$

In the following subsections we will analytically determine the leading monopole operators ($\mathcal{M}_\pm$) for a subset of the possible cases, including all cases where the leading operator does not involve derivatives or $(\phi_i \bar{\psi}^i)$ factors.

For convenience we tabulate the relevant charges of the basic supersymmetric letters and the bare monopole operators in these theories in table 5.

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Operator	U(1)	$\epsilon + j_3$	SU(N_f)
V_+	$-k - \frac{1}{2}N_f$	$\frac{1}{2}(1-r)N_f - N_c + 1$	$\mathcal{I}$
V_-	$k - \frac{1}{2}N_f$	$\frac{1}{2}(1-r)N_f - N_c + 1$	$\mathcal{I}$
$\phi_1, \lambda_1^i \phi_i$	$+1$	$1 + r$	N_f
$\bar{\psi}^1, \lambda_i^1 \bar{\psi}^i$	-1	$3 - r$	$\bar{N}_f$
$\phi_i \bar{\psi}^i$	0	2	$N_f \times \bar{N}_f$

Table 5. Some basic U($N_c - 1$)-invariant supersymmetric letters and their gauge and global charges in U(N_c)$_k$(N_f, 0) theories.

A.2.1 $\mathcal{M}_-$ for $k > \frac{1}{2}N_f$

In this subsection we analyze the operators of the schematic form $\mathcal{M}_- = V_- \bar{\psi}_1^{k - \frac{1}{2}N_f}$, possibly with derivatives sprinkled over the $\bar{\psi}$'s. This case can be divided into subcases, depending on the comparison between $k - \frac{1}{2}N_f$ and N_c.

$N_c - 1 \geq k - \frac{1}{2}N_f > 0$. The Index contribution at the level of the lowest possible $\mathcal{M}_-$ is given by the following series

$$\sum_{l=\max\left(0, k-\frac{3}{2}N_f\right)}^{k-\frac{1}{2}N_f} (-1)^l (\bar{\psi}_1)^{k-\frac{1}{2}N_f - l} (\lambda_1^i \bar{\psi}_i)^l. \tag{A.18}$$

Again, as in previous subsections, since the "x charge" of the operator in each term is the same, we can just work with the fermion number and SU(N_f) representation of the operators. This is given by

$$\sum_{l=\max\left(0, k-\frac{3}{2}N_f\right)}^{k-\frac{1}{2}N_f} (-1)^l \left({}_{k-\frac{1}{2}N_f-l}\left\{ \begin{array}{c}\boxed{}\\\vdots\end{array}\right. \otimes \overbrace{\boxed{\cdots}}^{l} \right)$$

$$= \sum_{l=\max\left(0, k-\frac{3}{2}N_f\right)}^{k-\frac{1}{2}N_f} (-1)^l \left({}_{k-\frac{1}{2}N_f-l+1}\left\{ \overbrace{\boxed{\cdots}}^{l}\begin{array}{c}\\\vdots\end{array}\right. \oplus {}_{k-\frac{1}{2}N_f-l}\left\{ \overbrace{\boxed{\cdots}}^{l+1}\begin{array}{c}\\\vdots\end{array}\right. \right) = 0. \tag{A.19}$$

Since the level considered above vanishes, we have go to higher levels by sprinkling derivatives over the ψ's, and/or attaching gauge-invariants ($\phi_i \bar{\psi}^i$) on top of the above operators. We will not perform this analysis, but in section 4 we give a conjecture for the

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leading operators in this case (Case 1a) based on Mathematica, and show its consistency with the dualities discussed in [11].

$N_f + N_c - 1 \geq k - \frac{1}{2}N_f \geq N_c$. In this case (Case 1b) one of the operators at the level discussed in the previous case survives cancellation. The Index contribution is given by

$$
\sum_{l=\max\left(0,k-\frac{3}{2}N_f\right)}^{N_c-1} (-1)^l (\bar{\psi}_1)^{k-\frac{1}{2}N_f-l}(\lambda_1^i \bar{\psi}_i)^l
$$

$$
= \sum_{l=\max\left(0,k-\frac{3}{2}N_f\right)}^{N_c-1} (-1)^l \left({}_{k-\frac{1}{2}N_f-l}\!\!\left\{ \begin{array}{c}\square \\ \vdots \end{array} \right. \otimes \overbrace{\square\square\cdots\square}^{l} \right)
$$

$$
= \sum_{l=\max\left(0,k-\frac{3}{2}N_f\right)}^{N_c-1} (-1)^l \left({}_{k-\frac{1}{2}N_f-l+1}\!\!\left\{ \overbrace{\begin{array}{c}\square\square\cdots\square\\ \square \\ \vdots \end{array}}^{l} \oplus {}_{k-\frac{1}{2}N_f-l}\!\!\left\{ \overbrace{\begin{array}{c}\square\square\cdots\square\\ \square \\ \vdots \end{array}}^{l+1} \right. \right. \right)
$$

$$
= (-1)^{N_c-1} {}_{k-\frac{1}{2}N_f-N_c+1}\!\!\left\{ \overbrace{\begin{array}{c}\square\square\cdots\square\\ \square \\ \vdots \end{array}}^{N_c} . \right. \tag{A.20}
$$

$k - \frac{1}{2}N_f = (nN_f + m) + (N_c - 1)$ with $n \geq 1$ and $N_f > m \geq 0$. In this case (Case 1c) there is a unique operator present at the minimal level,

$$
\mathcal{M}_- = V_-(\lambda_1^i \bar{\psi}_i)^{N_c-1}(\bar{\psi}_1)^{N_f}(\partial \bar{\psi}_1)^{N_f} \dots (\partial^{n-1}\bar{\psi}_1)^{N_f}(\partial^n \bar{\psi}_1)^m. \tag{A.21}
$$

Notice that in this operator none of the $\bar{\psi}_1$'s can be replaced with $\lambda_1^i \bar{\psi}_i$, as the resulting operator would vanish due to antisymmetry of more than $(N_c - 1)$ λ_1^i's. Furthermore, none of the ∂'s can be replaced with $\phi^i \bar{\psi}_i$, as the resulting operators would vanish due to antisymmetry of more then N_f $\bar{\psi}_1$'s. This proves that this is the unique leading operator in this case.

To determine the flavour representation of this operator note that each of the $(\partial^l \bar{\psi}_1)^{N_f}$ factors forms a flavour singlet, while the remaining factors give the representation

$$
\left(\overbrace{\square\square\cdots\square}^{N_c-1} \otimes m\!\!\left\{ \begin{array}{c}\square \\ \vdots \end{array} \right. \right). \tag{A.22}
$$

A.2.2 $\mathcal{M}_-$ for $k < \frac{1}{2}N_f$

Since in this case $k - \frac{1}{2}N_f < 0$, we need to dress V_- by ϕ_1's (as opposed to $\bar{\psi}_1$'s in the previous cases) to make it gauge-invariant. Schematically we have

$$
\mathcal{M}_- = V_-(\phi_1)^{\frac{1}{2}N_f-k}. \tag{A.23}
$$

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A straightforward application of the arguments presented in section A.1.1 gives us the following results for the operator of this form contributing to the index in this case:

$$\mathcal{M}_- = 0 \qquad \text{for} \quad \frac{1}{2}N_f - k < N_c,$$

$$\mathcal{M}_- = {}_{N_c}\Bigg\{ \overbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\\hline \end{array}}^{\frac{1}{2}N_f - k - N_c + 1} \qquad \text{for} \quad \frac{1}{2}N_f - k \ge N_c. \tag{A.24}$$

The second case is Case 2a, for which we have found the leading monopole operator. For the first case (Case 2b) we need to add derivatives and/or gauge-invariants on top of the operator (A.23). We will not do this here, but we give a general conjecture based on results obtained using Mathematica for low values of (k, N_c, N_f) in table 3.

A.2.3 $\mathcal{M}_+$ for $k > \frac{1}{2}N_f$

From table 5 we see that the naive lowest $\mathcal{M}_+$ in this case is of the schematic form $V_+(\phi_1)^{k+\frac{1}{2}N_f}$. A straightforward application of the arguments presented above gives the contribution at this level

$$\mathcal{M}_+ = {}_{N_c}\Bigg\{ \overbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\\hline \end{array}}^{k+\frac{1}{2}N_f - N_c + 1} \qquad \text{for} \quad N_c \le N_f, \ \ k + \frac{1}{2}N_f - N_c \ge 0. \tag{A.25}$$

Notice that the second condition in (A.25) above is the same as the condition for the existence of a supersymmetric vacuum in these theories. Thus, within the set of theories possessing a supersymmetric vacuum, this level survives in the Index for $N_f \ge N_c$ (Case 1a, as we saw in section 6).

For $N_f < N_c$ (Cases 1b and 1c), this level vanishes and we need to consider operators with derivatives and/or gauge invariants $(\phi^i \bar{\psi}_i)$. The analytic analysis for this gets complicated and we will not pursue it here. Instead we present a conjecture for these cases in table 2, based on Mathematica evaluations at low values of k, N_c and N_f.

A.2.4 $\mathcal{M}_+$ for $k < \frac{1}{2}N_f$

For $k < \frac{1}{2}N_f$, the condition for the existence of a supersymmetric vacuum is $N_f \ge N_c$. Thus the condition for the $V_+(\phi_1)^{k+\frac{1}{2}N_f}$ level to survive is $k + \frac{1}{2}N_f - N_c \ge 0$ in (A.25) (this is Case 2b that we analyzed already in section 6). For $k + \frac{1}{2}N_f - N_c < 0$ (Case 2a), this level vanishes in the Index and we need to consider operators with derivatives and/or gauge-invariants $(\phi^i \bar{\psi}_i)$. Again, we will not pursue this exercise here, but present a conjecture in table 2, based on Mathematica evaluations for low values of the parameters.

B Dimensions of the lowest monopole operators in a Chern-Simons-fermion theory

In this section we discuss the lowest monopole operators in (non-supersymmetric) theories of fermions in the fundamental representation coupled to a $\mathrm{U}(N_c)_k$ Chern-Simons theory. For simplicity we focus on the case of a single flavour, for which k must be half-integer. As mentioned in the introduction, in a theory with only fundamental fermions, we expect the lowest monopole operator to arise from a product of a bare $(1, \vec{0})$ monopole operator with $|k| - \frac{1}{2}$ fermions. In this appendix we will compute the naive scaling dimension of such an operator. We are mostly interested in how this dimension scales for large $|k|$.

The main point is that in the case of fermions, because of Fermi statistics, one cannot just add $\psi^{k-\frac{1}{2}}$ to a bare monopole operator. One necessarily has to include fermions dressed with derivatives to construct a product with more than two fermions. If we needed to construct an operator of the form $\psi^{k-\frac{1}{2}}$ without the monopole background (ignoring the fact that this would not be gauge-invariant), we would use the fact that the fermion operators with n derivatives $D_{\alpha_1\beta_1} \cdots D_{\alpha_n\beta_n}\psi_\alpha$ form a spin $(n+\frac{1}{2})$ representation.[24] Hence their number is given by $2(n + \frac{1}{2}) + 1 = 2n + 2$. Thus the schematic operator is

$$(\psi)^2 \cdots (D^n\psi)^{2n+2} \cdots . \tag{B.1}$$

We see that in an operator with order k fermions, we must have factors $D^{n_{\max}}\psi$ with $n_{\max}$ at least of order $\sqrt{k}$. The total number of derivatives acting on all k fermions in such an operator is then at least of order $n_{\max}^3 \sim k^{\frac{3}{2}}$. Each operator $D^n\psi$ has classical dimension $(n + 1)$. Hence, the naive scaling dimension of such an operator is $O(k^{\frac{3}{2}})$.

The only modification in the monopole background is that now the spectrum of fermions is shifted down by a half, namely they have spins $n = 0, 1, 2, \cdots$, with the energy on S^2 equal to the spin [28]. First, this means that there is now a fermionic zero mode, so there are two bare monopole operators with the lowest dimension and with charges $k \pm \frac{1}{2}$. Second, this means that the product in (B.1) involves operators with multiplicity $2n + 1$ for $n = 1, 2, \cdots$. However, this does not modify the analysis above for large k, so we still find that the dimension of the lowest monopole operator is naively of order $k^{\frac{3}{2}}$.

As discussed in section 8, these naive dimensions may have large corrections that we do not know how to control.

C Additional charges of monopoles and their matching

In this appendix we give the global charges of the chiral monopole operators presented in sections 3 and 4, beyond the $\mathrm{SU}(N_f)$ charges discussed there. We also show that the global charges of the dual operators match across the GK duality. We have already presented one combination of the global charges $2j_3 + R$, appearing in the index, in sections 3 and 4.

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[24]Antisymmetrizations between derivative and fermion indices are removed by the equation of motion which fixes $\epsilon^{\beta\gamma}D_{\alpha\beta}\psi_\gamma$. Spin singlet derivatives $D^2\psi_\alpha$ vanish for the same reason. Since the derivatives commute, the only remaining representation is the symmetrized product of fermions and derivatives that has spin $n + \frac{1}{2}$.

	Region	(j_3, R, A)
1	$N_f > N_c, k \geq N_c$	$\left(\frac{k-N_c+1}{2},\ N_f + r(k - N_f),\ k - N_f\right)$
2	$k \geq N_c \geq N_f$	$\left(\frac{k-N_f+1}{2},\ N_c(1-r) + N_c^d r,\ N_c^d - N_c\right)$
3	$N_c \geq N_f, N_c > k$	$\left(\frac{N_c-N_f+1}{2},\ k + r(N_f - k),\ N_f - k\right)$
4	$N_f \geq N_c \geq k$	$\left(0,\ N_c^d(1-r) + rN_c + 1,\ N_c - N_c^d\right)$

Table 6. Global charges of the leading $\mathcal{M}_+$ monopole operators in $\mathrm{U}(N_c)_k(N_f, N_f)$ theories in various parameter ranges.

	Region	$\mathcal{N} = 2$ primary charges: $\epsilon, (j_3, R, A)$
1	$N_f > N_c, k \geq N_c$	$\epsilon = j_3 + R + 1,\ \left(\frac{k-N_c}{2},\ N_f - 1 + r(k - N_f),\ k - N_f\right)$
2	$k \geq N_c \geq N_f$	$\epsilon = j_3 + R + 1,\ \left(\frac{k-N_f}{2},\ N_c(1-r) + N_c^d r - 1,\ N_c^d - N_c\right)$
3	$N_c \geq N_f, N_c > k$	$\epsilon = j_3 + R + 1,\ \left(\frac{N_c-N_f}{2},\ k + r(N_f - k) - 1,\ N_f - k\right)$
4	$N_f \geq N_c \geq k$	$\epsilon = R,\ \left(0,\ N_c^d(1-r) + rN_c + 1,\ N_c - N_c^d\right)$

Table 7. Global charges of the $\mathcal{N} = 2$ superconformal primary corresponding to the $\mathcal{M}_+$ monopole operator in $\mathrm{U}(N_c)_k(N_f, N_f)$ theories in various parameter ranges.

Let us start with the nonchiral case. The global charges are (j_3, R, A) as mentioned in section 2. For the monopole operators listed in table 1, we give these global charges in table 6. Here $N_c^d = k + N_f - N_c$ is the rank of the dual group. Note that we cannot read off the j_3 and R-charges just from the index. For Cases 1, 2 and 4, the value of j_3 is computed from the form of the leading monopole operator that we found analytically. One can verify that the charges of Cases 2 and 4 map correctly under the duality. For Case 3 we use the operator that we conjectured in table 1, and one can check that this is consistent with the duality to Case 1. For completeness, we mention that for $k = 0$, the naive chiral operator V_+ survives, whose charges are $(j_3, R, A) = (0, N_f(1-r) - N_c + 1, -N_f)$. We also note the charges of the superconformal primary from which the corresponding monopole operator descends in table 7, using the rules mentioned in [48].

Let us now consider the chiral case. Some of the charges of the two monopole operators $(\mathcal{M}_+, \mathcal{M}_-)$ which survive in this case were given in table 2 and table 3. With the same conventions for regions, we give the global charges (j_3, R) in table 8. Again, in the cases in which we computed the monopole operator explicitly, the charge we give is based on

	$\mathcal{M}_+$ charges (j_3, R)	$\mathcal{M}_-$ charges (j_3, R)
1a	$\left(\frac{\tilde{N}_c+1}{2}, \frac{N_f}{2}+kr\right)$	$\left(\frac{N_c+1}{2}, k(1-r)\right)$
1b	$\left(\frac{k+N_c-\frac{3}{2}N_f+1}{2}, \frac{1}{2}N_f+kr\right)$	$\left(k-\frac{1}{2}N_f-\frac{1}{2}N_c+\frac{1}{2}, k(1-r)\right)$
1c	$\left(\frac{N_c^d-1}{2}+\frac{n(n+1)}{2}N_f+m(n+1), \frac{1}{2}N_f+kr\right)$	$\left(\frac{N_c-1}{2}+\frac{n(n+1)}{2}N_f+m(n+1), k(1-r)\right)$
2a	$\left(\frac{N_c-k-\frac{1}{2}N_f+1}{2}, \frac{1}{2}N_f+kr\right)$	$\left(\frac{\frac{1}{2}N_f-k-N_c+1}{2}, \frac{1}{2}N_f-kr\right)$
2b	$\left(\frac{k+\frac{1}{2}N_f-N_c+1}{2}, \frac{1}{2}N_f+kr\right)$	$\left(\frac{k+N_c-\frac{1}{2}N_f+1}{2}, \frac{1}{2}N_f-kr\right)$

Table 8. Global charges of the leading $\mathcal{M}_+, \mathcal{M}_-$ monopole operators in $\mathrm{U}(N_c)_k(N_f, 0)$ theories.

	$\mathcal{N}=2$ primary (j_3, R); $\epsilon = j_3 + R + 1$	$\mathcal{N}=2$ primary charges (j_3, R); $\epsilon = j_3 + R + 1$
1a	$\left(\frac{\tilde{N}_c}{2}, \frac{N_f}{2}+kr-1\right)$	$\left(\frac{N_c}{2}, k(1-r)-1\right)$
1b	$\left(\frac{k+N_c-\frac{3}{2}N_f}{2}, \frac{N_f}{2}+kr-1\right)$	$\left(k-\frac{1}{2}N_f-\frac{1}{2}N_c, k(1-r)-1\right)$
1c	$\left(\frac{N_c^d-2}{2}+\frac{n(n+1)}{2}N_f+m(n+1), \frac{N_f-2}{2}+kr\right)$	$\left(\frac{N_c-2}{2}+\frac{n(n+1)}{2}N_f+m(n+1), k(1-r)-1\right)$
2a	$\left(\frac{N_c-k-\frac{1}{2}N_f}{2}, \frac{N_f}{2}+kr-1\right)$	$\left(\frac{\frac{1}{2}N_f-k-N_c}{2}, \frac{N_f}{2}-kr-1\right)$
2b	$\left(\frac{k+\frac{1}{2}N_f-N_c}{2}, \frac{N_f}{2}+kr-1\right)$	$\left(\frac{k+N_c-\frac{1}{2}N_f}{2}, \frac{N_f}{2}-kr-1\right)$

Table 9. Global charges of the $\mathcal{N}=2$ superconformal primary corresponding to the $\mathcal{M}_+, \mathcal{M}_-$ monopole operators in $\mathrm{U}(N_c)_k(N_f, 0)$ theories in various parameter ranges.

this operator. In the other cases the charge we give is based on our conjectures in tables 2 and 3, and one can verify that in all cases it is consistent with the duality. We also note the charges of the superconformal primary from which the corresponding monopole operator descends in table 9, using the rules mentioned in [48].

Note that j_3 matches straightforwardly under the duality, while for the R charge one needs an extra shift by $-\frac{N_f}{2}$ for Cases 1a, 1b, 1c, and by $-k$ in Cases 2a, 2b, as explicitly given in (4.5).[25]

As discussed in the main text, $k = \frac{1}{2}N_f$ is a special case and to find the lightest $\mathcal{M}_+$ monopole operator on the dual side one needs to take into account the contribution of the

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[25]These shifts in R-charge, required for duality matching, are from the side of the original theory. The corresponding shifts from the dual side are just the negatives of these.

gauge singlet chiral multiplet V_+. Here we present the charges of the operator in (4.6) of the dual $\mathrm{U}(N_c^d)_{-\frac{1}{2}N_f}(0, N_f)$ theory, which are

$$\left(\frac{1}{2}(N_c^d + 1), N_f(1 - r)\right). \tag{C.1}$$

With a shift of $\frac{1}{2}N_f$ in the R-charge these match precisely with those of $\mathcal{M}_+$ in Case 1a (or 2b) for $k = \frac{1}{2}N_f$. Further it is easily verified that our proposed dual operator (4.6) contains, in its flavour decomposition, the flavour representation of the corresponding operator in the original theory.

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