arXiv:1104.0680v2 [hep-th] 12 Apr 2011

Supersymmetric States in Large N Chern-Simons-Matter Theories

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ABSTRACT: In this paper we study the spectrum of BPS operators/states in $\mathcal{N} = 2$ superconformal $U(N)$ Chern-Simons-matter theories with adjoint chiral matter fields, with and without superpotential. The superconformal indices and conjectures on the full supersymmetric spectrum of the theories in the large N limit with up to two adjoint matter fields are presented. Our results suggest that some of these theories may have supergravity duals at strong coupling, while some others may be dual to higher spin theories of gravity at strong coupling. For the $\mathcal{N} = 2$ theory with no superpotential, we study the renormalization of R-charge at finite 't Hooft coupling using "$\mathcal{Z}$-minimization". A particularly intriguing result is found in the case of one adjoint matter.

Contents

1. Introduction

The coupling constant of a four dimensional gauge theory coupled to matter generically runs under the renormalization group. While it is sometimes possible to choose the matter content and couplings of the theory so that the gauge β function vanishes, such choices are very special. In three dimensions, on the other hand, gauge fields are naturally self coupled by a Chern-Simons type action. As the coefficient of the Chern-Simons term in the action is forced by gauge invariance to be integrally quantized, the low energy gauge coupling (inverse of coefficient of the Chern-Simons term) cannot be continuously renormalized and so does not run under the renormalization group. All these statements are for every choice of matter content and couplings of the theory. As a consequence CFTs are much easier to construct starting with Chern-Simons coupled gauge fields in $d=3$ than with Yang Mills coupled gauge fields in $d=4$ [1, 2, 3, 4, 5].

Precisely because the coefficient, k, of the Chern Simons term is an integer, the Chern-Simons coupling cannot be varied continuously. The set of Chern-Simons CFTs obtained, by varying a given Lagrangian over the allowed values of k, yields a sequence rather than

a fixed line of CFTs. Consider, however an $SU(N)$ Chern Simons theory at level k. Such a theory admits a natural 't Hooft limit in which we take $N \to \infty$, $k \to \infty$ with $\lambda = \frac{N}{k}$ held fixed. As explained by 't Hooft, λ is the true loop counting parameter or coupling constant in this limit. Several physical quantities - like the spectrum of operators with finite scaling dimension- are smooth functions of λ. Now a unit change in k changes λ by $-\frac{\lambda^2}{N}$, a quantity that is infinitesimal in the large N limit. As a consequence, even though k and N are both integers, λ is an effectively continuous parameter in the large N limit. Effectively, the discretum of Chern-Simon-matter CFTs at finite N merges into an effective fixed line of Chern-Simon-matter theories at large N, parameterized by the effectively continuous variable λ.

Lines of fixed points of large N CFTs map to families of theories of quantum gravity, under the AdS/CFT correspondence [6]. CFTs at weak or finite 't Hooft coupling λ are generically expected to map to relatively complicated higher spin theories of gravity [7, 8, 9] or string theories on AdS spaces of string scale radii. In many examples of explicit realization of AdS/CFT, the bulk description simplifies in some manner at strong λ. It is then natural to ask whether the large class of fixed lines of Chern-Simons-matter theories admit simple dual descriptions at large λ [5]. The first explicit realization of the gravity dual of a large N Chern-Simons-matter theory, as a critical string theory, was achieved by ABJM [10]. At infinitely strong coupling the ABJM Chern-Simons-matter theory develops a supergravity dual description, which is a considerable simplification over the highly curved stringy dual description at finite coupling. A direct field theoretic hint for the nature of the dual of ABJM theory [10] at strong coupling comes from the observation that the set of single trace supersymmetric states in ABJM theory have spins ≤ 2 (and in fact match the spectrum of supergravitons of IIA theory on $AdS_4 \times CP^3$).

While many examples of gravity duals of supersymmetric Chern-Simons-matter theories have been proposed following the work of ABJM (see for instance [11, 12, 13, 14, 15, 16]), essentially all such proposals involve quiver type matter content in the field theory. The gravity duals of seemingly simpler Chern-Simons-matter fixed points, both with and without supersymmetry, remain unknown (and may well be most interesting in the non supersymmetric context). On the other hand, it is of significant interest to find the CFT duals of gravity theories in AdS_4 with as few four-dimensional bulk fields (apart from gravity itself) as possible, and one may hope that the Chern-Simons theories with simple matter content are good candidates.

In order to maintain a degree of technical control, however, in this paper we study only supersymmetric theories with at least four supercharges. We will consider large N $\mathcal{N} = 2$ and $\mathcal{N} = 3$ Chern Simons theories with a single $U(N)$ gauge group and g adjoint chiral multiplets (for all integer g). Such theories have been studied perturbatively in [5, 17] (see also [18]). We will study theories both with and without superpotentials. We address and largely answer the following question: what is the spectrum of supersymmetric operators as a function of the 't Hooft coupling λ? In the rest of this introduction we elaborate on our motivation for asking this question. In the next section we briefly summarize our principal results.

In this paper we will compute (or present conjectures for) the supersymmetric spec-

trum of a large class of large N Chern-Simons-matter CFTs. As we will now describe in some detail, the results we find for the supersymmetric spectrum of several fixed lines is quite simple, and has several intriguing features. As we describe in more detail below, an important difference between some theories with $\mathcal{N} = 2$ supersymmetry and theories with $\mathcal{N} = 3$ and higher supersymmetry is that the R-charge of the chiral multiplets of the $\mathcal{N} = 2$ theories (and hence the charges and dimensions of supersymmetric operators) may sometimes be continuously renormalized as a function of λ [5] (and sometimes not [17]). Luckily, Jafferis [19] has presented a proposal that effectively allows the computation of the R-charge as a function of λ in many of these theories. In the rest of this paper we assume the correctness of Jafferis' proposal; we proceed to use a combination of analytic and numerical techniques to present a complete qualitative picture of this R-charge as a function of λ and the number of chiral multiplets g.

While we hope that our results will eventually inspire conjectures for relatively simple large λ descriptions of some of the theories we study, in no case that we have studied have our results proven familiar enough to already suggest a concrete conjecture for the dual description of large λ dynamics. Of the supersymmetric spectra we encounter in this paper, the one that appears most familiar is the spectrum of the $\mathcal{N} = 3$ theory with two chiral multiplets. As we will describe in more detail below, this spectrum includes only states of spins ≤ 2, and so might plausibly agree with the spectrum of some supergravity compactification: however a detailed study of the spectrum as a function of global charges reveals some unexpected features that has prevented us (as yet) from finding a supergravity compactification with precisely this spectrum. The spectrum of supersymmetric states in $\mathcal{N} = 2$ deformations of the $\mathcal{N} = 3$ theory also has similar features. We hope to return to an investigation into the possible meanings of these spectra in the future.

In the next section we will present a more detailed description of the theories we have studied and our results.

Note added in proof: Upon completion of this work, we received [20] which overlaps with section 4 of this paper.

2. Summary of results

2.1 Theories with a vanishing superpotential

2.1.1 R-charge as a function of λ

The first class of theories studied in this paper consists of $\mathcal{N} = 2$ $U(N)$ Chern Simons theories at level k, coupled to g adjoint chiral multiplets with vanishing superpotential. This class of theories was studied, and demonstrated to be superconformal (for all N, k and g) in [5]. In the free limit the R-charge of the chiral fields in this theory equals $\frac{1}{2}$. However, it was demonstrated in [5] that this R-charge is renormalized as a function of λ; indeed at first nontrivial order in perturbation theory [5] demonstrated that the R-charge of a chiral field is given by

$$h(\lambda) = \frac{1}{2} - (g+1)\lambda^2 \tag{2.1}$$

where $\lambda = \frac{N}{k}$. As the R-charge of a supersymmetric operator plays a key role in determining its scaling dimension (via the BPS formula), the exact characterization of the spectrum of supersymmetric states in this theory at large λ clearly requires control over the function $h(\lambda)$ at large λ. Such control cannot be achieved by perturbative techniques, but is relatively easily obtained by an application of the extremely powerful recent results of Jafferis [19] to this problem. In [19] Jafferis used localization methods to derive a formula (in terms of an integral over r variables, where r is the rank of the gauge group) for the partition function on S^3 of the CFT in question, as a function of h_i the R-charges of all the chiral fields in the theory. He then demonstrated that the modulus of this partition function is extremized by the values of h_i at the conformal fixed point, assuming the absence of accidental global symmetries. In the large N limit of interest to this paper, Jafferis' matrix integral is dominated by a saddle point. Using a combination of analytic and numerical techniques, it is not difficult to solve the relevant saddle point equations, extremize the action with respect to h, and thereby evaluate $h(\lambda)$. It turns out that $h(\lambda)$ is a monotonically decreasing function for all g. In fact at large g (but all values of λ)

$$h(\lambda) = \frac{1}{2} - \frac{g\pi^2}{\left(\frac{g\pi^2}{2}\right)^2 + \left(\frac{\pi}{\lambda}\right)^2} + \mathcal{O}\left(\frac{1}{g^2}\right). \tag{2.2}$$

Note that $h(\lambda)$ tends to a constant value at $\lambda = \infty$. At large g this constant value is barely below the free value $\frac{1}{2}$; it is given by

$$\frac{1}{2} - \frac{4}{\pi^2 g} + \mathcal{O}\left(\frac{1}{g^2}\right).$$

Although there is no a priori reason for this formula to apply at g of order unity, we have used numerical techniques to find that it appears to work at the 10-15 percent level even down to $g = 2$. More specifically, at $g = 3$, our numerics indicates $h(\infty) \approx 0.35$, and at $g = 2$, $h(\infty) \approx 0.27$ (see Figs. 1 and 2 below); these do not compare badly with 0.365 and 0.3 as predicted by (2.2).

Most interestingly, however, at $g = 1$, $h(\infty) = 0$ (see Fig. 4 below); i.e. the R-charge of the chiral multiplet decreases without bound in this special (extreme) case (this was anticipated in [18]) , raising several interesting questions that we will come back to later.[1]

2.1.2 Spectrum of single trace operators

Given the function $h(\lambda)$, it is not difficult to evaluate the superconformal index [21] of these theories as a function of $h(\lambda)$. As was already noted in [21], this index demonstrates that

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[1]We emphasize that these results have been obtained using large N saddle point techniques on Jafferis's integral. The fact that our results agree with perturbation theory at small λ give us confidence that we have identified the correct saddle point at small λ. It is possible, however, that the integral undergoes a large N phase transition to another saddle point at a finite value of λ. In this case the field theory would undergo a phase transition at that λ, and our results above apply only below the phase transition. Though this seems unlikely given the analysis of [18], it does not seem completely ruled out that the theory develops pathology, and ceases to be completely well behaved at a critical λ. If this is the case, the results of this paper apply only below this critical coupling λ.

the spectrum of supersymmetric single trace operators grows exponentially with energy for $g \geq 3$. In the absence of a superpotential one can actually compute a slightly more refined Witten index (adding in a chemical potential that couples to the global symmetry generators). Below we demonstrate that this refined index implies an exponentially growing density of states for the supersymmetric spectrum (in the theory without a superpotential) even for $g = 2$. This immediately suggests that the simplest possible dual description for all theories with $g \geq 3$ (and the theory without a superpotential at $g = 2$) is a string theory, with an exponential growth in *supersymmetric* string oscillator states.

However, the index indicates a sub exponential growth of supersymmetric states for all theories with $g = 1$ and theories with a nontrivial superpotential when $g = 2$. This leaves open the possibility of a simpler dual (one with a field theory's worth of degrees of freedom) in these cases. In this subsection we focus on theories without a superpotential, and so consider only the case $g = 1$. Making the assumption that the spectrum of supersymmetric states in this theory is isomorphic to the cohomology of the *classical* action of the susy operator, we have computed the full spectrum of single trace supersymmetric operators in this theory. Our explicit results are listed in Table (12). While the states listed in this table do grow in number with energy in a roughly Kaluza-Klein fashion, notice that the primaries listed in Table (12) include states of arbitrarily high spins, ruling out a possible dual supergravity dual description.

The supersymmetric states in Table (12) of course include the states in the chiral ring $\mathrm{Tr}\,\phi^n$ for all n where ϕ is the scalar component of the chiral field. The scaling dimension of these chiral ring operators is given by $nh(\lambda)$ ($h(\lambda)$ was defined in the previous subsection). Unitarity, however, requires that every scalar operator in any 3 d CFT has scaling dimension $\geq \frac{1}{2}$, and that an operator with dimension $\frac{1}{2}$ is necessarily free (i.e. decoupled from the rest of the theory). Recall that $h(\lambda)$ decreases monotonically to zero as λ is increased. Let λ_n^f denote the unique solution to the equation

$$h(\lambda_n^f) = \frac{1}{2n}.$$

For $\lambda > \lambda_n^f$, the scaling dimension of $\mathrm{Tr}\,\phi^n$ descends below its unitarity bound $\frac{1}{2}$. (For later use we will also find it useful to define

$$h(\lambda_n^m) = \frac{2}{n}.$$

λ_n^m is the value of the coupling at which a superpotential deformation by $\mathrm{Tr}\,\Phi^n$ becomes marginal. Note that $\lambda_n^m < \lambda_n^f$.)

Assuming Jafferis' proposal, it follows from unitarity that our theory must either cease to exist [2] or must undergo a phase transition at a critical value, $\lambda = \lambda_c \leq \lambda_2^f$. While many possibilities are logically open, one attractive scenario (which is close to the scenario suggested in [18]) is the following. As λ is increased past λ_2^f then $\mathrm{Tr}\,\phi^2$ becomes free and decouples from the theory. As λ is further increased past λ_3^f then $\mathrm{Tr}\,\phi^3$ also becomes free and decouples. This process continues ad infinitum, leading to an infinite number of phase transitions.

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[2]However Niarchos's study of this theory in [18] makes this possibility unlikely

The picture oulined above for the $g = 1$ theory, namely that each of the $\text{Tr}\phi^{n+1}$ decouple at successively larger values of n can be subject to a consistency check. It was demonstrated in [18], using brane constructions, that the deformation of the zero superpotential system by the operator $\text{Tr}\phi^{n+1}$ breaks supersymmetry precisely at $\lambda = n$ (and in particular susy is not broken at smaller values of λ).[3] However the deformation of a superpotential by a free field always breaks supersymmetry. Consistency with the scenario outlined above therefore requires that $\lambda_n^f > n - 1$. In other words

$$h(\lambda) \geq \frac{1}{2(\lambda + 1)}.$$

Our data (see Fig. 4) seems consistent with this bound, and moreover suggests (and we conjecture that) $h(\lambda)$ asymptotes to $\frac{1}{2(\lambda+1)}$ from above. It would be very interesting to establish this conjecture by analytic methods, but we leave that for future work.

If this picture outlined in the last two paragraphs is correct, then, in the limit $\lambda \to \infty$ we have an effective continuum of chiral primaries, with scaling dimension $\geq \frac{1}{2}$ (all primaries with lower dimension have decoupled). The higher spin fields listed in Table (12) also reduce to a continuum at large λ. All this suggests that the large λ behavior of this theory is intriguing, and possibly singular.[4]

2.2 Theories with a superpotential

Let us now turn to the study of superconformal theories with a single $U(N)$ gauge group, only adjoint fields (as above) but with appropriate superpotentials. For theories that reduce to free systems as $\lambda \to 0$, the superconformal index is independent of the details of the superpotential, other than the fact that the index cannot now be weighted with respect to a chemical potential for any global symmetry under which the superpotential is charged, and depends only on the R-charge of matter fields which may be renormalized.[5] So in particular, the index demonstrates the presence of an exponentially growing spectrum of supersymmetric states for $g \geq 3$, exactly as above. For this reason we focus our study on $g \leq 2$. Let us first start with $g = 1$.

2.2.1 $\text{Tr}\,\Phi^4$ at $g = 1$

The superpotential deformation $\text{Tr}\,\Phi^4$ is marginal at $\lambda = 0$, but is relevant at finite λ (this follows because $h(\lambda) < \frac{1}{2}$ for all finite λ). It has been argued in [5] that the beta function for this superpotential term vanishes when its coefficient is of order λ (at small λ) leading to a weakly coupled CFT with a $\text{Tr}\,\Phi^4$ superpotential turned on. The presence of the superpotential forces the R-charge of the field ϕ to be fixed at $h = \frac{1}{2}$ at all values of λ in this new fixed line. While the superconformal index of this theory is blind to the presence of the superpotential, the spectrum of single trace supersymmetric operators is

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[3]We are grateful to Ofer Aharony and Zohar Komargodski for bringing this to our attention.

[4]Note, on the other hand, that when $g \geq 2$, $h(\lambda) > \frac{1}{4}$ for all λ. The large λ behaviour for these theories shows no hint of any singular behaviour.

[5]At fixed points that are described by a generic quartic superpotential, or in fact, points that lie on the "conformal manifold", the R-charge is not renormalized [17].

not. We present a conjecture for this spectrum in Table (13) below (this conjecture is based on the same assumption described above, namely that the supersymmetric spectrum is accurately captured by the classical supercharge cohomology at all λ.) As in the case of theories without a superpotential, our conjectured supersymmetric spectrum grows with energy in a manner expected of Kaluza-Klein compactification, but continues to include states of arbitrarily high spin.

2.2.2 $\text{Tr}\,\Phi^3$ at $g = 1$

$\text{Tr}\,\Phi^3$ is another relevant deformation of the weakly coupled theory at $g = 1$. This deformation leads to a ϕ^4 term in the scalar potential of the theory. At least at weak coupling we would expect the RG flow seeded by this operator to end at the supersymmetric large N analogue of a Wilson Fisher fixed point. At this fixed point the scaling dimension (and superconformal R-charge) of ϕ is fixed to be $\frac{2}{3}$. This fixed point can then be continued to large λ by varying the gauge coupling. For this reason the superconformal index of this theory cannot be calculated from a free path integral; however it can be computed using the techniques of localization using the results of [22] (following the original work of [23, 24]). We have performed this computation in section 10 below. While the computation of this index is exact at all N and k, the final result simplifies dramatically in the large N 't Hooft limit of interest to this paper, and in fact reduces to the index of the free theory with appropriate charge renormalizations (in order to account for the fact that the R-charge of ϕ is $\frac{2}{3}$ rather than $\frac{1}{2}$). We have also computed the full spectrum of single trace supersymmetric primaries in this theory (using assumptions similar to those described above). Our rather simple final results are listed in Table (14) .

2.2.3 $\text{Tr}\,\Phi^n$ at $g = 1$

At small λ, operators of the form $\text{Tr}\,\Phi^n$ are irrelevant for $n \geq 5$. However if our description of the $g = 1$ theory without a superpotential in the previous subsection is indeed correct then for each n the operator $\text{Tr}\,\Phi^n$ is in fact relevant for $\lambda > \lambda_n^m$. It seems likely that the RG flows seeded by these operators end in new lines of CFTs in which the scaling dimension of ϕ is fixed at $\frac{2}{n}$. We expect the index of these theories (in the 't Hooft limit) to once again be given by the formula for non-interacting theory but with renormalized charges for all fields. Using methods similar to those described above, it should be easy to compute the full spectrum of single trace superconformal primaries in this theory. We leave this computation to future work.

2.2.4 $\mathcal{N} = 3$ theory at $g = 2$

Let us now turn to $g = 2$ theories with a superpotential. First consider superpotentials of the form $\text{Tr}\,[\Phi_1, \Phi_2]^2$. Like any quartic superpotential in this theory, this superpotential is marginal at $\lambda = 0$, but is relevant at finite λ (regarded as a deformation about the theory with no superpotential). It was argued in [5] that the RG flow seeded by this operator ends with the coefficient of this superpotential stabilized at that finite value (of order λ) that enhances the supersymmetry of the theory to $\mathcal{N} = 3$. This $\mathcal{N} = 3$ theory enjoys invariance under an enhanced $SU(2)$ R symmetry group, and also enjoys invariance under an $SU(2)$

flavour symmetry group. We have computed the spectrum of supersymmetric states in this theory; our results are presented in Table (4). Interestingly, it turns out that the spins of supersymmetric states in this theory grow roughly in the manner one would expect of a Kaluza-Klein compactification of a supergravity theory on $AdS_4 \times S^3$. In particular the spins of supersymmetric states in this theory never exceed two. We have not, however, managed to identify a specific supergravity compactification that could give rise to this spectrum (there seems to be a qualitative difficulty in making such an identification, as we describe in more detail in section 8.2).

2.2.5 Superconformal $\mathcal{N} = 2$ deformations of the $\mathcal{N} = 3$ theory

There exists a manifold of exactly marginal $\mathcal{N} = 2$ deformations [17] of the $\mathcal{N} = 3$ theory described in the previous paragraph. This manifold can be characterized rather precisely in the neighbourhood of the $\mathcal{N} = 3$ fixed point using the recent results of [25]. We have computed the spectrum of supersymmetric states in these deformed theories. Our results are presented in Tables (17), (18) and (19). Qualitatively, our results for these deformed theories are similar to those described in the previous paragraph. The spins of all supersymmetric states are less than or equal to two, potentially describing the supersymmetric spectrum of a Kaluza-Klein compactification.

2.2.6 Other superpotentials at $g = 2$

Finally, as λ is increased superpotentials of up to 7th order in the chiral fields eventually turn relevant (Here we use $h(\infty) \simeq 0.27$ at explained above). Just as in the case of $g = 1$, this suggests the existence of new fixed lines with superpotentials of up to 7th order in chiral fields turned on. In addition, at every value of λ there probably exist superconformal field theories with cubic superpotentials. We leave the investigation of these theories and their supersymmetric spectra to future work.

3. $\mathcal{N} = 2, 3$ superconformal algebras and their unitary representations [6]

In order to lay out the background (and notation) for our analysis of supersymmetric states, in this section we present a brief review of the structure of the $\mathcal{N} = 2, 3$ superconformal algebras, their unitary representations and their Witten indices. We also explicitly decompose every representation of the relevant superconformal algebras into irreducible representations of the conformal algebra. The paper [21] is useful background material for this section. The reader who is familiar with the superconformal algebras and their representation theory may wish to skip to the next section.

3.1 The superconformal algebras and their Witten indices

In this section we briefly review the representation theory of the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ superconformal algebras. The bosonic subalgebras of these Lie super algebras is given by $SO(3, 2) \times SO(2)$ (for $\mathcal{N} = 2$) and $SO(3, 2) \times SO(3)$ (for $\mathcal{N} = 3$). Primary states of these

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[6]This section was worked out in collaboration with Jyotirmoy Bhattacharya.

algebras are labeled by (Δ, j, h) where Δ is the scaling dimension of the primaries, j is its spin and h is its R-charge (or R symmetry highest weight). h can be any positive or negative real number for $\mathcal{N} = 2$, but is a positive half integer for $\mathcal{N} = 3$.

The labels of unitary representations of the superconformal algebra obey the inequalities forced by unitarity. When $j \neq 0$ the condition

$$\Delta \geq |h| + j + 1$$

is necessary and sufficient for unitarity. When $j = 0$ unitary representations occur when

$$\Delta = |h|, \quad (|h| \geq \frac{1}{2})$$

or when

$$\Delta \geq |h| + 1$$

The isolated representations, and those that saturate this bound, are all short.

The Witten index $\mathrm{Tr}(-1)^F x^{\Delta+j}$ vanishes on all long representations of the supersymmetry algebra but is nonzero on short representations. This index captures information about the state content of a conformal field theory. The only way that the Witten index of a CFT can change under continuous variations of parameters (like the parameter λ in our theory), is for the R-charge to be renormalized as a function of that parameter. Note that the R-charge is fixed to be a half integer at $\mathcal{N} = 3$, but can in principle be continuously renormalized at $\mathcal{N} = 2$.

In the case of $\mathcal{N} = 2$ theories, we have glossed over a detail. At the purely algebraic level, in this case, there are really two independent Witten indices; $\mathcal{I}_+$ and $\mathcal{I}_-$. These are defined as

$$\mathcal{I}_+ = \mathrm{Tr}(-1)^F x^{H+J} e^{-\beta(H-J-R)}$$

and

$$\mathcal{I}_- = \mathrm{Tr}(-1)^F x^{H+J} e^{-\beta(H-J+R)}$$

respectively. We used the notation H for the dilatation operator, J the third component of the spin, and R the R symmetry generator. The indices above are distinct, even though they both evaluate to quantities independent of β. The first index receives contributions only from states with $\Delta = j+h$; all such states are annihilated by and lie in the cohomology of the supercharge with charges $(\frac{1}{2}, -\frac{1}{2}, 1)$. The second index receives contributions only from states with $\Delta = j - h$; all such states are annihilated by and lie in the cohomology of the supercharge with charges $(\frac{1}{2}, \frac{1}{2}, 1)$. The existence of two algebraically independent Witten Indices is less useful than it might, at first seem, in the study of quantum field theories, as the two indices are closely linked by the requirement of CPT invariance.

3.2 State content of all unitary representations of the $\mathcal{N} = 2$ superconformal algebra

In the rest of this section we will list the conformal representation content of all unitary representations of the superconformal algebra, and use our listing to compute the index of all short representations of this algebra.

Operator	States
I	$(0,0,0)$
Q	$(\frac{1}{2},\frac{1}{2},1),\ (\frac{1}{2},\frac{1}{2},-1)$
Q^2	$(1,0,2),\ (1,0,-2),\ (1,0,0),\ (1,1,0)$
Q^3	$(\frac{3}{2},\frac{1}{2},1),\ (\frac{3}{2},\frac{1}{2},-1)$
Q^4	$(2,0,0)$

Table 1: A decomposition of the antisymmetrized products of supersymmetries into irreducible representations of the maximal compact bosonic subgroup of the relevant superalgebras. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation. The same labeling convention is used in all tables in this section.

Primary	Conformal content	Index
(Δ, j, h)	$(\Delta, j, h),$ $(\Delta+\frac{1}{2}, j+\frac{1}{2}, h+1),\ (\Delta+\frac{1}{2}, j-\frac{1}{2}, h+1),$ $(\Delta+\frac{1}{2}, j+\frac{1}{2}, h-1),\ (\Delta+\frac{1}{2}, j-\frac{1}{2}, h-1),$ $(\Delta+1, j, h+2),\ (\Delta+1, j, h-2),\ (\Delta+1, j+1, h),$ $2(\Delta+1, j, h),\ (\Delta+1, j-1, h),$ $(\Delta+\frac{3}{2}, j+\frac{1}{2}, h+1),\ (\Delta+\frac{3}{2}, j-\frac{1}{2}, h+1),$ $(\Delta+\frac{3}{2}, j+\frac{1}{2}, h-1),\ (\Delta+\frac{3}{2}, j-\frac{1}{2}, h-1),$ $(\Delta+2, j, h)$	0
$(\Delta, 0, h)$	$(\Delta, 0, h),$ $(\Delta+\frac{1}{2}, \frac{1}{2}, h+1),\ (\Delta+\frac{1}{2}, \frac{1}{2}, h-1),$ $(\Delta+1, 0, h+2),\ (\Delta+1, 0, h-2),\ (\Delta+1, 1, h),$ $2(\Delta+1, 0, h),\ (\Delta+\frac{3}{2}, \frac{1}{2}, h+1),$ $(\Delta+\frac{3}{2}, \frac{1}{2}, h-1),\ (\Delta+2, 0, h)$	0

Table 2: Conformal primary content of long representations. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

To start with, we present a group theoretic listing of the state content of an antisymmetrized product of supersymmetries. This is given in Table (1).

The conformal primary content of a long representation of the superconformal algebra is given by the Clebsh Gordon product of the state content of the product of susy generators above with that of the primary. We list the conformal primary content of an arbitrary long representation in Table (2).

Note that a long representation of the susy algebra decomposes into 16 long representations of the conformal algebra when $j \neq 0$; when $j = 0$ we must delete the representations with negative values for the $SO(3)$ highest weight ($j - \frac{1}{2}$ and $j - 1$) from the generic j result leaving us with a total of 11 conformal representations. The Witten index of all long representations automatically vanishes.

Let us now turn to the short representations. To start with consider representations with $h \neq 0$, $j \neq 0$ and $\Delta = |h| + j + 1$. These representations are short because they include

Primary	Conformal content	Indices $(h > 0; h < 0)$
$(j+h+1, j, h)$,	$(j+h+1, j, h)$,	$\mathcal{I}_- = 0$,
$(j \neq 0,\ h \neq 0)$	$(j+h+\frac{3}{2}, j+\frac{1}{2}, h+1),\ (j+h+\frac{3}{2}, j+\frac{1}{2}, h-1),$	$\mathcal{I}_+ = (-1)^{2j+1}\frac{x^{2j+h+2}}{1-x^2}$;
	$(j+h+\frac{3}{2}, j+\frac{1}{2}, h-1),\ (j+h+2, j, h-2),$	$\mathcal{I}_+ = 0,$
	$(j+h+2, j+1, h),\ (j+h+2, j, h),$	$\mathcal{I}_- = (-1)^{2j+1}\frac{x^{2j-h+2}}{1-x^2}$
	$(j+h+\frac{5}{2}, j+\frac{1}{2}, h-1)$	

Table 3: Conformal primary content and index of generic short representation. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

a family of null states. These null states themselves transform in a short representation of the superconformal algebra, with quantum numbers $(|h| + j + \frac{3}{2}, j - \frac{1}{2}, h + 1)$ (when h is positive) and $(|h| + j + \frac{3}{2}, j - \frac{1}{2}, h - 1)$ (when h is negative). It is not too difficult to verify that the conformal primary content of such a short representation (represented by $\chi_S(j, h)$) and the Witten index of these representations is as given in Table (3) (we list the result for positive h; the result for negative h follows from symmetry).[7] Note that all 8 conformal primaries that occur in this decomposition are long (recall we have assumed $h \neq 0$).

It is not difficult to verify that $\chi_L(h + j + 1, j, h) = \chi_S(j, h) + \chi_S(j - \frac{1}{2}, h + 1)$. This expresses the fact that the state content of a long representation just above unitarity is equal to the sum of the state content of the short representation it descends to plus the state content of the short representation of null states.

Let us now turn to the special case of short representation $(j + 1, j, 0)$. The null states of this representation consist of a sum of two irreducible representations with quantum numbers $(j + \frac{3}{2}, j - \frac{1}{2}, 1)$ and $(j - \frac{3}{2}, j - \frac{1}{2}, -1)$. It is not too difficult to convince oneself that the primary content of such a short representation is as given in Table (4). Note that all 4 conformal representations that appear in this split are short.[8] It may be verified[9] that $\chi_L(j + 1, j, 0) = \chi_S(j + 1, j, 0) + \chi_S(j + \frac{3}{2}, j - \frac{1}{2}, 1) + \chi_S(j + \frac{3}{2}, j - \frac{1}{2}, -1)$.

Let us next turn to the special case of a short representation with $j = 0$ and with quantum numbers $(h + 1, 0, h)$. Such representations are often referred to as semi short,

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[7]The Witten index of these representations may be evaluated as follows. When $h > 0$ there are no states with $\Delta = j - h$ and so $\mathcal{I}_- = 0$. States with $\Delta = j + h$ occur only in the representation $(j + h + \frac{3}{2}, j + \frac{1}{2}, h + 1)$ and we find $\mathcal{I}_+ = (-1)^{2j+1}\frac{x^{2j+h+2}}{1-x^2}$ where we have used the spin statistics theorem to assert that the fermion number of a primary of angular momentum j is $(-1)^{2j}$. Similarly when $h < 0$ we have $\mathcal{I}_+ = 0$ and $\mathcal{I}_- = (-1)^{2j+1}\frac{x^{2j-h+2}}{1-x^2}$.

[8]The Witten index of these representations may be evaluated as follows. States with $\Delta = j + h$ occur only in the representation $(j + \frac{3}{2}, j + \frac{1}{2}, 1)$ while states with $\Delta = j - h$ occur only in the representation $(j + \frac{3}{2}, j + \frac{1}{2}, -1)$. The Witten indices of this representation are given by $\mathcal{I}_+ = \mathcal{I}_- = (-1)^{2j+1}\frac{x^{2j+2}}{1-x^2}$.

[9]In order to perform this verification, it is important to recall that $\chi_L(j + 1, j, 0)$ is the sum of 16 *long* characters of the conformal group, 4 of which are at the unitarity threshold. Equivalently we may write this as the sum of $12 + 4 = 16$ long conformal characters and 4 short conformal characters (where we have used the fact that a long conformal character, at its unitarity bound, decomposes into the sum of a short and a long character). The 4 short characters in this decomposition simply yield $\chi_S(j + 1, j, 0)$ above, while the 16 long characters constitute $\chi_S(j + \frac{3}{2}, j - \frac{1}{2}, 1) + \chi_S(j + \frac{3}{2}, j - \frac{1}{2}, 1)$.

Primary	Conformal content	Index
$(j+1, j, 0)$	$(j+1, j, 0),$ $(j+\frac{3}{2}, j+\frac{1}{2}, 1),\ (j+\frac{3}{2}, j+\frac{1}{2}, -1),$ $(j+2, j+1, 0)$	$\mathcal{I}_{+} = \mathcal{I}_{-} = (-1)^{2j+1}\frac{x^{2j+2}}{1-x^2}$

Table 4: Conformal primary content and index for $(j+1, j, 0)$ representation. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

Primary	Conformal content	Index($h > 0$; $h < 0$)
$(h+1, 0, h)$	$(h+1, 0, h),$ $(h+\frac{3}{2}, \frac{1}{2}, h+1),\ (h+\frac{3}{2}, \frac{1}{2}, h-1),$ $(h+2, 0, h-2),\ (h+2, 1, h),\ (h+2, 0, h),$ $(h+\frac{5}{2}, \frac{1}{2}, h-1)$	$\mathcal{I}_{-} = 0,\ \mathcal{I}_{+} = -\frac{x^{h+2}}{1-x^2}$; $\mathcal{I}_{+} = 0,\ \mathcal{I}_{-} = -\frac{x^{-h+2}}{1-x^2}$

Table 5: Conformal content and index of representation $(h+1, 0, h)$. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

to distinguish them from the 'short' $j = 0$ representations we will deal with next. We will deal with the case $h > 0$ (the results for $h < 0$ can then be deduced from symmetry). The primary for the null states of this representation has quantum numbers $(h+2, 0, h+2)$. Note that the null states transform in an isolated short representation. The state content and Witten index of a semishort $j = 0$ representation are listed in Table (5). Note that $\chi_L(h+1, 0, h) = \chi_S(h+1, 0, h) + \chi_S(h+2, 0, h+2)$; this formula captures the split of a long representation into the short representation and null states.

The short representation with primary labels $(1, 0, 0)$ is a bit special; its null states have primaries with quantum numbers $(2, 0, 2)$ and $(2, 0, -2)$ (these are isolated short representations, see below). The state content and Witten index of this representation are given in Table (6). Of the 5 conformal primaries that appear in this split, only the representation with quantum numbers $(2, 1, 0)$ is short.[10] Using the results we present below, it is possible to verify the character decomposition rule:[11]

$$\chi_L(1, 0, 0) = \chi_S(1, 0, 0) + \chi_S(2, 0, 2) + \chi_S(2, 0, -2).$$

Now let us turn to the isolated short representations $(h, 0, h)$ for $|h| \geq 1$. The primaries for the null states of these representations have quantum numbers $(h+\frac{1}{2}, \frac{1}{2}, h+1)$. The null states transform in a (short) non-unitary representation, reflecting the fact that the isolated representations cannot be regarded as a limit of unitary long representations but

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[10] The conformal representation $(1, 0, 0)$ and $(\frac{3}{2}, \frac{1}{2}, \pm 1)$ are not short as spin 0 and spin $\frac{1}{2}$ are exceptions to the general rule.

[11] The character on the LHS is a sum of 10 long conformal representations or $11 = 5 + 3 + 3$ short conformal representations (the extra representation is the conformal shortening vector of $(2, 1, 0)$ and is given by $(3, 0, 0)$). States with $\Delta = j + h$ occur only in the representation $(\frac{3}{2}, \frac{1}{2}, 1)$ while states with $\Delta = j - h$ occur only in the representation $(\frac{3}{2}, \frac{1}{2}, -1)$.

Primary	Conformal content	Index
$(1,0,0)$	$(1,0,0)$ $(\frac{3}{2},\frac{1}{2},1),\ (\frac{3}{2},\frac{1}{2},-1)$ $(2,0,0),\ (2,1,0)$	$\mathcal{I}_+ = \mathcal{I}_- = -\frac{x^2}{1-x^2}$

Table 6: Conformal content and index for representation $(1,0,0)$. Representations are labeled by (Δ, j, h) where Δ is the scaling dimension, j the angular momentum (a positive half integer) and h the R-charge of the representation.

Primary	Conformal content	Index($h > 0$; $h < 0$)
$(h,0,h)$	$(h,0,h),$ $(h+\frac{1}{2},\frac{1}{2},h-1),$ $(h+1,0,h-2)$	$\mathcal{I}_- = 0,\ \mathcal{I}_+ = \frac{x^h}{1-x^2}$; $\mathcal{I}_+ = 0,\ \mathcal{I}_- = \frac{x^{-h}}{1-x^2}$

Table 7: Conformal content and index for representation $(h,0,h)$.

Primary	Conformal content	Index
$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2}),$ $(1,\frac{1}{2},-\frac{1}{2})$	$\mathcal{I}_- = 0,\ \mathcal{I}_+ = \frac{x^{\frac{1}{2}}}{1-x^2}$
$(\frac{1}{2},0,-\frac{1}{2})$	$(\frac{1}{2},0,-\frac{1}{2})$ $(1,\frac{1}{2},\frac{1}{2})$	$\mathcal{I}_+ = 0,\ \mathcal{I}_- = \frac{x^{\frac{1}{2}}}{1-x^2}$

Table 8: Conformal content and index for representations $(\frac{1}{2},0,\frac{1}{2})$ and $(\frac{1}{2},0,-\frac{1}{2})$

can be regarded as the limit of non-unitary long reps. The conformal primary content and Witten indices for these representations are given in Table (7). Here we have written conformal content for h positive; the result for negative h is given by symmetry. Recall that $h \geq \frac{1}{2}$ for the representations we have just discussed. The lower bound of this inequality, $h = \frac{1}{2}$, is a special case. The conformal decomposition and index for the $h = \frac{1}{2}$ and $-\frac{1}{2}$ cases are given in Table (8).

3.3 Decomposition of all unitary representations of the $\mathcal{N} = 3$ algebra into $\mathcal{N} = 2$ representations

In this subsection we record the decomposition of all $\mathcal{N} = 3$ representations into $\mathcal{N} = 2$ representations. Representations of the $\mathcal{N} = 3$ algebra are labeled as (Δ, j, h), where h is the highest weight under the Cartan of the $SO(3)$ R symmetry (normalized to be a half integer). A generic $\mathcal{N} = 3$ long representation with $j \neq 0$ breaks as follows

$$(\Delta, j, h)_3 = \bigoplus_{m=-h}^{h} \left[(\Delta, j, m)_2 \oplus (\Delta+\frac{1}{2}, j+\frac{1}{2}, m)_2 \oplus (\Delta+\frac{1}{2}, j-\frac{1}{2}, m)_2 \oplus (\Delta+1, j, m)_2 \right]. \quad (3.1)$$

where $\oplus$ denotes a direct sum and $\bigoplus_{m=-h}^{h}$ denotes the direct sum of representations.

Here the subscript denotes $\mathcal{N}$ of the algebra; i.e. $()_3$ denotes a representation of the $\mathcal{N} = 3$ algebra, while $()_2$ denotes a representation of the $\mathcal{N} = 2$ algebra.

The summation outside the brackets on the RHS of (3.1) reflects the fact that a primary that transforms in a given irreducible $SO(3)$ (R symmetry in $\mathcal{N} = 3$ algebra) representation consists of several different $SO(2)$ primaries (with distinct R-charges). The four terms in the bracket on the RHS of (3.1) represent the states obtained by acting on the $\mathcal{N} = 3$ primary with supercharges that belong to the $\mathcal{N} = 3$ algebra, but are absent in the $\mathcal{N} = 2$ algebra.

The decomposition (3.1) may be rewritten as follows:

$$
(j + h + 1 + \epsilon, j, h)_3 = \bigoplus_{m=-h}^{h} \left[(j + h + 1 + \epsilon, j, m)_2 \oplus (j + h + \frac{3}{2} + \epsilon, j + \frac{1}{2}, m)_2 \right]
$$

$$
\oplus \bigoplus_{m=-(h+1)}^{h+1} \left[(j + h + \frac{3}{2} + \epsilon, j - \frac{1}{2}, m)_2 \oplus (j + h + 2 + \epsilon, j, m)_2 \right]. \qquad (3.2)
$$

In this equation we have grouped together terms on the RHS for the following reason. Recall that the decomposition of a long representation - with $j \neq 0$ - at the unitarity bound, into short unitary representations of the superconformal algebra, is given both at $\mathcal{N} = 3$ and at $\mathcal{N} = 2$ by

$$
(j + h + 1 + \epsilon, j, h) \xrightarrow{\epsilon \to 0} (j + h + 1, j, h) \oplus (j + h + \frac{3}{2}, j - \frac{1}{2}, h + 1) \qquad (3.3)
$$

This formula should apply to (3.2). Comparing (3.3) and (3.2), it is plausible (and correct) that the generic short $\mathcal{N} = 3$ representation decomposes into $\mathcal{N} = 2$ representations according to the formula

$$
(j + h + 1, j, h)_3 = \bigoplus_{m=-h}^{h} \left[(j + h + 1, j, m)_2 \oplus (j + h + \frac{3}{2}, j + \frac{1}{2}, m)_2 \right] \qquad (3.4)
$$

where all representations that saturate the unitarity bound, on the RHS of (3.4), are short.

We may deduce the split of a generic $j = 0$ short $\mathcal{N} = 3$ representation into representations of the $\mathcal{N} = 2$ algebra using identical reasoning. To start with we note that $\mathcal{N} = 3$ long representation at $j = 0$ splits up into long $\mathcal{N} = 2$ representations as follows

$$
(\Delta, 0, h)_3 = \bigoplus_{m=-h}^{h} \left[(\Delta, 0, m)_2 \oplus (\Delta + \frac{1}{2}, \frac{1}{2}, m)_2 \oplus (\Delta + 1, 0, m)_2 \right] \qquad (3.5)
$$

We next note that, both in the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$ algebras,

$$
(h + 1 + \epsilon, 0, h) \xrightarrow{\epsilon \to 0} (h + 1, 0, h) \oplus (h + 2, 0, h + 2)
$$
$$
(h + \frac{3}{2} + \epsilon, \frac{1}{2}, h) \xrightarrow{\epsilon \to 0} (h + \frac{3}{2} + \epsilon, \frac{1}{2}, h) \oplus (h + 2, 0, h + 1) \qquad (3.6)
$$

These two equations allow us to deduce that, for short representations,

$$
(h + 1, 0, h)_3 = \bigoplus_{m=-h}^{h} \left[(h + 1, 0, m)_2 \oplus (h + \frac{3}{2}, \frac{1}{2}, m)_2 \right] \oplus \bigoplus_{m=-(h+2)}^{h+2} (h + 2, 0, m)_2 \qquad (3.7)
$$

Spin j	$\mathcal{N}=3$ primary	$\mathcal{N}=2$ primaries
$j \neq 0$	(Δ, j, h)	$\bigoplus_{m=-h}^{h}\Big[(\Delta, j, m) \oplus (\Delta+\tfrac{1}{2}, j+\tfrac{1}{2}, m)\oplus$ $(\Delta+\tfrac{1}{2}, j-\tfrac{1}{2}, m) \oplus (\Delta+1, j, m)\Big]$
$j = 0$	$(\Delta, 0, h)$	$\bigoplus_{m=-h}^{h}\Big[(\Delta, 0, m) \oplus (\Delta+\tfrac{1}{2}, \tfrac{1}{2}, m)\oplus$ $(\Delta+1, 0, m)\Big]$

Table 9: Decomposition of long $\mathcal{N}=3$ representations into $\mathcal{N}=2$ representations.

Spin j	$\mathcal{N}=3$ primary	$\mathcal{N}=2$ primaries
$j \neq 0$	$(j+h+1, j, h)$	$\bigoplus_{m=-h}^{h}\Big[(j+h+1, j, m)\oplus$ $(j+h+\tfrac{3}{2}, j+\tfrac{1}{2}, m)\Big]$
Generic short $j=0$	$(h+1, 0, h)$	$\bigoplus_{m=-h}^{h}\Big[(h+1, 0, m) \oplus (h+\tfrac{3}{2}, \tfrac{1}{2}, m)\Big]$ $\oplus \bigoplus_{m=-(h+2)}^{h+2}(h+2, 0, m)$
Isolated short $j=0$	$(h, 0, h)$	$\bigoplus_{m=-h}^{h}(h, 0, m)$

Table 10: Decomposition of short $\mathcal{N}=3$ representations into $\mathcal{N}=2$ representations.

In the equation above, representations that saturate the BPS bound are short.

The breakup of the $\mathcal{N}=3$ isolated short representations needs slightly more indirect reasoning to deduce; we simply present the final result:

$$(h, 0, h)_3 = \bigoplus_{m=-h}^{h} (h, 0, m)_2 \tag{3.8}$$

The complete decomposition of the $\mathcal{N}=3$ algebra into $\mathcal{N}=2$ representations is given in Table (9) for long representations and Table (10) for short representations.

4. The R-charge as a function of λ in the absence of a superpotential

It was explained in [5] (see the introduction) that the $\mathcal{N}=2$ $U(N)$ Chern Simons theory with g chiral multiplets and no superpotential, is superconformally invariant at every value of N and k, and so at every value of λ, in the large N limit. In the free limit the R-charge of each of the chiral multiplets in this theory is equal to half. As was explained in [5], however, this R-charge is renormalized as a function of λ. As the R-charge of an operator appears in the BPS formula that determines its scaling dimension, the determination of the charge of a chiral field, as a function of λ, is perhaps the most elementary characteristic

of the supersymmetric spectrum of the theory. In this section we will adopt a proposal by Jafferis [19] to perform this determination.

4.1 The large N saddle point equations

According to the prescription of [19], the superconformal R-charge of the theories we study is determined by extremizing the magnitude of their partition function on S^3 with respect to the trial R-charge,[12] h, assigned to a chiral multiplet. The partition function itself is determined by the method of supersymmetric localization to be given by the finite dimensional integral

$$\mathcal{Z}(h) = \int \prod_{i=1}^{N} du_i \, \exp\left\{ N^2 \left[\frac{i\pi}{\lambda} \frac{1}{N} \sum_i u_i^2 + \frac{1}{N^2} \sum_{i \neq j} \log \sinh (\pi u_{ij}) + \frac{g}{N^2} \sum_{i,j} \ell(1 - h + iu_{ij}) \right] \right\}$$
(4.1)

where $\lambda = N/k$ is the 't Hooft coupling, u_i $(i = 1 \ldots N)$ are real numbers (and the integration range is from $-\infty$ to ∞), $u_{ij} \equiv u_i - u_j$, and the function $\ell(z)$ is given by

$$\ell(z) = -z \log \left(1 - e^{2\pi i z} \right) + \frac{i}{2} \left[\pi z^2 + \frac{1}{\pi} \mathrm{Li}_2(e^{2\pi i z}) \right] - \frac{i\pi}{12}$$
(4.2)

where Li_2 is the dilogarithm function. While the function $\ell(z)$ is complicated looking, its derivative is elementary

$$\partial_z \ell(z) = -\pi z \cot(\pi z)$$

and is all we will need in this paper.

According to [19], once the partition function $\mathcal{Z}$ is obtained by performing the integral in (4.1), the R-charge of the chiral fields is determined (up to caveats we will revisit below) by solving the equation $\partial_h |\mathcal{Z}(h)|^2 = 0$. This gives the exact superconformal R-charge.

In the large N limit the integral in (4.1) may be determined by saddle point techniques. The saddle point equations, together with the equation $\partial_h |\mathcal{Z}(h)|^2 = 0$ (which determines h, given the saddle point) are given by

$$0 = \frac{iu_k}{\lambda} + \frac{1}{N} \sum_{j(\neq k)}^{N} \left\{ \coth (\pi u_{kj}) - \frac{i}{2} g \left[(1 - h + iu_{kj}) \cot \pi (1 - h + iu_{kj}) \right. \right. \qquad (4.3)$$

$$\left. \left. -(1 - h - iu_{kj}) \cot \pi (1 - h - iu_{kj}) \right] \right\},$$

$$0 = \mathrm{Re} \left[\sum_{i,j=1}^{N} (1 - h + iu_{ij}) \cot \pi (1 - h + iu_{ij}) \right]. \qquad (4.4)$$

---

[12]More precisely, as shown in [19], a supersymmetric theory on S^3 can be defined with an arbitrary choice of R-charge h, and the partition function of this theory on S^3 is $\mathcal{Z}(h)$. The superconformal R-charge is such that $|\mathcal{Z}(h)|^2$ is minimized.

4.2 Perturbative solution at small λ

While we have been unable to solve the equations (4.3) in general even at large N, it is not difficult to solve these equations either at small λ (at all g) or at large g (for all λ). In this subsection we describe the perturbative solution to these equations at small λ (for all g). In the next subsection we will outline the perturbative procedure that determines $h(\lambda)$ at all λ but large g.

It is apparent from a cursory inspection of (4.3) that the eigenvalues u_i must become small in magnitude (in fact must scale like $\sqrt{\lambda}$) at small λ. It follows that complicated functions of u simplify to their Taylor series expansion in a power expansion in λ. This is the basis of the perturbative technique described in this subsection.

More quantitatively, at small λ we expand u_i and h as

$$u_k = \sqrt{\lambda}\left(u_k^{(0)} + \lambda u_k^{(1)} + \cdots\right), \tag{4.5}$$

$$h = h^{(0)} + \lambda h^{(1)} + \lambda^2 h^{(2)} + \cdots , \tag{4.6}$$

and attempt to solve our equations order by order in λ. At leading nontrivial order, $\mathcal{O}(\lambda^0)$, equation (4.4) reduces to

$$(1 - h^{(0)})\cot\pi(1 - h^{(0)}) = 0 \Rightarrow h^{(0)} = \frac{1}{2} \tag{4.7}$$

which tells us that $h = \frac{1}{2}$ at leading order in λ. On the other hand, equation (4.3) at its leading nontrivial order, namely $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$, reduces to

$$iu_i^{(0)} = -\frac{1}{\pi}\frac{1}{N}\sum_{j(\neq i)}\frac{1}{u_i^{(0)} - u_j^{(0)}}. \tag{4.8}$$

Apart from an unusual factor of i, this is precisely the large N saddle point equations of the Wigner model. The extra factor of i may be dealt with by working with the rescaled variable

$$y_j = e^{-\frac{\pi i}{4}}u_j$$

in terms of which

$$y_i^{(0)} = \frac{1}{\pi}\frac{1}{N}\sum_{j(\neq i)}\frac{1}{y_i^{(0)} - y_j^{(0)}}. \tag{4.9}$$

The solution to this equation is well known in the large N limit. The eigenvalues $y_i^{(0)}$ cluster themselves into a "cut" along the interval $(-a, a)$ with

$$a = \sqrt{\frac{2}{\pi}}\;.$$

The density of eigenvalues, $\rho(y) = \sum_{i=1}^{N}\delta(y - y_i^i)$, in this interval is given by

$$\rho(y) = \frac{2}{\pi a^2}\sqrt{a^2 - y^2}. \tag{4.10}$$

Using the fact that $u \approx e^{\frac{\pi i}{4}} \sqrt{\lambda} y$, we see that, at leading order in λ, the saddle point is given by the eigenvalues u_i clustering along a straight line of length of order $\sqrt{\lambda}$, oriented at 45 degrees in the complex plane.

Note that the distribution of eigenvalues has $u \to -u$ symmetry and in particular the average value of eigenvalues is zero. The $u \to -u$ symmetry is an exact symmetry of the saddle point equations, and we will assume that it is preserved in the solution (i.e. not spontaneously broken) in the rest of this paper.

Let us now proceed beyond the leading order. (4.4) is automatically satisfied at $\mathcal{O}(\sqrt{\lambda})$ (this is because $\mathrm{Im}\left[\sum_{i \neq j}(u_i^{(0)} - u_j^{(0)})\right] = 0$). At order λ the same equation reduces to

$$h^{(1)} = -2\,\mathrm{Re}\left[\frac{1}{N^2}\sum_{i \neq j}(u_i^{(0)} - u_i^{(0)})^2\right] \tag{4.11}$$

Now recall that the phase of u_i is $e^{\frac{\pi i}{4}}$. As a consequence the real part vanishes and hence, from (4.11), $h^{(1)} = 0$.

In order to compute the correction to $h(\lambda)$ at $\mathcal{O}(\lambda^2)$ we need to find the first correction u_i^1 to the eigenvalue distribution. We now turn to that task. At order $\mathcal{O}(\lambda^0)$, (4.3) reduces to

$$iu_k^{(1)} - \frac{1}{N}\sum_{j(\neq k)}\left\{\frac{\pi}{6}(3g-2)(u_k^{(0)} - u_j^{(0)}) + \frac{u_k^{(1)} - u_j^{(1)}}{\pi(u_k^{(0)} - u_j^{(0)})^2}\right\} = 0 \tag{4.12}$$

Now, part of the second term on the RHS of (4.3) is easily simplified. It follows immediately from (4.8) (differentiating that equation with respect to $u_i^{(0)}$) that

$$\frac{1}{\pi N}\sum_{j(\neq k)}\frac{1}{(u_k^{(0)} - u_j^{(0)})^2} = i.$$

Inserting this relation into part of the RHS of (4.12) gives us a piece that cancels the LHS, and (4.12) simplifies to

$$\frac{1}{\pi N}\sum_{j(\neq k)}\frac{1}{(u_k^{(0)} - u_j^{(0)})^2}u_j^{(1)} = \frac{\pi}{6N}(3g-2)\sum_{j \neq k}(u_k^{(0)} - u_j^{(0)}). \tag{4.13}$$

The RHS of this equation may be further simplified using $\sum_j u_j = 0$. Retaining only terms that contribute at leading order in N, we find

$$\frac{1}{\pi N}\sum_{j(\neq k)}\frac{1}{(u_k^{(0)} - u_j^{(0)})^2}u_j^{(1)} = \frac{\pi}{6}(3g-2)u_k^{(0)}. \tag{4.14}$$

In order to solve this equation we once again move to the "real" variable y. That is we define $u_j^{(0)} = e^{\frac{\pi i}{4}}y_j$ as above. Let us also define $u_j^{(1)} = e^{\frac{\pi i}{4}}v_j(y_j)$. In the large N limit $u_i^{(0)}$ is effectively a continuous variable on the 45 degree cut on the complex plane, and

$$u^{(1)} = e^{\frac{\pi i}{4}}v(y)$$

for a continuous function $v(y)$ that we now determine. The equation for $v(y)$ is given by

$$\frac{1}{\pi}\mathcal{P}\int \frac{v(y)\rho(y)}{(y_1-y)^2}dy = \frac{i\pi}{6}(3g-2)y_1.\tag{4.15}$$

Integrating both sides of this equation with respect to y_1 we find

$$\mathcal{P}\int \frac{v(y)\rho(y)}{(y_1-y)}dy = -\frac{i\pi^2}{12}(3g-2)y_1^2 + k_1,\tag{4.16}$$

where k_1 is the constant of integration.

In order to proceed we must solve the integral equation (4.16). We will now explain how, in more generality, it is possible to solve the equation

$$\mathcal{P}\int \frac{z(y)}{(y_1-y)}dy = g_n(y_1)\tag{4.17}$$

where $g_n(y)$ is a complex polynomial in y, and $z(y)$ is a function defined over the range $(-a,a)$. The solution may be found by constructing an analytic function $P(y)$ whose only singularities on the real axis are a cut in the range $(-a,a)$, and whose real part on this interval is given by $g_n(y)$. An obvious ansatz for such a function is

$$P(y) = g_n(y) - h_n(y)\sqrt{y^2 - \frac{2}{\pi}}\tag{4.18}$$

where $h_n(y)$ is a yet to be determined polynomial in y. Let us now choose $h_n(y)$ to ensure that the leading behaviour of $P(y)$ at infinity is $P(y) \sim \frac{1}{y}$. If we manage to achieve this then it follows from an application of Cauchy's theorem that, for any complex valued w,

$$P(w) = -\frac{1}{2\pi i}\int_{-a}^{a} \frac{\mathrm{disc}\, P(x)}{x-w}dx$$

where disc$P(x)$ is the discontinuity of $P(x)$ across the branch cut which is on the real axis. We can now apply this equation to $w = y + i\epsilon$ and also to $w = y - i\epsilon$ (where $y \in (a,-a)$), take the average of the two, and take the limit $\epsilon \to 0$. This gives the equation

$$g_n(y) = -\frac{1}{2\pi i}\mathcal{P}\int_{-a}^{a} \frac{\mathrm{disc}\, P(x)}{x-w}dx$$

where we have used the fact that the real part of $P(y)$ is $g_n(y)$ in the range $y \in (-a,a)$. Comparing with (4.17) we conclude that

$$z(y) = \frac{1}{2\pi i}\mathrm{disc}\, P(y) = \frac{1}{\pi}h_n(y)\sqrt{\frac{2}{\pi} - y^2}.$$

If, as is the case in our situation, that

$$z(y) = v_n(y)\rho(y)$$

then the solution above for $z(y)$ implies that

$$v_n(y) = \frac{1}{\pi}h_n(y)\tag{4.19}$$

Applying this general formalism to the equation (4.16) is straightforward. We have

$$g_1(y) = -\frac{i\pi^2}{12}(3g-2)y^2 + k_1 \tag{4.20}$$

where k_1 is an as yet arbitrary constant. In order to be able to find a suitable function $P(y)$ we need to choose $k_1 = \frac{i\pi}{12}(3g-2)$ and

$$h_1(y) = -\frac{i\pi^2 y}{12}(3g-2).$$

These choices lead to the analytic function

$$P(y) = -\frac{i\pi^2}{12}(3g-2)\left(y^2 - \frac{1}{\pi} - y\sqrt{y^2 - \frac{2}{\pi}}\right) \tag{4.21}$$

and yield[13]

$$v(y) = -\frac{i\pi}{12}(3g-2)y. \tag{4.22}$$

With the first nontrivial correction to the eigenvalue distribution in hand, it is now a simple matter to compute the shift in the scaling dimension $h(\lambda)$ at leading order nontrivial order in λ. (4.4) is automatically obeyed at $\mathcal{O}(\lambda^{\frac{3}{2}})$.[14] However at $\mathcal{O}(\lambda^2)$, the same equation yields

$$\text{Re}\left[\sum_{k\neq j} -2\pi^3(u_i^{(0)} - u_j^{(0)})^4 + 3\pi\left(4(u_i^{(0)} - u_j^{(0)})(u_i^{(1)} - u_j^{(1)}) + h^{(2)}\right)\right] = 0 \tag{4.24}$$

In other words

$$\begin{aligned}
3h^{(2)} &= \text{Re}\left[\int dy_1 dy_2 \rho(y_1)\rho(y_2)\left\{-2\pi^2(y_1 - y_2)^4 - 12i(y_1 - y_2)(v(y_1) - v(y_2))\right\}\right] \\
&= \text{Re}\left[-2\pi^2\left(2\langle y^4\rangle + 6\langle y^2\rangle^2\right) - 24i\langle y\ v(y)\rangle\right]
\end{aligned} \tag{4.25}$$

where we have defined

$$\langle O(y)\rangle \equiv \int O(y)\rho(y)dy.$$

Evaluating the integrals we find

$$h^{(2)} = -(1+g) \tag{4.26}$$

This exactly matches the explicit perturbative result of Gaiotto and Yin [5]. Similar agreement was also found with [26].

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[13]This correction to the eigenvalue distribution tilts the eigenvalue cut - originally at 45 degrees in the complex plane - a little nearer to the real axis.

[14]The identity in question is

$$\text{Im}\sum_{i\neq j}\left[\pi^3(u_i^{(0)} - u_j^{(0)})^3 - 3\pi(u_i^{(1)} - u_j^{(1)})\right] = 0 \tag{4.23}$$

The method presented here is easily iterated to higher orders in λ. It turns out that at each order the correction to the eigenvalue distribution is determined by the solution to an equation of the form (4.17). This solution may be obtained, order by order, using the method described above. The new correction to the eigenvalue distribution yields a new term in the correction to the anomalous dimension.

We have explicitly implemented this perturbative procedure to a few orders in perturbation theory. At $g = 1$ we find

$$h = \frac{1}{2} - 2\lambda^2 + \frac{13\pi^2}{3}\lambda^4 - \left(\frac{207\pi^4}{10} - 32\pi^2\right)\lambda^6 + \left(\frac{339019\pi^6}{2520} - \frac{3355\pi^4}{9} + \frac{160\pi^2}{3}\right)\lambda^8 + \cdots$$
(4.27)

while for general g we have

$$\begin{aligned}
h = {} & \frac{1}{2} - (1+g)\lambda^2 + \frac{1}{12}(1+g)\left[-24(-1+g) + \pi^2(3g^2 + 15g + 8)\right]\lambda^4 \\
& + \left[-8 - 25\frac{\pi^2}{3} - \frac{61\pi^4}{60} + g(-\frac{4\pi^2}{3} - \frac{637\pi^4}{120}) + g^2(8 + \frac{64\pi^2}{3} - \frac{395\pi^4}{48})\right. \\
& \left. + g^3(\frac{52\pi^2}{3} - \frac{239\pi^4}{48}) + g^4(3\pi^2 - \frac{53\pi^4}{48}) - g^5\frac{\pi^4}{16}\right]\lambda^6 + \cdots
\end{aligned}$$

4.3 Perturbative solution at large g

As we have described in the previous subsection, in the limit of small 't Hooft coupling the solution to the saddle point equation, (4.3), is obtained by balancing the first term (large because of the inverse power of λ) against the second (large because of the singularity at small u), and treating the third term as a perturbation.

Let us now consider another limit; one in which the number of flavours g becomes large, without making any assumptions on λ. In this case the small u singularity of the second term in (4.3) balances against either the largeness of g (in the third term) or the smallness of λ (in the first term). The important point is that u is necessarily small for this balance to work, so perturbative techniques apply. The most interesting regime is one in which $\lambda = \mathcal{O}(1/g)$. In order to focus in on this regime we formally scale $\lambda \to \lambda x^2$, $g \to \frac{g}{x^2}$ and work in a power series expansion in x. We make the expansion

$$u_i = x\left(u_i^{(0)} + x^2 u_i^{(1)} + \ldots\right)$$

$$h = \frac{1}{2} + x^2 h^{(1)} + x^4 h^{(2)} + \ldots$$

and solve the equations order by order in the formal expansion parameter x. The perturbative procedure proceeds exactly as in the previous subsection. At lowest order we obtain the equation

$$\frac{1}{N}\sum_{j \neq k} \frac{1}{(u_k^{(0)} - u_j^{(0)})} = u_k^{(0)}\left(\frac{-i\pi}{\lambda} + \frac{g\pi^2}{2}\right)$$
(4.28)

so that, once again, the eigenvalue distribution is given by a Wigner type cut at leading nontrivial order. More specifically, the eigenvalue distribution for this is given by

$$\rho(u) = \frac{2}{\pi a^2}\sqrt{a^2 - u^2}; \quad a^2 = \frac{2}{\frac{-i\pi}{\lambda} + \frac{g\pi^2}{2}}$$
(4.29)

The lowest order correction to R-charge can be calculated as before by expanding the $\mathcal{Z}$ extremization equation (4.4) to lowest order in x. This gives

$$h^{(1)} = -4\,\mathrm{Re}\langle u^2\rangle \tag{4.30}$$

where $\langle\ \rangle$ denotes expectation value as earlier. Using the zeroth order eigenvalue distribution this gives

$$h^{(1)} = -\mathrm{Re}\langle a^2\rangle = -\frac{g\pi^2}{\left(\frac{g\pi^2}{2}\right)^2 + \left(\frac{\pi}{\lambda}\right)^2} \tag{4.31}$$

In summary, we have, to leading order

$$h(\lambda) = \frac{1}{2} - \frac{g\pi^2}{\left(\frac{g\pi^2}{2}\right)^2 + \left(\frac{\pi}{\lambda}\right)^2} + \text{ higher order.} \tag{4.32}$$

4.4 Numerical study of R-charge for small g and large λ

In the previous subsections we used perturbative techniques to establish the following. The function $h(\lambda)$ starts out at the value $\frac{1}{2}$ at $\lambda = 0$. It always decreases at small λ; at leading order $h(\lambda) = \frac{1}{2} - (g+1)\lambda^2$. What happens at larger values of λ? This question turned out to be easy to answer at large g. In this limit the decrease of $h(\lambda)$ as a function of λ stops at $\lambda \sim \frac{1}{g}$, after which $h(\lambda)$ settles down at its asymptotic value

$$h(\lambda) = \frac{1}{2} - \frac{4}{\pi^2 g} + \mathcal{O}(\frac{1}{g^2}). \tag{4.33}$$

Neither of the analyses we have performed, however, reliably predict the behaviour of $h(\lambda)$ at large λ when g is of order unity. An unjustified extrapolation of (4.33) suggests that $h(\lambda)$ always monotonically decreases, asymptoting to a constant value at $\lambda = \infty$. In order to check whether all this is really the case we have numerically solved a discretized version of the saddle point equations with Ne eigenvalues using Mathematica. [15]
. We give details of our numerical procedure in Appendix A. In this section we merely present our results.

At $g = 3$ the function is as given by the graph in Fig. 1. Note that $h(\lambda)$ asymptotes to almost a constant value by $\lambda = 2$, and varies only slightly in the λ range 2 to 7 (this constant is approximately 0.354).

At $g = 2$ we find the function $h(\lambda)$ given by Fig. 2. Note that $h(\lambda)$ asymptotes to almost a constant value by $\lambda = 2$, and varies only slightly in the λ range 2 to 7 (the asymptote value is approximately 0.274).

As is apparent, in both these cases the function $h(\lambda)$ displays the qualitative behaviour predicted by the large g formula (4.32); $h(\lambda)$ monotonically decreases from $h = \frac{1}{2}$ at $\lambda = 0$ to a finite value of h (greater than $\frac{1}{4}$) at $\lambda = \infty$. Notice that at $g = 3$, $\frac{1}{2} - h(\infty) = \frac{1}{2} - .354 \approx 0.146$. This shift from $\frac{1}{2}$ agrees to 10 percent with the first order prediction of

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[15]In the rest of this paper Ne denotes the number of eigenvalues used for the purposes of discretized numerics. Ne is a composite symbol - (it is not equal to product of N with e).
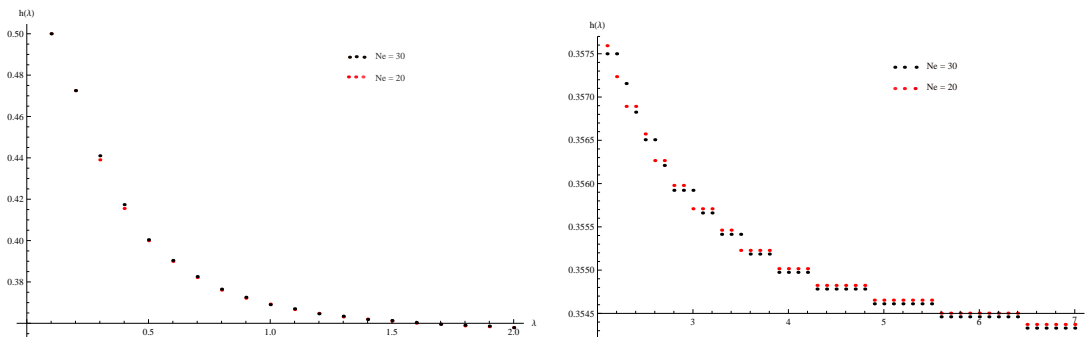
Figure 1: h vs. λ for $g = 3$, $Ne = 20$ and $Ne = 30$. In the figure on the left, λ varies from 0 to 2. In the figure on the right λ varies from 2 to 7. Note that $h(\lambda)$ scale is different in the two figures. R-charge saturates to around 0.354. While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.
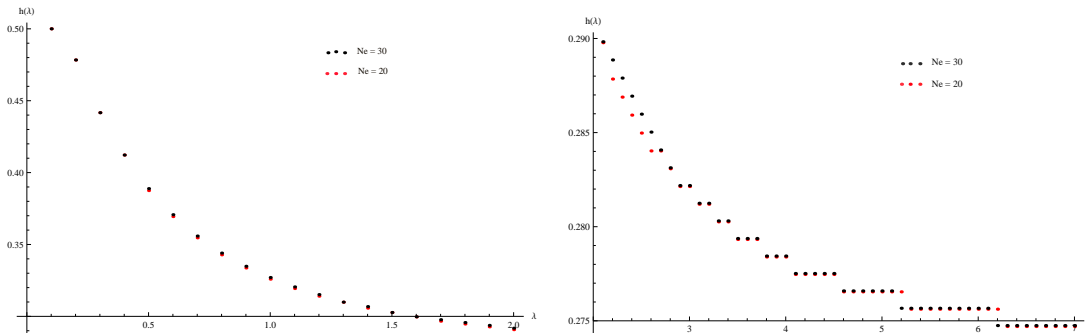

Figure 2: h vs. λ for $g = 2$, $Ne = 20$ and $Ne = 30$. In the figure on the left, λ varies from 0 to 2. In the figure on the right λ varies from 2 to 7. Note that $h(\lambda)$ scale is different in the two figures. R-charge saturates to around 0.274.While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.

large g perturbation theory (4.33), $\frac{4}{3\pi^2} = 0.135$. At $g = 2$, $1 - h(\infty) = \frac{1}{2} - .274 = 0.226$ which agrees to 12 percent with the first order prediction of large g perturbation theory (4.33), $\frac{4}{2\pi^2} = 0.203$. It thus appears that, even quantitatively, the results of large g perturbation theory are not too far off the mark from the correct answer all the way down to $g = 2$. In order to see this more clearly, in Fig 3 we replot our results for $h(\lambda)$ versus λ at $g = 3$ and compare with the predictions of large g perturbation theory at first order in the perturbative expansion. Note the semi quantitative agreement between the curves.

On the other hand our numerical result for the function $h(\lambda)$ at $g = 1$ is presented in Fig. 4 for $\lambda \in (0, 10)$ [16] As we explain in more detail in Appendix A, at every λ we have solved the saddle point equations at $Ne = 20, 30, \ldots 100$ and bestfitted our results to the

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[16]It would of course be possible to obtain data at larger values of λ as well. However this process becomes computationally increasingly expensive, as $\frac{1}{Ne}$ errors appear to increase upon increasing λ. In order to generate reliable data at larger and larger λ requires solving the equations at larger and larger Ne.

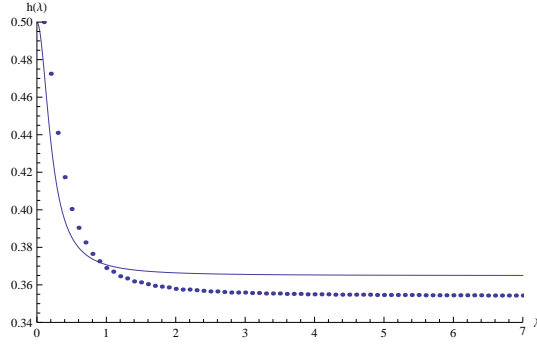**Figure 3:** h vs. λ for $g = 3$ and $Ne = 30$. The blue line is large g perturbation theory prediction. R-charge saturates to around 0.354. While we have not performed a serious error estimate, it seems unlikely to us that the error in this asymptote value exceeds ± 0.01.
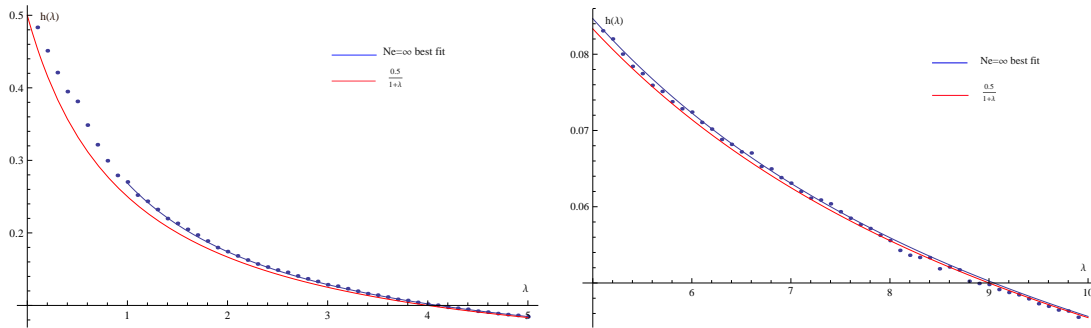

Figure 4: h vs. λ for $g = 1$. Blue line is best fit of the data points to the form $\frac{\alpha}{\beta + \lambda}$. The red curve is $\frac{1}{2(1+\lambda)}$.

form

$$h(\lambda) = h(\lambda) + \frac{b(\lambda)}{c^2(\lambda) + (Ne)^2}. \tag{4.34}$$

In the plot in Fig. 4 the data points represent the values of $h(\lambda)$ (obtained out of this best fit procedure) versus λ. In Appendix A we have also presented a very crude estimate of the likely magnitude of the error in $h(\lambda)$; we estimate that this error is not larger that a few (conservatively, say, 5) percent. In Appendix A we also demonstrate that our curve of $h(\lambda)$ versus λ agrees quite well at small λ with the perturbative predictions of previous subsections.

In Fig. 4 we have also presented two curves. The blue (upper) curve represents the bestfit of $h(\lambda)$ versus λ to the form $h(\lambda) = \frac{\alpha}{\beta + \lambda}$. The bestfit values turn out to be $\alpha = 0.495$ and $\beta = 0.841$. The red (lower) curve in Fig. 4 is simply a graph of the function $f(\lambda) = \frac{1}{2(\lambda+1)}$. Notice that our data (the blue curve) always lies above the red curve, but appears to asymptote rather accurately to the later at large λ. As explained in the introduction, the red curve is a theoretical lower bound for $h(\lambda)$. Based on our data we conjectrure that $h(\lambda)$ asymptotes to from above to the curve $\frac{1}{2(\lambda+1)}$ at large λ. It would

	λ_2^f	λ_3^f	λ_4^f	λ_5^f	$\lambda_6 f$	λ_7^f	λ_8^f	λ_9^f	λ_{10}^f
$N = 100$ corrected(linear interpolation)	1.14	2.12	3.17	4.13	5.17	6.14	7.08	8.02	8.97
$N = \infty$(linear interpolation)	1.14	2.13	3.14	4.09	5.09	6.08	7.06	7.98	8.91
$N = 100$ corrected(fit to $\frac{\alpha}{\beta+\lambda}$)	1.15	2.14	3.14	4.13	5.13	6.12	7.12	8.11	9.10
$N = \infty$ (fit to $\frac{\alpha}{\beta+\lambda}$)	1.14	2.13	3.12	4.10	5.09	6.08	7.07	8.06	9.05

Table 11: Values of λ_n^f. $Ne = 100$ corrected is $Ne = 100$ data corrected for $\frac{b(\lambda)}{c^2(\lambda)+(Ne)^2}$
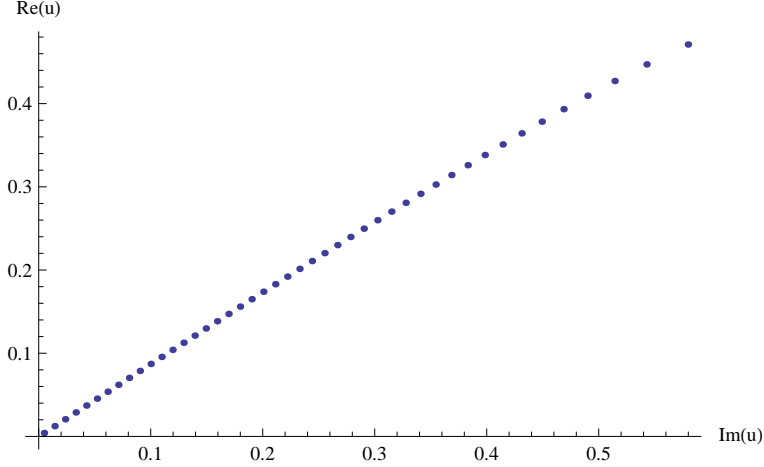

Figure 5: Eigenvalue distribution for Ne=90 at $\lambda = 11$.

be interesting (and may be possible) to establish this fact by a direct analytic study of the saddle point equations at large λ; however we leave this exercise for future work.

As we have explained above, in order to obtain $h(\lambda)$ above we have to solve a saddle point eigenvalue equation. In Fig 5 we present a scatter plot of the saddle point value of the eigenvalues at $\lambda = 11$ and $Ne = 90$. Note that the eigenvalues appear to orient themselves along a curve that does not deviate too far from a straight line (in the complex plane). The magnitude of this 'cut' is approximately 0.7469 and its angle with the real axis in the complex plane is approximately 39.69 degrees.

To study the variation of eigenvalue distribution with λ we plot the length and angle of the eigenvalue distribution for $Ne = 50$ from λ=1 to 10 in Fig. 6. These plot show that the length of the eigenvalue distribution first increases with increasing λ (we know from small λ perturbation theory that this length scales like $\sqrt{\lambda}$ at extremely small λ) but then reaches a maximum at λ somewhere between 5 and 6, and then decreases again. On the other hand the angle made by the cut continues to decrease upon increasing λ (see the second graph in 6. It would be interesting to continue this analysis to larger λ, but we leave that for future work.

In the introduction we had defined λ_n^f as the root of the equation

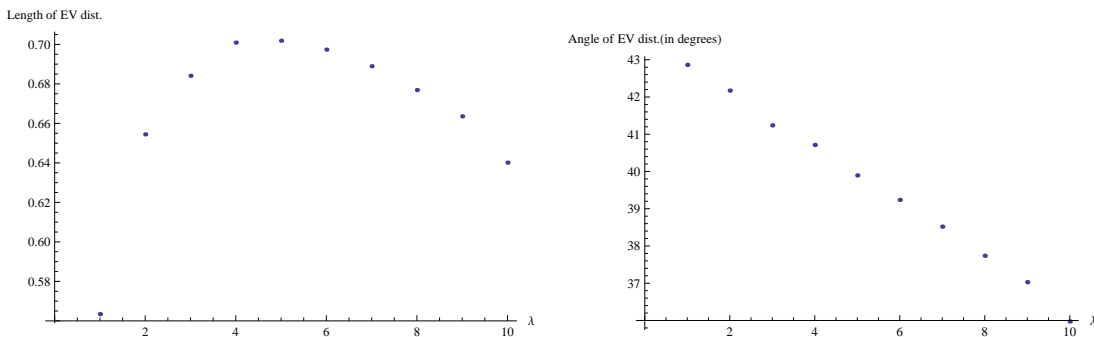$$h(\lambda_n^f) = \frac{1}{2n}$$

Figure 6: Variation of size and angle(with real axis) of eigenvalue distribution for Ne=50 from λ=1 to 10.

Recall that consistency with known results appears to require that

$$\lambda_n^f \geq n - 1.$$

Note also that, were to be given by $h(\lambda) = \frac{1}{2(\lambda+1)}$ then λ_n^f would saturate that inequality. In Table 4.4 we present a table of λ_n^f for $n \in (2, ...10)$ as obtained from our numerical data. In the second row we have computed λ_n^f using the numerical function $h(\lambda)$ obtained by extrapolating our results to large Ne as explained above equation (4.34). In the first row of Table 4.4 we have computed λ_n^f using $h(\lambda)$ obtained out of corrected $Ne = 100$ data, as explained around equation (A.1) in Appendix A. The difference between these two results gives a crude estimate of the likely order of error in our results. Note that λ_n^f starts approaching $\lambda_n^f \approx n - 1$ at large n. This is, of course, a restatement of the fact that the graph of $h(\lambda)$ appears to asymptote to $\frac{1}{2(\lambda+1)}$ from above at large λ.

To end this section let us recall the significance of λ_n^f. At $\lambda_2^f \approx 1.13$, the R-charge of ϕ is renormalized to 1/4; $\mathrm{Tr}\,\phi^2$ then saturates the unitarity bound 1/2 and becomes a free field and (presumably) decouples from the theory. There is a newly emergent $U(1)$ global symmetry which rotates this free field. When $\lambda > \lambda_2^f$, there seems to be no clear argument that $\mathcal{Z}$-minimization should give the correct superconformal R-charge due to possible mixing of the R-charge with this accidental $U(1)$. It is nonetheless plausible that the naive result obtained by applying $\mathcal{Z}$-minimization still holds at large N, since only one field decouples from the theory at a time (on the other hand, there could also be decoupled topological sectors, which may well contribute to the $\mathcal{Z}$ function at leading order in N). If we assume this, then at each λ_n^f an operator $\mathrm{Tr}\,\phi^n$ becomes free and decouples from the theory, at outlined in the introduction.

5. Supersymmetric states of a theory with a single adjoint in the absence of a superpotential

5.1 Superconformal index

5.1.1 The free theory

We begin by analyzing the index of the free theory of a single $\mathcal{N} = 2$ $U(N)$ adjoint chiral

multiplet Φ (the $U(N)$ is gauged), in the large N limit. Let the generator of the global flavour symmetry of this theory - corresponding to the rephasing of Φ - be denoted by G. G is normalized so that the field ϕ has charge $\frac{1}{2}$ under G. In this case the refined Witten index

$$\mathcal{I}_+ = \text{Tr}\left[(-1)^F x^{H+J} e^{-\beta(H-J-R)} y^G\right] \tag{5.1}$$

is easily computed in the free theory. The letter index relevant to this computation is

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}} y^{\frac{1}{2}} - x^{\frac{3}{2}} y^{-\frac{1}{2}}}{1 - x^2}.$$

Note that the two supersymmetric letters in the basic multiplet are ϕ and $\bar{\psi}_+$ (from now on denoted as $\bar{\psi}$) and that the flavour charge of $\bar{\psi}$ is opposite to that of ϕ. It follows that the refined index is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{(1 - x^{\frac{n}{2}} y^{\frac{n}{2}})(1 + x^{\frac{3n}{2}} y^{-\frac{n}{2}})}. \tag{5.2}$$

Once again it is possible to rewrite this index in terms of an index over single trace primaries as

$$\mathcal{I}_+ = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n, y^n)}{n(1 - x^{2n})}\right], \tag{5.3}$$

where $\mathcal{I}_{st}$ is found to be [17]

$$\mathcal{I}_{st} = (xy)^{\frac{1}{2}} + xy + (xy)^{\frac{3}{2}} - x^{\frac{3}{2}} y^{-\frac{1}{2}} + \frac{1}{1 - (xy)^3}\left[(xy)^2 - x^2 + (xy)^{\frac{5}{2}} - x^{\frac{5}{2}} y^{\frac{1}{2}} + (xy)^3 - x^3 y\right.$$

$$\left. + (xy)^{\frac{7}{2}} - x^{\frac{7}{2}} y^{\frac{3}{2}} + (xy)^4 - x^4 y^2 + (xy)^{\frac{9}{2}} - x^{\frac{9}{2}} y^{\frac{5}{2}}\right] + \frac{1}{1 - x^3 y^{-1}}\left(x^{\frac{7}{2}} y^{-\frac{1}{2}} - x^{\frac{9}{2}} y^{-\frac{3}{2}}\right). \tag{5.4}$$

Note that (5.4) receives contributions from (effectively) either two or three states at each energy level[18]. At every energy level we see the contribution of the chiral ring (in the form of $(xy)^E$ for every E). At every level we also, however, see the contribution of either one or two additional "particles".

5.1.2 The large N theory at finite 't Hooft coupling

Now consider the $\mathcal{N} = 2$ $U(N)$ CS theory at level k with one adjoint matter field and no superpotential, in the 't Hooft limit. As we have seen previously, the R-charge of this theory is renormalized as a function of λ. The renormalization of the R-charge is, really, more accurately a mixing of the R-charge and the flavour charge; the extent of this mixing varies with λ.

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[17]Similar techniques were used to compute the Index of $\mathcal{N} = 4$ Yang Mills theory in [27], ABJM theory in [21], and the partition function of free gauge theories in, for instance, [28]

[18]Note that (5.4) reduces to (6.1) below upon setting $y = 1$.

The effect of this mixing on the superconformal index may be dealt with very simply. If we perform the replacement

$$y \to yx^{2h(\lambda)-1}$$

on all the formulas for the (gauged) free theory in the previous subsubsection, then they apply to the interacting theory at arbitrary λ (and infinite N).

5.2 Conjecture for the supersymmetric spectrum at all couplings

In this section we (conjecturally) enumerate all single trace primary operators that are annihilated by the supercharge Q that was preserved in the superconformal index. This enumeration will allow us to enumerate the full single trace supersymmetric operator spectrum of our theory. We perform our enumeration of operators in the classical (though nonlinear) theory; and assume without proof that the same result continues to hold in the full quantum theory. We initially ignore the effect of the renormalization of R-charge in this theory as a function of coupling; as in the previous section, it will prove extremely easy to insert this effect into our final answer right at the end.[19]

We now describe the procedure we will use for the enumeration in this subsection in more detail. Every supersymmetric operator of the sort we seek must be built out of ϕ, $\bar{\psi}_+$ and D_{++} where ϕ and $\bar{\psi}_+$ both have R-charge $\frac{1}{2}$ and the $+$ subscript denotes $SO(3)$ charge (these are the only letters with $\Delta = h + j$). As we have explained above, the letters ϕ and $\bar{\psi}_+$ have flavour charge $\pm\frac{1}{2}$ respectively.

In the free theory any operator constructed out of these elements is annihilated by our special supercharge Q (with R-charge unity and $SO(3)$ charge $-\frac{1}{2}$). In the interacting theory, on the other hand, while it continues to be true that $[Q, \phi] = \{Q, \bar{\psi}\} = 0$ we now have

$$[Q, D_{++}O] \sim [[\phi, \bar{\psi}_+], O]$$

As illustrated by this equation, the derivative D carries the same x and y charges as the combination of letters $\phi\bar{\psi}_+$. Consequently a given term in the index counts an infinite number of distinct possible operator structures. For instance, at order x^5y the cohomology potentially has operators of the form $\mathrm{Tr}\,(\phi^4\bar{\psi}^2)$, $\mathrm{Tr}\,(\phi^3 D\bar{\psi})$ and $\mathrm{Tr}\,(\phi D^2\phi)$. In any given charge sector we refer to the *number of derivatives* in the operator as its *level*. As we see from the equation above, the operator Q preserves charge but maps states of level l to states of level $l-1$. At any given fixed value of the charge let the number of states in the free theory at level l be $n(l)$. Let the number of states in the kernel of the action of Q (those that are annihilated by Q) at level l be denoted by $c(l)$. It follows that the

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[19]The reason that the index is insufficient to completely characterize the supersymmetric spectrum is that it is blind to Bose-Fermi pairs of supersymmetric states, whose contribution to the index cancel. The reader may feel that such conspiratorial cancellations are unlikely, but that is far from the truth. In fact it is immediately clear on general grounds that our result for the index in the previous section must include some important cancellations. In order to see this recall that the R current and stress tensor appear in the supersymmetry multiplet with quantum numbers $(\Delta = 2, J = 1, R = 0, G = 0)$. It follows that the contribution of this multiplet to the index is $-x^4y^0$: however such a term is absent in (5.4). It must be that the full susy spectrum of the theory includes a bosonic state whose contribution to the index cancels that of the stress tensor multiplet. Below we will see in some detail how this works.

number of states in Q cohomology at level l is given by $c(l) - (n(l+1) - c(l+1))$ (because $n(l+1) - c(l+1)$ gives the number of exact states at level l).

The precise value of this cohomology, as defined above, contains some redundant information. This is because a total D_{++} derivative (outside the trace) of any member of cohomology obviously itself belongs to the cohomology. Such descendant elements of cohomology are trivial (they do not map to new particles under the AdS/CFT map) and should be removed from the analysis. This is easy to do in a self consistent manner.

We have written a Mathematica routine that computes the numbers $c(l)$ and $n(l)$ and hence the cohomology at low orders. Our routine then proceeds to eliminate descendant. All the results we have obtained so far are consistent with the following conjectures for the structure of primary states in cohomology:

- The cohomology (obviously) contains only states with x and y quantum numbers of $\text{Tr}\,(\phi^m \bar{\psi}^n)$ for positive m and n.

- We conjecture that, at level zero, all such charges admit a unique (conformal primary) state in cohomology unless $m = 0$ and n is even. This state in cohomology may be thought of as the trace of the completely symmetric combination of $\bar{\psi}$ and ϕ. The exception ($m = 0$ and n even) is a consequence of the fact that $\text{Tr}\,\bar{\psi}^{2k}$ vanishes because of the fermionic nature of $\bar{\psi}$.

- We conjecture that, at level one, there also always exists a unique (conformal primary) state in the cohomology subject to the following exceptions. There are no states at level one when $n = 0$ or when $n = 1$.[20] There are also no level one states when $m = 0$ or when $m = 1$ and n is even.[21]

- We conjecture that the cohomology has no (conformal primary) states at levels higher than one.

- That our conjecture is consistent with the index listed above as may be seen as follows. A pair of a level zero and level one state gives a vanishing contribution to the index. It follows that we have a contribution to the index only in those cases in which level zero and level one states are unpaired. According to our conjecture, unpaired states occur at charges at which there exists a level one state but no level zero state. This occurs for states of the charges $\text{Tr}\,\phi^m$, $\text{Tr}\,(\phi^m \bar{\psi})$, $\text{Tr}\,\bar{\psi}^{2m+1}$ and $\text{Tr}\,(\bar{\psi}^{2m}\phi)$. This precisely accounts for the index computed above.

We summarize the conjecture described above in Table (12). This is found to agree with the index calculated for this theory as calculated in section 5.1.1. In that table we have also used the fact that every short representation of the $\mathcal{N} = 2$ superconformal algebra

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[20]This statement is obvious when $n = 0$ because then there do not exist any level one states with the right quantum numbers. Moreover when $n = 1$ the unique level one state with the right quantum numbers is a descendant of $\text{Tr}\,\phi^{m-1}$.

[21]When $m = 0$ there are no level one states with the right quantum numbers. When $m = 1$ and n is even, the unique such state is a descendant of $\text{Tr}\,\bar{\psi}^{n-1}$. Of course no such statement can be (or is) true when n is odd because $\text{Tr}\,\bar{\psi}^{n-1}$ vanishes.

Cohomology state	#	$\mathcal{N}=2$ Primary	Allowed h and j
$(h,0,h,h)$	1	$(h,0,h,h)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(j+h,j,h,h-2j)$	1	$(j+h-\frac{1}{2},j-\frac{1}{2},h-1,h-2j)$	$j,h \in \frac{1}{2}\mathbb{Z}^+,\ h \geq j$ $h=j \in \mathbb{Z}$ not allowed.
$(j+h,j,h,h-2j+2)$	1	$(j+h-\frac{1}{2},j-\frac{1}{2},h-1,h-2j+2)$	$h,j \in \frac{1}{2}\mathbb{Z}^+,\ j \geq \frac{3}{2},\ h \geq j-\frac{1}{2}$
$(2h+1,h+1,h,-2h)$	1	$(2h+\frac{1}{2},h+\frac{1}{2},h-1,-2h)$	$h \in \frac{1}{2}\mathbb{Z}^+$

Table 12: $\mathcal{N}=2$ primary content of single adjoint matter theory with zero superpotential. The second column stands for the multiplicity of the cohomology states. The notation is (Δ,j,h,g) which respectively are scaling dimension, spin, R-charge and $U(1)$ flavor charge. The flavor charges are normalized to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. The results above apply when $h(\lambda) = \frac{1}{2}$. The results for the general case are obtained from the results of the table above by the replacement $\Delta \to \Delta - (1-2h(\lambda))g$, $h \to h - (1-2h(\lambda))g$ for every primary in the table.

has a unique (conformal primary) state in Q cohomology to read off the full spectrum of the short (or supersymmetric) single trace primary operators of the theory implied by our conjecture for Q cohomology.

6. Supersymmetric states of theories with a single adjoint field with nonzero superpotential

6.1 Space of theories

One may construct several superconformal field theories with a single chiral multiplet by perturbing the theory with no superpotential. The simplest theory of this sort may be constructed by perturbing the superpotential of the theory of the previous subsection with a $\mathrm{Tr}\,\Phi^4$ term. While this perturbation is marginal at zero λ, as we have explained above it is relevant at every finite λ (this follows as $h(\lambda) < \frac{1}{2}$ for all finite λ). The RG flow seeded by this operator may be argued to terminate at a new fixed point at which the coefficient c of $\mathrm{Tr}\,\Phi^4$ in the superpotential is of order unity, in units in which a uniform factor of $k = \frac{N}{\lambda}$ sits outside the whole action.[22] This line of CFTs also reduces to the free theory as $\lambda \to 0$. The $\mathrm{Tr}\,\Phi^4$ superpotential in this theory breaks the flavour symmetry of the zero superpotential theory down to Z_4. The requirement of the invariance of the superpotential under R symmetry transformations forces the R-charge of this system to equal $\frac{1}{2}$ at every value of the coupling constant λ.

There exist no other lines of CFTs with this matter content that reduce to the free CFT in the limit $\lambda \to 0$. However there plausibly exist many other lines of superconformal fixed points that cannot be deformed to the free theory. To start with the superpotential term $\mathrm{Tr}\,\Phi^3$ is relevant at all values of λ. At small λ the RG flow seeded by this term presumably terminates at a large N and supersymmetric analogue of the Wilson Fisher

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[22]The argument that the RG flow ends at a finite value of c is simple. When $c \gg 1$, the gauge interaction in the theory is negligibly weak compared to the $\mathrm{Tr}\,\Phi^4$ interaction and may be ignored. The model is then effectively the Wess Zumino theory whose β function towards the IR is known to be negative. Consequently the sign of the β function flips from positive for small c to negative at large c, and so must have a zero at c of order unity.

fixed point. It thus seems plausible that the new fixed point exists at every value of λ. The R-charge of ϕ is fixed at $\frac{2}{3}$ at this fixed point.

More exotically, as we have explained in the introduction, the fact that $h(\lambda)$ decreases without bound (as λ is increased) plausibly suggests the existence of fixed points seeded by $\text{Tr}\,\Phi^n$ induced RG flows, for all values of n, at sufficiently large λ. The R-charge of the operator ϕ is fixed at $\frac{2}{n}$ along these fixed lines of theories.

6.2 Superconformal index of the theory with a $\text{Tr}\,\Phi^4$ superpotential

Let us compute the Witten index

$$\text{Tr}(-1)^F x^{H+J}$$

for the theory with the $\text{Tr}\,\Phi^4$ superpotential. The letter index of this theory is the Witten index of the sum of the $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, 0, -\frac{1}{2})$ representations. We find the single letter contribution[23]

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}}}{1+x}.$$

It follows that the Witten index of the theory - in the large N limit - is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1}{1 - \mathcal{I}_L(x^n)} = \prod_{n=1}^{\infty} \frac{1 + x^n}{1 - x^{\frac{n}{2}} + x^n}.$$

The formula above gives the Witten index of the full theory (including all multi trace operators) in the large N limit. It is interesting to inquire about the single trace index for the same theory. Now the full index is obtained from the single trace index by Bose exponentiation. Now the single trace index actually receives contributions both from single trace conformal primaries and single trace conformal descendants. It is of most interest to isolate the index over all single trace primaries (as this gives the index over the particle spectrum of the dual theory). Let us define the index over single trace primaries as $\mathcal{I}_{st}$. It then follows that

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1 + x^n}{1 - x^{\frac{n}{2}} + x^n} = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n)}{n(1 - x^{2n})}\right]$$

This equation completely determines $\mathcal{I}_{st}$. It is possible to show that

$$
\begin{aligned}
\mathcal{I}_{st} &= x^{\frac{1}{2}} + x + x^{\frac{7}{2}} - x^{\frac{9}{2}} + x^{\frac{13}{2}} - x^{\frac{15}{2}} + x^{\frac{21}{2}} - x^{\frac{23}{2}} + \dots \\
&= x^{\frac{1}{2}} + x + \frac{x^{\frac{7}{2}}}{1 + x + x^2}.
\end{aligned}
\tag{6.1}
$$

Note that the spectrum of single traces is periodic at large enough energies, with one new boson and one new fermion being added at energy intervals of three.

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[23]Note that this letter partition function is finite (and in fact evaluates to half) in the limit $x = 1$.

6.3 Conjecture for the supersymmetric spectrum of the theory with a $\mathrm{Tr}\,\Phi^4$ superpotential

In this subsection we present a conjecture for the partition function over single trace supersymmetric operators in the theory with a $\mathrm{Tr}\,\Phi^4$ term in the superpotential. Our method is the same as that employed in the previous section: we list the classical cohomology of the special supercharge Q (the supercharge that annihilates the superconformal index).

The only difference between the cohomology of Q in this section, and the cohomology of Q in the absence of a superpotential (computed in the previous section) is that Q no longer annihilates $\bar\psi$, but we instead have

$$Q\bar\psi \sim \phi^3$$

.

As in the previous section the cohomology of interest can be calculated separately for operators with distinct values of $\Delta + j$. At any given value of $\Delta + j$, we grade operators by their *level*, defined to be *twice the angular momentum* of the operator in question. As in the previous section, we work with the full set of single trace operators constructed out of ϕ, $\bar\psi$ and D_{++}. States at level 0 are constructed entirely out of ϕ's, and are all annihilated by Q. There is a unique level zero state at every value of $\Delta + j$. States at level 1 are built out of several ϕ's but one $\bar\psi$. There is also a unique state at level 1 at every value of $\Delta + j$.

The situation is more complicated at higher levels. As in the previous section, we have written a Mathematica code to compute the cohomology for all states with $\Delta + j \leq \frac{23}{2}$. As in the previous subsection, our Mathematica routine automatically removes the contributions of descendants. Our results are consistent with following conjecture.

- At every $\Delta + j$ the number of (conformal primary) states in cohomology is either 0 or 1.

- When $\Delta + j = \frac{1}{2}$ there exists a single conformal primary state in cohomology at level 0 and no states at any other level.

- When $\Delta + j = 1$ there exists a single conformal primary state in cohomology at level 0 and no state at any other level.

- When $\frac{3}{2} \geq \Delta + j \geq 3$ there are no conformal primary states in cohomology.

- When $\Delta + j = \frac{6k+7}{2}$ for $k = 0, 1 \ldots \infty$ there is a single conformal primary state in cohomology at level $2(k+1)$

- When $\Delta + j = \frac{6k+8}{2}$ for $k = 0, 1 \ldots \infty$ there are two conformal primary states in cohomology. The first occurs at level $2(k+1)$ while the second occurs at level $2(k+1)+1$.

- When $\Delta + j = \frac{6k+9}{2}$ for $k = 0, 1 \ldots \infty$ there is a single conformal primary state in cohomology at level $2(k+1)+1$

Cohomology states	Multiplicity	Protected primaries
$(\frac{1}{2}, 0, \frac{1}{2})$	1	$(\frac{1}{2}, 0, \frac{1}{2})$
$(1, 0, 1)$	1	$(1, 0, 1)$
$(2k + \frac{5}{2}, k+1, k+\frac{3}{2})$	1	$(2k+2, k+\frac{1}{2}, k+\frac{1}{2})$
$(2k+3, k+1, k+2)$	1	$(2k+\frac{5}{2}, k+\frac{1}{2}, k+1)$
$(2k+\frac{5}{2}, k+\frac{3}{2}, k+1)$	1	$(2k+2, k+1, k)$
$(2k+3, k+\frac{3}{2}, k+\frac{3}{2})$	1	$(2k+\frac{5}{2}, k+1, k+\frac{1}{2})$

Table 13: Supersymmetric spectrum for $\mathcal{N} = 2$ single adjoint with superpotential $\text{Tr}\,\Phi^4$, $k \in 0, \mathbb{Z}^+$. The notation again is (Δ, j, h).

- There are no conformal primary states in cohomology when $\Delta + j = \frac{6k+10}{2}$ or $\frac{6k+11}{2}$ or $\frac{6k+12}{2}$ (for $k = 0, 1 \ldots \infty$)

This conjecture has been summarized in Table (13) in terms of charges (Δ, j, h) of the cohomology states. This is found to agree with the index calculated for this theory in section 6.2. As states of the form $(j + h, j, h)$ (for $j > 0$) are descendants of the superconformal primary $(j + h - \frac{1}{2}, j - \frac{1}{2}, h - 1)$, the corresponding superconformal primaries are also listed out in the table.

6.4 Conjecture for the supersymmetric spectrum of the theory with a $\text{Tr}\,\Phi^3$ superpotential

In this subsection we present a conjecture for the classical cohomology of the particular supercharge Q for the theory with a $\text{Tr}\,\Phi^3$ superpotential. A theory with such a superpotential is always strongly coupled, and so we cannot use a free calculation to compute its superconformal index. However, the computation of Q cohomology in such a theory is easily performed (under the same assumption of the previous subsection, i.e. that it is sufficient to use the classical supersymmetry transformation rules), using the same method as in the previous subsection.

The difference between the computation of this subsection and that of the previous one is as follows. In this case the action of Q on $\bar{\psi}$ is given by $Q\bar{\psi} \sim \phi^2$. Further the fact that we have a superconformal theory with a Φ^3 superpotential forces the following charge assignments: the R-charges of $\phi, \bar{\psi}$ are $\frac{2}{3}, \frac{1}{3}$ respectively, while the scaling dimensions of $\phi, \bar{\psi}$ as determined by the BPS relation $\Delta = j + h$, are $\frac{2}{3}, \frac{5}{6}$ respectively.

As in the previous subsection, the cohomology must be computed separately at every value of $\Delta + j$. At any given value of $\Delta + j$, we grade operators by their *level* defined to be *twice the angular momentum* of the operator in question. As in the previous subsection, we work with the full set of single trace operators constructed out of ϕ, $\bar{\psi}$ and D_{++}. States at level 0 are constructed entirely out of ϕ's, and are all annihilated by Q.

The calculation is formulated exactly as in the previous section. This calculation has been done up to $\Delta + j = 12$ on Mathematica. Our results are consistent with following conjecture:

Cohomology states	Multiplicity	Protected primaries
$\left(\frac{2}{3}, 0, \frac{2}{3}\right)$	1	$\left(\frac{2}{3}, 0, \frac{2}{3}\right)$
$\left(\frac{5k+2}{3}, k, \frac{2}{3}(k+1)\right)$	1	$\left(\frac{5k}{3} + \frac{1}{6}, k - \frac{1}{2}, \frac{2k-1}{3}\right)$
$\left(\frac{5}{6}(2k+1), k + \frac{1}{2}, \frac{2k+1}{3}\right)$	1	$\left(\frac{5k+1}{3}, k, \frac{2}{3}(k-1)\right)$

Table 14: Supersymmetric spectrum for $\mathcal{N} = 2$ single adjoint with superpotential Φ^3, $k \in \mathbb{Z}^+$. The notation is (Δ, j, h)

- At $\Delta + j = \frac{2}{3}$, there is a single (conformal primary) state in the cohomology, and it occurs at level 0.

- For every allowed value of $\Delta + j$ and at any given level, the number of conformal primary states in the cohomology is either 0 or 1.

- The full conformal primary cohomology content of the theory is summarized as follows:

  a. $\Delta + j = \frac{4}{3}(2k + 1)$ one state at level $2k + 1$, for $k = 1, 2, 3 \ldots$

  b. $\Delta + j = \frac{4}{3}(2k + \frac{1}{2})$ one state at level $2k$ for $k = 1, 2, 3 \ldots$

This conjecture has been summarized in Table (14) in terms of charges (Δ, j, h) of the cohomology states. This is found to agree with the index calculated for this theory in section 10, by making the substitutions as given in (10.7). As states of the form $(j + h, j, h)$ (for $j > 0$) are descendants of the superconformal primary $(j + h - \frac{1}{2}, j - \frac{1}{2}, h - 1)$, the corresponding superconformal primaries are also listed out in the table.

7. Supersymmetric states in the theory with two adjoint fields and vanishing superpotential

The superconformal theory with two adjoints and no superpotential has a $U(2)$ flavour symmetry, realized as the rotation of the chiral multiplets Φ_1 and Φ_2 as a doublet. We denote the two $U(1)$ Cartan charges of this $U(2)$ by G_1 and G_2. Our conventions are that the field ϕ_1 has charges $(G_1, G_2) = (1, 0)$ while the charges of ϕ_2 are $(0, 1)$. We compute the refined Witten index defined by

$$\mathcal{I} = \text{Tr} \left[(-1)^F x^{H+J} y_1^{G_1} y_2^{G_2} \right]. \tag{7.1}$$

The letter index is given by

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}}(y_1 + y_2) - x^{\frac{3}{2}}(y_1^{-1} + y_2^{-1})}{1 - x^2}.$$

As usual the full multitrace index of the free theory is given by

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1}{1 - \mathcal{I}_L(x^n, y^n)}.$$

This index captures an exponential growth of density of supersymmetric states. In order to see that this must be the case, note that the number of chiral primaries at level L is of order 2^L. As there are no fermionic states with the quantum numbers of the chiral primaries, it is impossible for this contribution to be cancelled in the full index. It follows that the index, in this state, receives contributions from an exponentially growing number of states.

As an exponentially growing spectrum of supersymmetric states is presumably rather difficult to characterize more precisely, we postpone further analysis of this case to future work, and turn, instead, to the study of theories with two chiral multiplets and a superpotential.

8. Supersymmetric states in the $\mathcal{N} = 3$ theory with two adjoint chiral multiplets (i.e. one adjoint hypermultiplet)

Of all the possible superpotential deformations of the zero superpotential theory with two adjoint chiral multiplets, the deformation $W = \alpha \text{Tr} \, [\Phi_1, \Phi_2]^2$ has a special role. As we have explained above, this deformation is relevant at nonzero λ. The RG flow seeded by this deformation has a fixed point at $\alpha = \frac{2\pi}{k}$; at this value of the coefficient, the theory is conformally invariant; further its supersymmetry is enhanced to $\mathcal{N} = 3$ and the theory enjoys invariance under the full $\mathcal{N} = 3$ superconformal algebra.

$\mathcal{N} = 3$ superconformal symmetry forces the theory to have an $SU(2)$ R symmetry; ϕ_1 and ϕ_2^* transform as a doublet under this symmetry. ϕ_1 and ϕ_2 each have R-charge $\frac{1}{2}$ under the canonical $U(1)$ subgroup of this R symmetry group. Note of course that the value of this R-charge is protected by $SU(2)$ representation theory, and cannot be renormalized as a function of λ.

In addition the fact that the superpotential is proportional to $\text{Tr} \, (\epsilon^{ij} \Phi_i \Phi_j)^2$ reveals that the superpotential deformation preserves an $SU(2)$ flavour subgroup of the $U(2)$ flavour isometry group of the theory without a superpotential.

In this section we will compute the supersymmetric spectrum of this theory at large N.

8.1 Superconformal index

As the $\mathcal{N} = 3$ theory reduces to a free theory at $\lambda = 0$, its superconformal index is easily computed. The superconformal index is defined by

$$\mathcal{I} = \text{Tr} \left[(-1)^F x^{H+J} y^G \right] \tag{8.1}$$

where G is the $U(1)$ component of the $SU(2)$ flavor group (under which ϕ_1 has charge $\frac{1}{2}$ and ϕ_2 has charge $-\frac{1}{2}$). The relevant letter index is given by

$$\mathcal{I}_L = \frac{x^{\frac{1}{2}} y^{\frac{1}{2}} - x^{\frac{3}{2}} y^{-\frac{1}{2}} + x^{\frac{1}{2}} y^{-\frac{1}{2}} - x^{\frac{3}{2}} y^{\frac{1}{2}}}{1 - x^2} = \frac{x^{\frac{1}{2}} (y^{\frac{1}{2}} + y^{-\frac{1}{2}})}{1 + x},$$

so that the index over the theory is

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1 + x^n}{(1 - x^{\frac{n}{2}} y^{\frac{n}{2}})(1 - x^{\frac{n}{2}} y^{-\frac{n}{2}})}.$$

As in previous sections, it is possible to rewrite this index in terms of an index over single trace primaries as

$$\mathcal{I}_+ = \prod_{n=1}^{\infty} \frac{1 + x^n}{(1 - x^{\frac{n}{2}} y^{\frac{n}{2}})(1 - x^{\frac{n}{2}} y^{-\frac{n}{2}})} = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}_{st}(x^n, y^n)}{n(1 - x^{2n})}\right]. \tag{8.2}$$

We find

$$\mathcal{I}_{st} = x^{\frac{1}{2}}\left(y + \frac{1}{y}\right) + x\left(1 + y^2 + \frac{1}{y^2}\right) + x^{\frac{3}{2}}\left(y^3 + \frac{1}{y^3}\right) + x^2\left(y^4 + \frac{1}{y^4}\right)$$
$$+ \sum_{n=5}^{\infty} x^{\frac{n}{2}}\left(y^n + \frac{1}{y^n} - y^{n-4} - \frac{1}{y^{n-4}}\right). \tag{8.3}$$

This simple result describes a spectrum with 4 states - two bosons and two fermions - at every energy higher than a minimum value. Note that (8.3) reduces to (9.7) upon setting y to unity.

8.2 Supersymmetric cohomology

We now proceed to compute the single trace supersymmetric cohomology of the $\mathcal{N} = 3$ theory.[24] We are instructed to count traces built out of $D_+^n \phi_i$ and $D_+^n \bar{\psi}_i$ ($i = 1, 2$). We are only interested in Q cohomology, where the action of Q on the basic fields is given by

$$\begin{aligned}
Q\phi_i &= 0, \\
Q\bar{\psi}_i &= [\phi_i, [\phi_1, \phi_2]], \\
Q[D_{++}, \cdot] &= \left[[\phi_1, \bar{\psi}_1] + [\phi_2, \bar{\psi}_2], \cdot\right].
\end{aligned} \tag{8.4}$$

As in previous subsections, we have explicitly enumerated this cohomology (with the help of Mathematica) at low quantum numbers, and used the results of this numerical experiment to suggest a relatively simple conjecture for the conformal primary content of this cohomology. As in previous subsections, each conformal primary member of cohomology implies the existence of a single short $\mathcal{N} = 2$ superconformal representation. Unlike the situation in previous sections, however, this spectrum has to satisfy an additional consistency check, as $\mathcal{N} = 2$ representations must group together into $\mathcal{N} = 3$ representations. Our conjecture for Q cohomology passes this consistency check, and leads us to conjecture that the short supersymmetric operator content of the theory is given as in Table (15). This is found to agree with the index calculated for this theory in section 8.1. (Our notation for quantum numbers of states as well as primaries is (Δ, j, h, g) where g is the $SU(2)$ flavour charge.)

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[24]As in the previous subsubsection, we know that there must exist states that contribute to the partition function but are invisible in the index simply from the observation that the contribution of the stress tensor multiplet -$x^4 y^0$ - is not visible in (8.3).

Cohomology states ($\mathcal{N}=2$ quantum numbers)	Multiplicity	$\mathcal{N}=2$ Primary	$\mathcal{N}=3$ Primary	Allowed h
$(h,0,h,h)$	1	$(h,0,h,h)$	$(h,0,h,h)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+\frac{1}{2},\frac{1}{2},h,h)$	1	$(h,0,h-1,h)$		$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{1}{2},h+1,h)$	1	$(h+1,0,h,h)$	$(h+1,0,h,h)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+2,1,h+1,h)$	1	$(h+\frac{3}{2},\frac{1}{2},h,h)$		$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+1,1,h,0)$	1	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$h \in \mathbb{Z}^+$
$(h+\frac{3}{2},\frac{3}{2},h,0)$	1	$(h+1,1,h-1,0)$		$h \in \mathbb{Z}^+$

Table 15: Supersymmetric spectrum for 2 chiral adjoints at $\mathcal{N}=\ni$ fixed point with $SU(2)$ flavor symmetry. The flavor charges for ϕ and $\bar{\psi}$ are normalized to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively

A striking feature of this conjectured supersymmetric spectrum is that it includes no states with spin ≥ 2. This is unlike all the other supersymmetric spectra we have computed in this paper, and suggests that the $\mathcal{N}=3$ theory might admit a supergravity-like dual description at large λ.

As the full global symmetry group of our theory is $SU(2) \times SU(2)$ it is tempting to conjecture that the supergravity description in question is obtained by a compactification of a 7-dimensional supergravity on S^3. Indeed the states in the first 2 rows (of the second last column) of Table (15) have $SU(2) \times SU(2)$ quantum numbers that are strongly reminiscent of scalar and vector spherical harmonics on S^3. However the states in the last line of this table do not appear to fit well into this pattern; as the difference between the two $SU(2)$ quantum numbers of these states grows without bound; this never happens for S^3 spherical harmonics for states with a fixed (or bounded) value of spin. For this reason we are unsure whether our results for the supersymmetric spectrum of this theory are consistent with a possible dual description in terms of a higher dimensional supergravity theory. We leave further discussion of this question to future work.

9. Marginal $\mathcal{N}=2$ deformations of the $\mathcal{N}=3$ theory

It was demonstrated in [25] that the infinitesimal manifold of exactly marginal deformations of a given SCFT has a very simple characterization.[25] This space is simply given by modding out the space of marginal (but not necessarily exactly marginal) classical deformations by the complexified action of the global (non R) symmetry group $\mathcal{G}_C$ of the theory.

The space of marginal scalar deformations of the $\mathcal{N}=2$ theory was worked out in the previous section. At the level of the superpotential it is given by operators $\text{Tr}(\Phi^{a_1}\Phi^{a_2}\Phi^{a_3}\Phi^{a_4})$ with the indices $a_1 \ldots a_4$ completely symmetrized. These operators transform in the 5 dimensional (spin 2) representation of the global symmetry group $SU(2) \sim SO(3)$. In colloquial terms they constitute a complex traceless symmetric 3×3 matrix M on which complexified $SO(3)$ transformations O act according to the law

$$M \to OMO^T$$

---

[25]In the perturbative regime, a global characterization of the "conformal manifold" was given in [17].

Points on conformal manifold	Flavour symmetry	Charges
$\lambda_1 = 0 = \lambda_2$	$SU(2)$	Φ_1, Φ_2 form a doublet
$\lambda_1 = \lambda_2 \neq 0$	$U(1)$	Φ_1: 1, Φ_2: -1
$\lambda_1 = 0, \lambda_2 \neq 0$	$U(1)$	$\Phi_1 + i\Phi_2$: 1, $\Phi_1 - i\Phi_2$: -1
$\lambda_1 \neq 0, \lambda_2 = 0$	$U(1)$	$\Phi_1 + \Phi_2$: 1, $\Phi_1 - \Phi_2$: -1

Table 16: Symmetries on the conformal manifold

The classification of all 3×3 matrices M that are inequivalent under this transformation law is well studied [26]. Generic complex symmetric matrices M can be diagonalized by the action of O; consequently generic exactly marginal deformations of the $\mathcal{N} = 3$ theory are in one to one correspondence with complex diagonal traceless 3×3 matrices and are labeled by two complex eigenvalues. In addition to the generic case, however, there exist two special classes of matrices M that cannot be diagonalized. Instead, one of them can be put in the form [29]

$$\begin{pmatrix} \lambda_1 + \frac{i}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \lambda_1 - \frac{i}{2} & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix} \tag{9.1}$$

This gives a new one parameter set of exactly marginal deformations of the $\mathcal{N} = 3$ theory. The second possible form is

$$\begin{pmatrix} 0 & \frac{1+i}{2} & 0 \\ \frac{1+i}{2} & 0 & \frac{1-i}{2} \\ 0 & \frac{1-i}{2} & 0 \end{pmatrix} \tag{9.2}$$

Let us first focus on the generic marginal deformations of the $\mathcal{N} = 3$ theory. The generic deformation can be put in the form

$$W = \lambda_1 \text{Tr} \, (\Phi_1^2 + \Phi_2^2)^2 - \lambda_2 \text{Tr} \, (\Phi_1^2 - \Phi_2^2)^2 - (\lambda_1 + \lambda_2)\text{Tr} \, (\Phi_1\Phi_2 + \Phi_2\Phi_1)^2 \tag{9.3}$$

which is better written as

$$W = \lambda_1 \text{Tr} \, \left[(\Phi_1 + \Phi_2)^2(\Phi_1 - \Phi_2)^2\right] - \lambda_2 \text{Tr} \, \left[(\Phi_1 + i\Phi_2)^2(\Phi_1 - i\Phi_2)^2\right] \tag{9.4}$$

At a generic point on the conformal manifold the flavor symmetry is completely broken. For special values of $\lambda_{1,2}$, listed in Table (16), a $U(1)$ flavor symmetry is restored. The space of generic exactly marginal deformations of the $\mathcal{N} = 3$ theory is a two complex dimensional manifold.

Next, for the nongeneric deformation as parametrized by (9.1), the superpotential deformation is

$$W = \lambda_2 \left(\text{Tr} \, (\Phi_2^4) - \lambda_1 \text{Tr} \, (\Phi_1\Phi_2\Phi_1\Phi_2)\right) \tag{9.5}$$

There is no flavour symmetry in this case.

As for the nongeneric deformation as parametrized by (9.2), the superpotential deformation is

$$W = \lambda_2 \left((1 + i)\text{Tr} \, (\Phi_1^4 - \Phi_2^4) + 2(1 - i)\text{Tr} \, (\Phi_1^3\Phi_2 - \Phi_1\Phi_2^3)\right) \tag{9.6}$$

---

[26]We thank A. Mukherjee for discussions on this topic

There is no flavour symmetry in this case also.

The $U(1)$ R-charge of chiral multiplets in these theories is fixed to $\frac{1}{2}$ (merely by the observation that each of these theories has a space of exactly marginal deformations, labeled by different quartic superpotential deformations).

9.1 Superconformal index of these theories

The Witten index of the deformed theories that preserve $U(1)$ flavor symmetry is simply identical to the index of the $\mathcal{N} = 3$ theory determined in the previous section.

The Witten index of deformed theories that break the $U(1)$ symmetry is also equal to to that of the $\mathcal{N} = 3$ theory, but with $y = 1$ (as there is no flavor charge with respect to which states can be weighted in this case). We find the remarkably simple result

$$\mathcal{I}_{st} = 2x^{\frac{1}{2}} + 3x + 2x^{\frac{3}{2}} + 2x^2. \tag{9.7}$$

Thus the single trace index sees a total of only 9 conformal primaries!

9.2 Conjecture for the supersymmetric cohomology

For cohomology calculations we consider the general marginal superpotential as given in (9.3) along with the original $\mathcal{N} = 3$ superpotential with a coefficient normalized to one. With this superpotential the action of the special supercharge on the basic letters is as follows

$$Q(\bar{\psi}_1) = -\phi_1 \phi_2^2 - \phi_2^2 \phi_1 + 2\phi_2 \phi_1 \phi_2 + (\lambda_1 - \lambda_2)\phi_1^3 - 4(\lambda_1 + \lambda_2)(\phi_2 \phi_1 \phi_2) \tag{9.8}$$

$$Q(\bar{\psi}_2) = -\phi_2 \phi_1^2 - \phi_1^2 \phi_2 + 2\phi_1 \phi_2 \phi_1 + (\lambda_1 - \lambda_2)\phi_2^3 - 4(\lambda_1 + \lambda_2)(\phi_1 \phi_2 \phi_1) \tag{9.9}$$

$$Q[D_{++}, \cdot] = \left[[\phi_1, \bar{\psi}_1] + [\phi_2, \bar{\psi}_2], \cdot\right] \tag{9.10}$$

Although it is not obvious, it (experimentally) appears that the cohomology is largely independent of the complex ratio $\frac{\lambda_1}{\lambda_2}$ but instead depends only on whether the flavour symmetry of the theory is broken or restored. Using the methods described in earlier sections we have generated data that suggests that the cohomology of these theories takes the following form.

For the generic $\mathcal{N} = 2$ deformations (9.4), the conformal primary states in the cohomology, and the corresponding $\mathcal{N} = 2$ superconformal representations, are given, in the case that a $U(1)$ flavor symmetry is preserved (($\lambda_1 = 0, \lambda_2 \neq 0$), ($\lambda_1 \neq 0, \lambda_2 = 0$), ($\lambda_1 = \lambda_2 \neq 0$)) in Table (17).

On the other hand the conformal primary cohomology and the superconformal primary content of $\mathcal{N} = 2$ cases when there is no flavour symmetry i.e. when $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \neq 0$ is given in Table (18). Also for the first non generic $\mathcal{N} = 2$ deformation (9.5) with $\lambda_1 \neq 0$ and for the second nongeneric $\mathcal{N} = 2$ deformation (9.6) the superconformal primary content is the same as in Table (18).

For the first non generic $\mathcal{N} = 2$ deformation (9.5), if $\lambda_1 = 0$, the superconformal primary content is given in Table (19)

Note that in each case the supersymmetric spectrum has no states with spins greater than two, suggesting again the possibility of a dual supergravity description for these theories at strong coupling.

Cohomology states ($\mathcal{N}=2$ quantum numbers)	Multiplicity	$\mathcal{N}=2$ Primary	Allowed h
$(h,0,h,h)$	1	$(h,0,h,h)$	
$(h,0,h,-h)$	1	$(h,0,h,-h)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(h,0,h,0)$	1	$(h,0,h,0)$	
$(\frac{3}{2},\frac{1}{2},1,0)$	1	$(1,0,0,0)$	
$(h+\frac{3}{2},\frac{1}{2},h+1,h)$	1	$(h+1,0,h,h)$	
$(h+\frac{3}{2},\frac{1}{2},h+1,-h)$	1	$(h+1,0,h,-h)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+\frac{3}{2},\frac{1}{2},h+1,0)$	1	$(h+1,0,h,0)$	
$(h+1,1,h,0)$	1	$(h+\frac{1}{2},\frac{1}{2},h-1,0)$	$h \in \frac{1}{2}\mathbb{Z}^+$
$(h+\frac{1}{2},\frac{3}{2},h-1,0)$	1	$(h,1,h-2,0)$	$h \in 2\mathbb{Z}^+$

Table 17: Supersymmetric spectrum for 2 chiral adjoints at $\mathcal{N}=2$ fixed point with $U(1)$ flavor symmetry. Notation is (Δ,j,h,g) with g normalized to be $\frac{1}{2}$ for ϕ.

Cohomology state	Primary	Multiplicity	Allowed values
$(\frac{k}{2},0,\frac{k}{2})$	$(\frac{k}{2},0,\frac{k}{2})$	3 if $\frac{k}{2}$ is odd, 2 if else	$k \in \mathbb{Z}^+$
$(\frac{k+1}{2},\frac{1}{2},\frac{k}{2})$	$(\frac{k}{2},0,\frac{k}{2}-1)$	3 if $\frac{k}{2}$ is even, 1 if $\frac{k}{2}$ is odd, 2 if $\frac{k}{2}$ is half an odd integer	$k \in \mathbb{Z}^+$
$(2k+\frac{1}{2},\frac{3}{2},2k-1)$	$(2k,1,2k-2)$	1	$k \in \mathbb{Z}^+$

Table 18: Cohomology and primary content of theories with no flavor symmetry. The notation is (Δ,j,h), since there is no flavor symmetry.

Cohomology states	Multiplicity	$\mathcal{N}=2$ Primary	Allowed k
$(\frac{1}{2},0,\frac{1}{2})$	2	$(\frac{1}{2},0,\frac{1}{2})$	
$(1,\frac{1}{2},\frac{1}{2})$	1	$(\frac{1}{2},0,-\frac{1}{2})$	
$(\frac{3}{2},\frac{1}{2},1)$	1	$(1,0,0)$	
$(2,\frac{1}{2},\frac{3}{2})$	3	$(\frac{3}{2},0,\frac{1}{2})$	
$(k,0,k)$	3	$(k,0,k)$	$k \in \frac{\mathbb{Z}^+}{2}, k \geq 1$
$(k+\frac{1}{2},\frac{1}{2},k)$	3 , if $k=$ even; 4 , if $k \neq$ even	$(k,0,k-1)$	$k \in \frac{\mathbb{Z}^+}{2}, k \geq 2$
$(2k+1,1,2k)$	2	$(2k+\frac{1}{2},\frac{1}{2},2k-1)$	$k \in \mathbb{Z}^+$
$(k+\frac{3}{2},1,k+\frac{1}{2})$	1	$(k+1,\frac{1}{2},k-\frac{1}{2})$	$k \in \mathbb{Z}^+$
$(2k+\frac{1}{2},\frac{3}{2},2k-1)$	1	$(2k,1,2k-2)$	$k \in \mathbb{Z}^+$

Table 19: Supersymmetric spectrum for $\mathcal{N}=2$ nongeneric deformation with $\lambda_1=0$

9.3 Theories with three or more chiral multiplets

In this case the letter partition function equals unity at a value of $x < 1$. It follows that the Witten index undergoes a Hagedorn transition at finite 'temperature'. In other words the number of supersymmetric operators protected by susy grows exponentially with energy

in these theories. Restated, our system has a stringy growth in its degrees of freedom; the effective string scale is the *AdS* scale (unity in our units). It is clearly impossible for such theories to have a gravitational description (in any dimension).

Note that in these theories the index undergoes a phase transition at a finite value of the chemical potential. In the 'high temperature' (more accurately small x) phase the logarithm of the index is of the order N^2. It seems possible that this index captures the entropy of supersymmetric black holes in the as yet mysterious bulk dual of these theories.

10. Superconformal index at finite k and N from localization

In the 't Hooft limit where $\lambda = N/k$ can be treated as a continuous parameter, the superconformal index is expected to be independent of λ, apart from possible renormalization of R-charge, as discussed in previous sections. There, the index was computed from the free limit of the theory. This is no longer the case at finite k, as k is quantized and generally the index can jump as k varies. Nonetheless, the exact superconformal index at finite k and N can be computed using the powerful technique of supersymmetric localization [22, 23, 24]. As a special case of a more general formula derived in [22], the superconformal index of $\mathcal{N} = 2\ U(N)$ CS theory with g adjoint chiral multiplets is given by the following expression involving a sum over magnetic flux sectors on the S^2 and integration over the maximal torus of $U(N)$,

$$
\begin{aligned}
\mathcal{I}_+(x,y) &= \mathrm{Tr}\left[(-)^F e^{-\beta(H-J-R)} x^{H+J} \prod_{I=1}^{g} y_I^{G_I}\right] \\
&= \sum_{s_i \in \mathbb{Z}} x^{\epsilon_0(s)} \prod_{I=1}^{G} y_I^{q(s)G_I} \frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{da_i}{2\pi} e^{-S_{CS}^{(0)}(s,a)} \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} f_{tot}(e^{ima}, x^m, y_I^m)\right] \\
&= \sum_{s_i \in \mathbb{Z}} \left[x^{g(1-h)-1} \prod_{I=1}^{G} y_I^{-G_I}\right]^{\sum_{i<j}|s_{ij}|} \frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{da_i}{2\pi} e^{-ik\sum_i s_i a_i} \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} f_{tot}(e^{ima}, x^m, y_I^m; s)\right]
\end{aligned}
$$

$$\tag{10.1}$$

where $S_{CS}^{(0)}(s,a)$ is the contribution from the supersymmetric CS action evaluated on the localized solutions, $\epsilon_0(s)$ and $q(s)G_I$ are the zero-point energy and global symmetry charges in the magnetic flux sector s. Their explicit expressions are given by

$$
\begin{aligned}
S_{CS}^{(0)}(s,a) &= ik\sum_{i=1}^{N} s_i a_i, \\
\epsilon_0(s) &= \left[\frac{g}{2}(1-h) - \frac{1}{2}\right]\sum_{1\le i,j\le N}|s_i - s_j|, \\
q(s) &= -\frac{1}{2}\sum_{1\le i,j\le N}|s_i - s_j|.
\end{aligned}
$$

$$\tag{10.2}$$

The exponential involving $f_{tot}(e^{ia}, x, y_I)$ is the 1-loop determinant from the vector multiplet

and the chiral multiplets. They are given by

$$f_{tot}(e^{ia}, x, y_I; s) = f_V(e^{ia}, x; s) + g f_\Phi(e^{ia}, x, y_I; s),$$

$$f_V(e^{ia}, x; s) = - \sum_{1 \le i \ne j \le N} e^{i(a_i - a_j)} x^{|s_i - s_j|},$$

$$f_\Phi(e^{ia}, x, y_I; s) = \sum_{1 \le i, j \le N} e^{i(a_i - a_j)} x^{|s_i - s_j|} \left[\frac{x^h}{1 - x^2} \prod_I y_I^{G_I} - \frac{x^{2-h}}{1 - x^2} \prod_I y_I^{-G_I} \right]$$

(10.3)

In the large k limit, the contribution to the index from operators with finite dimension (namely dimension that does not scale with k) comes from the $s = 0$ sector only. This part of the index is given by the integral formula

$$\mathcal{I}^{s=0}(x, y) = \frac{1}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{da_i}{2\pi} \exp \left[\sum_{m=1}^\infty \frac{1}{m} f_{tot}(e^{ima}, x^m, y_I^m; 0) \right]$$

$$= \frac{2^{N(N-1)}}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{da_i}{2\pi} \left[\prod_{i<j} \sin^2(\frac{a_{ij}}{2}) \right] \exp \left[g \sum_{m=1}^\infty \frac{1}{m} f_\Phi(e^{ima}, x^m, y_I^m; 0) \right]$$

(10.4)

where

$$f_\Phi(e^{ia}, x, y_I; 0) = \sum_{1 \le i, j \le N} e^{ia_{ij}} \left[\frac{x^h}{1 - x^2} \prod_I y_I^{G_I} - \frac{x^{2-h}}{1 - x^2} \prod_I y_I^{-G_I} \right].$$

(10.5)

Let us define $z \equiv x^{h-1} \prod_I y_I^{G_I}$, then the index in the large k limit depends on x and z only. We can write $\mathcal{I}^{s=0}(x, y)$ as

$$\widetilde{\mathcal{I}}(x|z) = \frac{2^{N(N-1)}}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{da_i}{2\pi} \left[\prod_{i<j} \sin^2(\frac{a_{ij}}{2}) \right] \exp \left[g \sum_{m=1}^\infty \frac{1}{m} \sum_{i,j} e^{ima_{ij}} \frac{x^m}{1 - x^{2m}} (z^m - z^{-m}) \right]$$

$$= \frac{2^{N(N-1)}}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{da_i}{2\pi} \left[\prod_{i<j} \sin^2(\frac{a_{ij}}{2}) \right] \prod_{n=0}^\infty \prod_{i,j} \left(\frac{1 - z^{-1} x^{2n+1} e^{ia_{ij}}}{1 - z x^{2n+1} e^{ia_{ij}}} \right)^g$$

(10.6)

One can verify that this indeed agrees with the index derived from the free limit of $\mathcal{N} = 2$ $U(N)$ CS theory with g adjoint chiral multiplets.

Let us focus on the $g = 1$ example, and set the flavor charge $G = 1/2$. In the theory with no superpotential, $W = 0$, $h = h(\lambda)$ is the renormalized R-charge given by $\mathcal{Z}$-minimization when there are no accidental global symmetries. In the case $W = \alpha \mathrm{Tr}\Phi^4$, $h = 1/2$, whereas in the case $W = \mathrm{Tr}\Phi^3$ the renormalized R-charge is $h = 2/3$. Let $\mathcal{I}_{free}(x, y)$ be the index of the $k = \infty$ $W = 0$ theory. Then by writing $\mathcal{I}_{free}(x, y) = \widetilde{\mathcal{I}}_{free}(x|x^{-1}y)$, we can relate the superconformal indices of the three theories in the 't Hooft limit to $\mathcal{I}_{free}(x, y)$ as

$$\mathcal{I}_{W=0}(x, y) = \mathcal{I}_{free}(x, x^{2h(\lambda)-1}y),$$

$$\mathcal{I}_{W=\mathrm{Tr}\Phi^4}(x) = \mathcal{I}_{free}(x, 1),$$

$$\mathcal{I}_{W=\mathrm{Tr}\Phi^3}(x) = \mathcal{I}_{free}(x, x^{\frac{1}{3}}).$$

(10.7)

where

$$\mathcal{I}_{free}(x, y) = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{(1 - x^{\frac{n}{2}}y^{\frac{n}{2}})(1 + x^{\frac{3n}{2}}y^{-\frac{n}{2}})}. \tag{10.8}$$

In particular, we have

$$\mathcal{I}_{W=\mathrm{Tr}\Phi^3}(x) = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{(1 - x^{\frac{2n}{3}})(1 + x^{\frac{4n}{3}})}. \tag{10.9}$$

We can rewrite it in terms of single trace conformal primary contribution $\mathcal{I}^{st}_{W=\mathrm{Tr}\Phi^3}(x)$, through

$$\mathcal{I}_{W=\mathrm{Tr}\Phi^3}(x) = \exp\left[\sum_{n=1}^{\infty} \frac{\mathcal{I}^{st}_{W=\mathrm{Tr}\Phi^3}(x^n)}{n(1 - x^{2n})}\right] \tag{10.10}$$

The explicit expression for $\mathcal{I}^{st}$ is

$$\mathcal{I}^{st}_{W=\mathrm{Tr}\Phi^3}(x) = \frac{1 + x^{\frac{2}{3}}}{2(1 + x^{\frac{4}{3}})} - \frac{1}{2(1 + x^{\frac{2}{3}})} + x^{\frac{4}{3}}$$
$$= x^{2/3} + x^{10/3} - x^4 + x^6 - x^{20/3} + x^{26/3} - x^{28/3} + x^{34/3} - x^{12} + x^{14} + \cdots \tag{10.11}$$

The above equation is the same as the index for the free theory (5.4) upon setting $y \to x^{1/3}$. The index is precisely reproduced by the conjectured BPS spectrum in the theory with $\mathrm{Tr}\,\Phi^3$ superpotential listed in Table (14).

As argued earlier, the $\mathcal{N} = 2$ $g = 1$ theory with no superpotential in the 't Hooft limit has a renormalized R-charge $h(\lambda)$ that approaches $1/4$ where $\mathrm{Tr}\,\Phi^2$ becomes a free field and decouples. Near this point, $\mathrm{Tr}\,\Phi^m$ for $m \leq 7$ are relative superpotential deformations, each of which gives rise to a strongly coupled critical point. More precisely, we have a fixed line parameterized by (a range of) λ for each superpotential deformation $W = \mathrm{Tr}\,\Phi^m$. At such a critical point, the R-charge is renormalized to $h = 2/m$. Its superconformal index is then given by

$$\mathcal{I}_m(x) = \mathcal{I}_{free}(x, x^{\frac{4}{m}-1}) = \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{(1 - x^{\frac{2n}{m}})(1 + x^{2n(1-\frac{1}{m})})}. \tag{10.12}$$

11. Discussion

So, what did we learn about the gravity dual of these large N Chern-Simons-matter theories?

Perhaps the "nicest" theories we studied are the $\mathcal{N} = 3$ theory with one adjoint hypermultiplet and the $\mathcal{N} = 2$ superpotential deformed theories with two adjoint chiral multiplets and with $U(1)$ or no flavor symmetry. We found that their supersymmetric spectrum consists of only operators of spin ≤ 2, suggesting a possible supergravity dual in the strong coupling limit. In the $\mathcal{N} = 3$ case, while part of the supersymmetric spectrum looks like the Kaluza-Klein spectrum of 7-dimensional supergravity compactified on S^3, there is an additional tower of states in spectrum that do not seem to come from standard

KK modes. In the $\mathcal{N} = 2$ deformed theories, the spectrum contains states of arbitrarily high $U(1)$ charges, suggesting that they could come from KK modes of S^1-compactification of supergravity theories, but to identify their duals appears difficult due to some unusual features of the spectrum.

The $\mathcal{N} = 2$ theories with one adjoint chiral multiplet are even more intriguing. With either $\text{Tr}\,\Phi^4$ or $\text{Tr}\,\Phi^3$ superpotential, there is a line of fixed points. At these fixed point theories, in the large N limit, the supersymmetric spectrum involves a single tower of operators/states of arbitrarily high spin as well as R-charge. This rules out the possibility of a supergravity dual, but leaves open the possibility that the duals of the strongly coupled SCFTs are higher spin theories of gravity in AdS_4.

The most mysterious case is the $\mathcal{N} = 2$ theory with one adjoint chiral multiplet and no superpotential. The R-charge of this theory is renormalized and decreases monotonically with the 't Hooft coupling λ. At some point, when $\lambda = \lambda_2^f \approx 1.23$, the operator $\text{Tr}\,\Phi^2$ becomes a free field and decouples from the theory. At this point, a new $U(1)$ global symmetry emerges and in principle the $\mathcal{Z}$-minimization prescription no longer determines the superconformal R-charge. If we assume that the naive $\mathcal{Z}$-minimization is still valid at large N for $\lambda > \lambda_2^f$, then we find that the renormalized R-charge approaches zero asymptotically at strong coupling. If this is true, apart from the decoupled free fields, the BPS spectrum involves a discretum of states starting at dimension $\Delta = 1/2$. While at general λ the BPS spectrum consists of towers of states of arbitrarily high spin and R-charge, the R-charge form a discretum at strong coupling, suggesting that a new noncompact dimension emerges in the higher spin gravity dual.

Finally, in the cases with more than two adjoint flavours, the number of supersymmetric states grow exponentially with the dimension. It suggests that their dual theories are string theories in AdS_4 with an exponentially growing tower of *supersymmetric* string oscillator excitations. The superconformal index of these theories as a functional of the chemical potential undergoes a phase transition. After this phase transition, these theories are likely dual to supersymmetric black holes in the yet to be determined dual string theories in AdS_4.

Let us comment briefly on brane constructions for the $\mathcal{N} = 2$ $U(N)$ Chern-Simons theory coupled to one adjoint chiral matter with no superpotential.[27] This theory can be embedded in type IIB string theory by suspending N D3-branes between an NS5-brane and a $(1, k)$ 5-brane. One takes the NS5-brane to extend in 012456 directions, and take the $(1, k)$ 5-brane to extend in 01245 directions and at an angle in the $6 - 9$ plane in order to preserve $\mathcal{N} = 2$ supersymmetry. The D3-branes extend in 0123 directions, and are free to move in the $4 - 5$ plane.

To connect this brane configuration to the more familiar setup of [30], one should deform it by rotating the $(1, k)$ 5-brane in the $4 - 7$, $5 - 8$ and $6 - 9$ planes in such a way that the $\mathcal{N} = 2$ supersymmetry is preserved; a superpotential mass term is then generated for the adjoint chiral matter multiplet. The s-rule [30] would indicate that supersymmetry is spontaneously broken if $N > k$, i.e. $\lambda > 1$. It has been observed in [31] that this is

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[27]We thank Ofer Aharony and Daniel Jafferis for discussions on this point.

consistent with supersymmetry being preserved by the undeformed theory.

Alternatively, the $\mathcal{N} = 2$, $W = 0$ theory may also be embedded as the world volume theory of N M5-branes wrapped on a special Lagrangian lens space $S^3/\mathbb{Z}_k$ in a Calabi-Yau 3-fold [5]. The M5-brane extends in an $\mathbb{R}^{1,2}$ in the $\mathbb{R}^{1,4}$. It has been noted in [5] that, however, finding the gravity dual by taking the decoupling limit from this brane construction is difficult.

Acknowledgments

Appendices

A. Details of numerics and plots

In this appendix we briefly describe the numerical technique we have used to determine the R-charge, $h(\lambda)$, of the chiral multiplets in the theory with g chiral multiplets and no superpotential.

As we have explained above, the function $h(\lambda)$ is determined by the solution of the equations (4.3) and (4.4). The basic idea is to determine $h(\lambda)$ by solving those equations numerically. This procedure has two possible pitfalls

- The equations (4.3) and (4.4) correctly determine $h(\lambda)$ only in the large N limit. Numerically, however, it is feasible to solve these equations only at finite N. It is important to check that our results do not change substantially upon increasing N.

- The equations (4.3) and (4.4) could admit multiple solutions; we need to help the numerical solving procedure to focus on the correct solution. We achieved this as follows. At small λ we used as an input guess the results of our perturbative computation as our initial guess for the equation solving technique, and then increased λ in small steps. At every subsequent step we used the result of the previous step as

our input guess. We ensure, by this procedure, that we always zoom into the correct saddle point, atleast in a finite neighbourhood of $\lambda = 0$. It is of course possible that Jafferis' matrix integral (and hence the field theory) undergoes a large N phase transition at finite $\lambda = \lambda_c$. If this indeed does happen then all results of this paper are valid only for $\lambda < \lambda_c$. We leave the investigation of possible phase transitions in this path integral to future work.

In actual practice we found it easier to solve (4.3) but not (4.4) numerically. This solution determines $|\mathcal{Z}|$ as a function of $h(\lambda)$. We actually proceeded to evaluate $|\mathcal{Z}|$ for 60 closely spaced trial values of $h(\lambda)$ and then estimated $h(\lambda)$ by the value (of our 60 trial point) that minimizes $|\mathcal{Z}|$, thus effectively solving (4.4). We performed all our numerics using Mathematica).

In the rest of this appendix we will present evidence that the results of our numerical routine are reliable. To start with, in Fig. 8 we present a plot of $h(\lambda)$ versus λ, obtained from our numerical routine, with $Ne = (10, 20, 30, ..., 100)$ in the range $\lambda \in (0, 4)$. As is apparent from the Fig 8, the result changes substantially from $Ne = 10$ (the lowest graph) to $Ne = 20$ (the second lowest graph), but appears to converge to a limit curve for $Ne \geq 30$ or so. The lesson of this exercise is that numerics with $Ne \geq 30$ are rather reliable for $\lambda < 4$.

In order to understand the convergence of $h(\lambda)$ as N is taken to ∞, we present a plot of $h(\lambda)$ vs. Ne at $\lambda = 4.0$ and we also best fit our data to

$$h = a + \frac{b}{c + N^2}.$$

Note that the best fit seems to agree rather well with the data indicating that the error in the $N \to \infty$ limit scales as $1/N^2$. As we have explained in the appendix, we have performed a similar best fit of our data (as a function of N) for all values of λ, and have used this best fit value to generate the curves presented in Section 4.4. This bestfitting procedure appears to work rather well for every $\lambda \in (1, 10)$.

It is important that generating accurate results at large λ requires larger values of Ne. This is seen in Fig.11 where $h(\lambda)$ is plotted against λ for $Ne = 20, 30, ...100$. Compairing Fig(7) and Fig(12) that while for $\lambda \sim 4$ Ne=30 seems good enough, for $\lambda \sim 10$ one has to go to $Ne \geq 60$ for results reliable upto a few percent of accuracy. This is also apparent from the curves displayed in Fig. 8 and 11.

Though we have not performed a serious estimate of errors in these calculations, we can crudely estimate the errors in our procedure as follows. Let us define

$$h_{Ne}^{corr}(\lambda) = h_{Ne}(\lambda) - \frac{a(\lambda)}{c^2(\lambda) + 100^2} \tag{A.1}$$

Here $h_{ne}(\lambda)$ is the raw data for h obtained from a numerical run with N_e eigenvalues, and the subtraction represents the best fit correction for finite N effects. $h_{Ne}^{corr}(\lambda)$ differs from $h(\lambda)$, the value of h obtained from bestfitting our results at $Ne = 20 \ldots 100$. The difference between $h_{100}^{corr}(\lambda)$ and h_λ may be taken as a crude estimate of the errors in our results. In
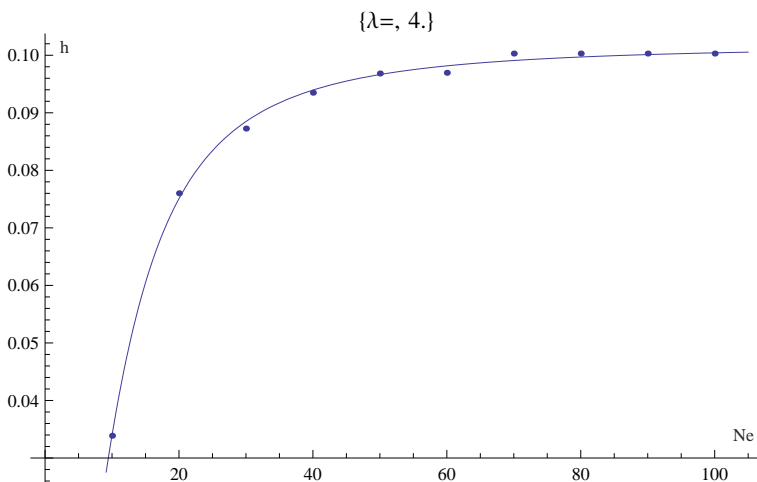
Figure 7: $h(\lambda)$ vs. Ne at $\lambda = 4.0$. Data best fit to $a + b/(c + N^2)$, a, b and c were found to be 0.102518, -16.9934 and 239.4509 respectively. Note that the fit seems rather good.
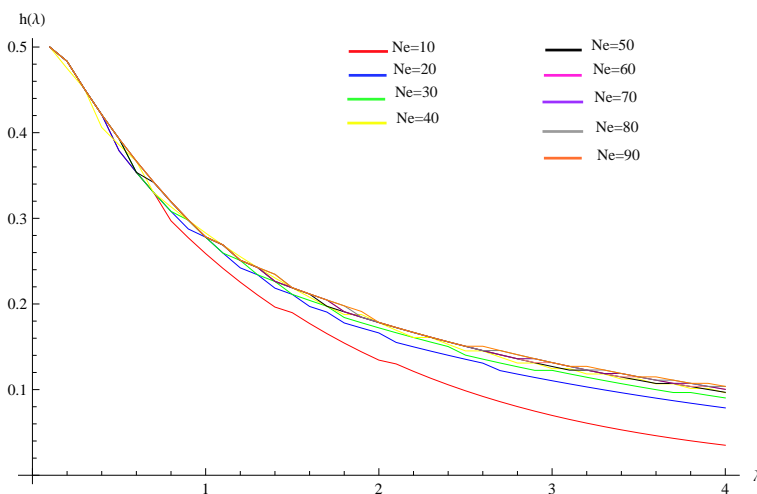

Figure 8: Numeric plot of h vs. λ for λ upto 4, at $g = 1$.

Fig 9 we present a plot of these two functions versus λ. Note that they agree very closely for $\lambda \in (1, 10)$. More quantitatively, in Fig. 10 we have plotted the fractional error.

$$\frac{h(\lambda) - h_{100}^{corr}(\lambda)}{h(\lambda)}.$$

Note that all errors lie within three percent. This is the basis of our belief that our results for $h(\lambda)$ are accurate to within a few percent.

As another comparison of our numerical results versus those of perturbation theory, in Fig. 13 we have plotted $h(\lambda)$ (both numerical and perturbative) against λ at small λ. We have plotted our numerical results at $Ne = 10, 14, 18, 22, 26, 30$. As is apparent from
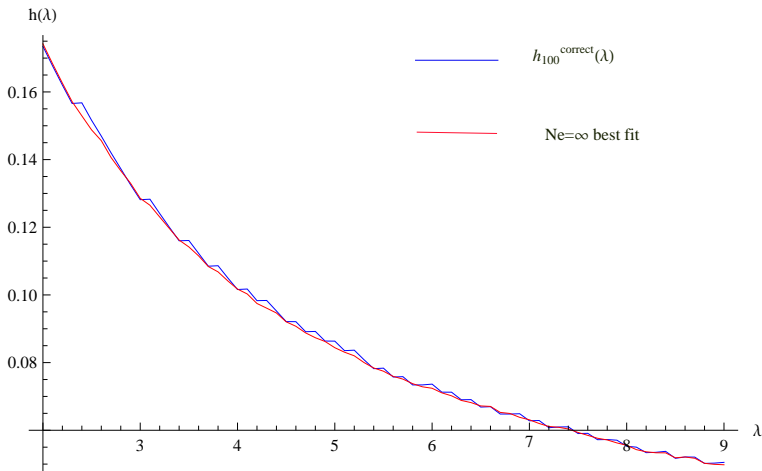
Figure 9: $h_{100}^{corr}(\lambda)$ data(Blue curve) and $h_{\infty}(\lambda)$(red curve) best fit as a function of λ
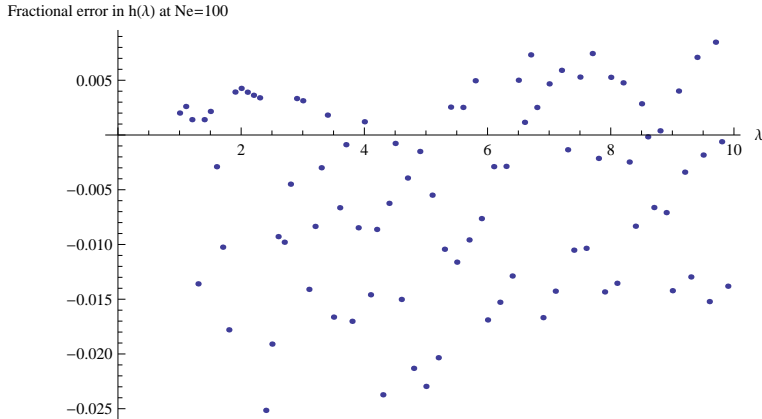

Figure 10: Fractional error in $Ne = 100$ data as a function of λ

the graph, the numerical results converge towards the perturbative values at large Ne; the agreement with perturbation theory is already rather good at $Ne = 20$.

As a more sensitive test of our numerics we next compare the *eigenvalue distributions* obtained from perturbation theory to those obtained from our numerics. A scatter plot of the eigenvalues (on the complex plane) generated by the numerics for $Ne = 30$ at $\lambda = 0.06$ is presented in Fig 14. As is visually apparent, the eigenvalues lie in a straight line. The angle of this cut turns out, numerically, to be $\frac{\pi}{4} - 0.014$ radians and its magnitude (crudely estimated by the distance of the largest eigenvalue from the origin plus a rough correction [28] is numerically given by approximately 0.17 (this value is obtained by fitting the observed eigenvalue density function to the Weigner form). This compares reasonably well with the perturbative prediction of the angle of the line ($\frac{\pi}{4} - 0.016$) and magnitude ($\sqrt{\frac{2}{\pi}\lambda} = 0.19$).

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[28]We estimate the correction as follows. Given 30 eigenvalues distributed according to the Wigner distribution. The last eigenvalue in such a distribution will not be located at $x = a$ by instead, most
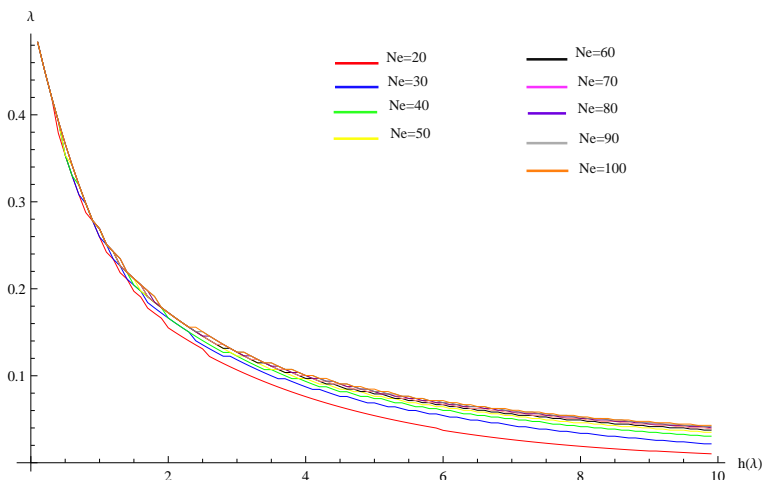
Figure 11: Numeric plot of h vs. λ for λ upto 10, at $g = 1$, with different Ne.
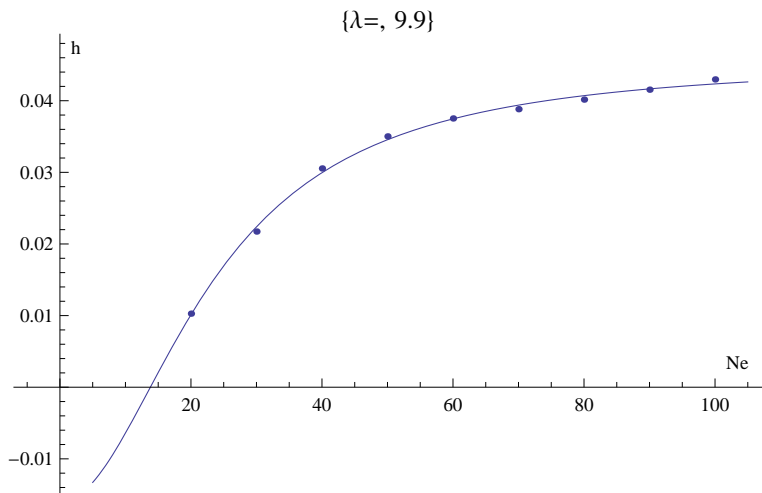

$\{\lambda=, 9.9\}$

Figure 12: $h(\lambda)$ vs. Ne at $\lambda = 9.9$. Data best fit to $a + b/(c + N^2)$, a, b and c were found to be 0.0455, -33.3054 and 541.42 respectively. Note again that the fit seems rather good.

The crude comparison reported above can be improved by bestfitting the results from various different value of Ne; we will not pause to do so here.

As a final check on our numerics, we have used our numerical routine to compute $h(\infty)$ at $g = 10$ with $Ne = 20$. Numerically we found $h = 0.458$. This compares rather well with the prediction of our large g perturbative expansion, $h = 0.46$.

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probably, at $x = a - \frac{y}{2}$, where y is the solution to the equation

$$\int_y^a \rho(y) = \frac{1}{30}.$$

In the situation at hand the distance of the largest eigenvalue form the origin was approximately 0.17 while we estimated the shift by
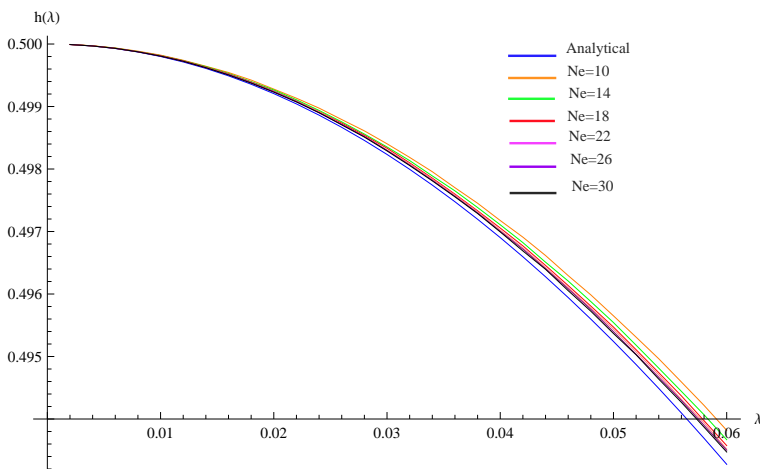
Figure 13: Comparison of perturbative result and numeric results (different Ne) for small λ.
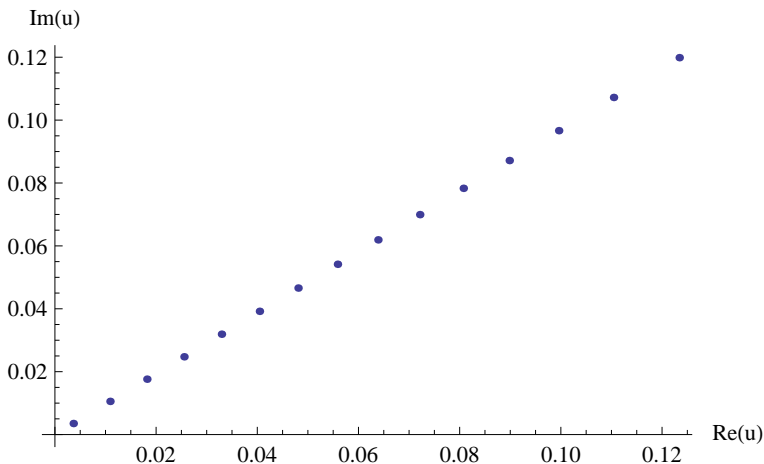

Figure 14: Scatter plot (on the complex plane) of the eigenvalue distribution obtained at $Ne = 30$ and $\lambda = 0.06$.

B. Cohomology calculation

In this appendix we explain the calculation of Q cohomology in more detail. For definiteness, consider the calculation of Q cohomology in the theory with a $U(N)$ gauge group with a single chiral adjoint matter with a superpotential $\operatorname{Tr} \Phi^4$ as given in section 6.3. Since Q carries quantum numbers $(\Delta, j, h) = (\frac{1}{2}, -\frac{1}{2}, 1)$, action of Q does not change the value of $\Delta + j$. Hence cohomology can be calculated independently for each $\Delta + j$ sector. For a fixed $\Delta + j$, operators can be arranged into "levels". The level of an operator is just twice the angular momentum. Level 0 operators are made only of ϕ and are of the form $\operatorname{Tr}(\phi^n)$. Level 1 operators are of the form $\operatorname{Tr}(\phi^n \bar{\psi})$ and so on.

By using Mathematica one can construct all the states at a given $\Delta + j$ and level. Also, by using Mathematica a state at level $k + 1$ say $|\xi\rangle$ can be acted upon by Q and

decomposed into a linear combination of states at level k. Thus a matrix $Q_{k+1,k}$ can be constructed whose rows correspond to level $k+1$ operators and columns correspond to operators at level k. Then the number of states in Q cohomology at level $k+1$ will be naively given by $N_1(k+1, \Delta + j) = (\# \text{ of states at level } k+1) - \text{rank}(Q_{k+1,k}) - \text{rank}(Q_{k+2,k+1})$. But to remove the conformal descendant operators, i.e operators ξ which are of the form $D_{++}|\xi'\rangle = |\xi\rangle$, one should further subtract by all allowed $|\xi'\rangle$ states. Therefore the total number of states in cohomology at level $k+1$, at a given $\Delta + j$ is $N_{\text{cohomology}}(k+1, \Delta + j) = N_1(k+1, \Delta + j) - N_1(k-1, \Delta + j - 2)$. This is because, in this theory D increases the level by 2 and has $\Delta = 1, j = 1$.

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