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A NOTE ON THE SMOOTHNESS OF THE MINKOWSKI FUNCTION

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Abstract. The Minkowski function is a crucial tool used in the study of balanced domains and, more generally, quasi-balanced domains in several complex variables. If a quasi-balanced domain is bounded and pseudoconvex then it is well-known that its Minkowski function is plurisubharmonic. In this short note, we prove that under the additional assumption of smoothness of the boundary, the Minkowski function of a quasi-balanced domain is in fact smooth away from the origin. This allows us to construct a smooth plurisubharmonic defining function for such domains. Our result is new even in the case of balanced domains.

1. Introduction

The study of holomorphic mappings between balanced and quasi-balanced domains pose an interesting challenge. As the automorphism group contains the circle, such domains possess symmetry that often confers strong rigidity on holomorphic mappings between these domains. Indeed, a classical result of Cartan exploits the circle action to show that any automorphism of a bounded balanced domain fixing the origin must be linear. One of the key tools that facilitate the study of balanced and quasi-balanced domains is the Minkowski function. Several generalizations of Cartan's theorem are now known ([Bel82, BP00, Kos14, YZ17]), and many of them use the Minkowski function as a central tool in the proofs. The demand of the presence of a circle action is also not too severe and there are several interesting classes of domains that are quasi-balanced. For instance, the symmetrized polydisk and related domains are quasi-balanced domains that have been extensively studied using the Minkowski function (see [Nik06, Kos11]).

Let $p_1, p_2, \ldots, p_n$ be relatively prime positive integers. We say that a domain $D \subset \mathbb{C}^n$ is $(p_1, p_2, \ldots, p_n)$-*balanced (quasi-balanced)* if

$$\lambda \bullet z \in D \ \forall \lambda \in \overline{\mathbb{D}} \ \ \forall z \in D,$$

where $\overline{\mathbb{D}}$ is the closed unit disk in $\mathbb{C}$ and for $z = (z_1, z_2, \ldots, z_n) \in D$, we define $\lambda \bullet z := (\lambda^{p_1} z_1, \lambda^{p_2} z_2, \ldots, \lambda^{p_n} z_n)$. If $p_1 = p_2 = \cdots = p_n = 1$ above, then we say D is a *balanced domain* (also known as a *complete circular domain* in the literature).

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Given a $(p_1, p_2, \ldots, p_n)$-balanced domain $D \subset \mathbb{C}^n$, we define the Minkowski function $\mathfrak{h}_D : \mathbb{C}^n \to \mathbb{C}$

$$\mathfrak{h}_D(z) := \inf\{t > 0 : \frac{1}{t} \bullet z \in D\}.$$

Clearly $D = \{z \in \mathbb{C}^n : \mathfrak{h}_D(z) < 1\}$ and $\mathfrak{h}_D(\lambda \bullet z) = |\lambda| \mathfrak{h}_D(z)$. It also turns out that $\mathfrak{h}_D$ is plurisubharmonic if D is additionally psuedoconvex. This fact has been a crucial ingredient in several results on balanced domains; see [Ham00, JP13], for instance.

One natural question that seems to be unanswered (to the best of the authors' knowledge) in the literature is the following:

Is the Minkowski function of a smoothly bounded psuedoconvex quasi-balanced domain a smooth function near the boundary?

In fact, we found a remark in [GK03, p. 190], with a reference to Hamada's paper [Ham00], stating that the answer to the above question is *no* if the domain has only a $\mathcal{C}^1$-boundary. That the Minkowski function of a balanced and bounded pseudoconvex domain with $\mathcal{C}^1$-smooth plurisubharmonic defining function is $\mathcal{C}^1$-smooth on $\mathbb{C}^n \setminus \{0\}$ has already been established in [Ham00, Proposition 1]. Using the recent work [NZZ17], we are able to prove smoothness of the Minkowski function on $\mathbb{C}^n \setminus \{0\}$ for any smoothly bounded quasi-balanced domains. The main result of this paper is the following

Theorem 1. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded quasi-balanced pseudoconvex domain. Then the Minkowski function $\mathfrak{h}_D$ is $\mathcal{C}^\infty$-smooth on $\mathbb{C}^n \setminus \{0\}$. Furthermore, the function $r(z) := \mathfrak{h}_D(z) - 1$ is a plurisubharmonic defining function for D.*

Remark 2. By a smoothly bounded domain, we shall mean a bounded domain whose boundary is $\mathcal{C}^\infty$-smooth.

Remark 3. The analogue of the above result for convex domains is well-known. The reader is referred to [KP99, Section 6.3] for details

2. Supporting results

Before we give the proof of Theorem 1, we first give a brief overview of the necessary tools.

We shall now consider the setting in [NZZ17, p. 518, p. 523]. Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain and let $G \subset \mathsf{Aut}(D) \cap \mathcal{C}^\infty(\overline{D})$ be a compact Lie subgroup of $\mathsf{Aut}(D)$ in the compact open topology. Consider a continuous representation $\rho : G \to GL(\mathbb{C}^n)$ of G and the set

$$\mathcal{O}(\mathbb{C}^n)^G := \{f \in \mathcal{O}(\mathbb{C}^n) : f \circ \rho(g) = f \text{ for all } g \in G\}$$

called the set of G-invariant entire functions.

A domain D is said to be G-invariant if $\rho(g) \cdot D = D$ for all $g \in G$. We will say that G acts transversely on D if for each $z_0 \in \partial D$ the image of the tangent map $d\Psi_{z_0} : T_e G \to T_{z_0} \partial D$ associated to the map $\Psi_{z_0} : G \to \partial D$ given by $g \mapsto g(z_0)$, is not contained in $T^{\mathbb{C}}_{z_0} \partial D$, the complex tangent space to ∂D at z_0. We have the following

Result 4 (Theorem 2.7 in [NZZ17])**.** *Let G be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^G = \mathbb{C}$. If D is a G-invariant smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ that contains the origin, then G acts transversely on D.*

Consider the representation of the compact lie group $\mathbb{S}^1$ given by

$$\rho(\lambda)(z) = \lambda \bullet z \text{ where } \lambda \in \mathbb{S}^1.$$

Proposition 5. *Under the above action, $\mathcal{O}(\mathbb{C}^n)^{\mathbb{S}^1} = \mathbb{C}$.*

Proof. Consider $f \in \mathcal{O}(\mathbb{C}^n)$ such that $f(\lambda \bullet z) = f(z)$ for every $\lambda \in \mathbb{S}^1$ and for all $z \in D$. Fix $z \in \mathbb{C}^n$ and define a function $g_z : \mathbb{C} \to \mathbb{C}$ given by $g_z(\lambda) = f(\lambda \bullet z)$. Then g is a holomorphic function that is constant on $\mathbb{S}^1$ and hence $g_z \equiv g(0)$. Since our choice of z was arbitrary, we have $f(z) = f(0)$. The constant functions clearly belong to $\mathcal{O}(\mathbb{C}^n)^{\mathbb{S}^1}$. □

That D is $\mathbb{S}^1$-invariant is a direct consequence of the fact that D is $(p_1, p_2, \ldots, p_n)$-balanced. Thus in our case, we can conclude the following

Corollary 6. *Under the hypotheses on D as in Theorem 1, for each $\xi = (\xi_1, \ldots, \xi_n) \in \partial D$, the vector*

$$(ip_1\xi_1, \ldots, ip_n\xi_n) \notin T^{\mathbb{C}}_{\xi}\partial D.$$

Proof. With $\Psi_\xi : \mathbb{S}^1 \to \partial D$ given by $\Psi_\xi(\lambda) = \lambda \bullet \xi$, the evaluation of the derivative map $d\Psi_\xi(1) = (ip_1\xi_1, \ldots, ip_n\xi_n) \in T_\xi\partial D$. By Result 4, $d\Psi_\xi(1) \notin T^{\mathbb{C}}_{\xi}\partial D$ as otherwise $d\Psi_\xi(T_e\mathbb{S}^1) \subset T^{\mathbb{C}}_{\xi}\partial D$. □

We will use the following version of Hopf's lemma in the proof of Theorem 1.

Lemma 7 (Lemma 3, p. 177, [KG89])**.** *Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain and let r be a negative plurisubharmonic function defined on D. Then there exists a constant $c > 0$ such that $|r(z)| > c \cdot dist(z, \partial D)$.*

3. Proof of Theorem 1

Let ψ be a defining function for D. Consider the map $g \in \mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{R} \setminus \{0\})$ given by

$$g(z,t) := \psi\left(\frac{1}{t} \bullet z\right)$$

Observe that $g(z, \mathfrak{h}_D(z)) = 0$. Let us fix a point $z_0 \in \mathbb{C}^n \setminus \{0\}$. We shall show that $\frac{\partial g}{\partial t}|_{(z_0, \mathfrak{h}_D(z_0))} \neq 0$.

Let us denote the coordinates of z_0 by $(z_1, \ldots, z_n)$. Then the point $\xi = (\xi_1, \ldots, \xi_n)$ defined to be $\frac{1}{\mathfrak{h}_D(z_0)} \bullet z_0$ belongs to ∂D. A direct calculation gives us that

$$\frac{\partial g}{\partial t}|_{(z_0, \mathfrak{h}_D(z_0))} = \frac{-1}{\mathfrak{h}_D(z_0)} \left(\frac{\partial \psi}{\partial z_1}, \ldots, \frac{\partial \psi}{\partial z_n}\right)|_\xi \cdot \begin{pmatrix} p_1\xi_1 \\ \vdots \\ p_n\xi_n \end{pmatrix}$$

If $\frac{\partial g}{\partial t}|_{(z_0,\mathfrak{h}_D(z_0))} = 0$, then $(p_1\xi_1, \dots, p_n\xi_n) \in T_\xi \partial D$. Consider the curve

$$\gamma(\theta) = e^{i\theta} \bullet \xi$$

in ∂D. Then the corresponding tangent vector $(ip_1\xi_1, \dots, ip_n\xi_n) \in T_\xi \partial D$ and hence is in the complex tangent space $T_\xi^{\mathbb{C}} \partial D$ which is a contradiction to Corollary 6. Now by the implicit function theorem, $\mathfrak{h}_D$ is $\mathcal{C}^\infty$-smooth on $\mathbb{C}^n \setminus \{0\}$.

We shall now prove that r is a defining function. We are left with observing that $dr \neq 0$ on ∂D. It is easy to see that the normal derivative at every point on the boundary ∂D is bounded below by the constant c by an application of Hopf's lemma (Lemma 7). Hence $dr \neq 0$ on ∂D. □

Our result implies that the main results in [Ham00, HK01] on balanced domains with $\mathcal{C}^1$-smooth plurisubharmonic defining function also hold for smoothly bounded balanced pseudoconvex domains.

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