Fibers on a graph with local load sharing

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Abstract

We study a random fiber bundle model with tips of the fibers placed on a graph having co-ordination number 3. These fibers follow local load sharing with uniformly distributed threshold strengths of the fibers. We have studied the critical behaviour of the model numerically using a finite size scaling method and the mean field critical behaviour is established. The avalanche size distribution is also found to exhibit a mean field nature in the asymptotic limit.

1 Introduction

Collapse of buildings, failure of networks and many other breakdown events have resulted into a great deal of research to understand the cause of such disasters and also forced the scientists to invent measures to prevent such mishappenings[1]. Fiber Bundle Model (FBM) is a step towards capturing the essential physics of breakdown phenomena [2, 3, 4].

A FBM consists of N parallel elastic fibers with identical elastic constant but randomly distributed threshold strengths chosen from a distribution. This random distribution of threshold strengths (σ_{th}) introduces randomness in the model and hence the model is called Random Fiber Bundle Model (RFBM). When the stress generated due to the externally applied force exceeds the threshold strength of a given fiber, the fiber breaks. The additional load due to the failure of fibers is distributed to the remaining intact fibers by a load sharing rule. In a Global Load Sharing (GLS) scheme, the broken fibers redistribute their stress to all the intact fibers. This is a mean field model with an effective long range interaction amongst the intact fibers. The critical exponents have been obtained for such a model[5]. We investigate here the other extreme load sharing rule, namely Local Load Sharing (LLS)[6, 7, 8]. In a local load sharing rule, the broken fiber gives its load only to the nearest intact fibers. Therefore, this load sharing depends upon the dimensionality of the system concerned. As the external force is increased, more fibers break. The redistribution of the stress of a broken fiber for a given applied load takes place until a fixed point is reached when no more failures take place. If the external load exceeds a particular load called *critical load*, the complete breakdown of the bundle takes

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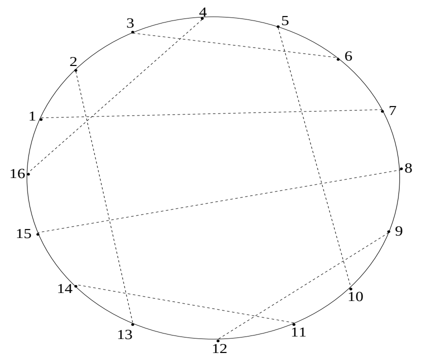

Figure 1: An example of a graph with coordination number 3 and 16 sites.

place. The determination of this critical load and the study of the associated phase transition are of prime interest.

It is well established that the fibers with LLS in a one-dimensional regular lattice do not show any critical behaviour and the critical stress σ_c (=critical load per fiber) goes to zero in the thermodynamic limit[6]. Models with load sharing rule interpolating between GLS and LLS have also been studied [9]. Kim, Kim and Jeong[10] studied fibers placed on different network models: namely, the Erdos-Renyi model of random network, Watts-Strogatz network and the static model of a scale free network with LLS. They established the mean field (GLS) critical behaviour of fiber bundles residing on these networks. In Ref.[11] the instability introduced in a large scale free network by the triggering of node-breaking avalanches is analyzed using fiber bundle model as the framework. It should be noted that FBM can be related to a fuse model where breakdown of a fuse is caused due to a large applied voltage and the system is eventually driven to a completely broken state as the external voltage is increased beyond the critical voltage. The breaking profiles of such a model has also been studied on a scale free network[12]. Under certain load modes, this breaking profile resembles the stress-strain curve of the dynamics of the fibers.

In this paper we look at the dynamics of fibers following local load sharing rule when placed on a graph with co-ordination number 3. We will point out how our model is different from the above models later in this paper.

2 The model

Dhar, Shukla and Sethna [13] studied a Random Field Ising Model on a Bethe lattice using a particular random graph[14] for simulating the Bethe lattice structure. We have used the same approach to generate a geometry which has connections as described below.

• Label the sites by integers from 1 to N where N is even.

• Connect site i to $i+1$ for all i such that site N is connected to site 1 (periodic boundary condition). We call this connection a direct connection. Thus we have a ring of N sites.

• Now connect each site i randomly to a unique site other than $i+1$ and $i-1$ thus forming $N/2$ pairs of sites. This connection is called random connection.

Random connection is unique for a given site. For example, in Fig. 1 where we show a particular configuration, site number 14 is directly connected to 13, 15 and randomly connected to 11 which is unique for site number 14. At the same time, random connection of site 11 then becomes 14. Thus any site will have two directly connected neighbours and one unique random neighbour making the coordination number of each site to be three at the beginning of the dynamics. In this construction, all sites are on the same footing. The graph resembles a Newman-Watts (NW) model where nearest neighbour connections are retained and each site is randomly connected to another site with a probability p[15]. But there is no condition of uniqueness of random connection in NW model as in our case. We do not allow the possibility of two random connections emerging from the same site. More technically speaking, it is a regular graph as each vertex has three incident edges. However, two of these edges are nearest and the third is random though unique.

The tips of the fibers are now placed on these N sites and are associated with a threshold strength chosen randomly from a uniform distribution. If p is the density of threshold strength of fibers, then a uniform distribution implies:

$$\begin{aligned} p &= 1 \qquad 0 \leq \sigma_{th} \leq 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

In the passing, let us comment that a graph as generated above resembles Bethe lattice in the thermodynamic limit ($N \to \infty$) [13, 16, 17]. The similarity arises due to the loopless structure of both, the Bethe lattice and the above construction as $N \to \infty$. Although our construction has loops, it can be shown that there are typically very few small loops as $N \to \infty$. But there is a crucial difference between the dynamics of Random Field Ising Model (RFIM) studied by Dhar *et.al* and that of RFBM. In the RFIM, no connection is lost in the course of dynamics whereas in RFBM, random connections disappear as the fibers start to fail under an external load (as explained below).

When an external force is applied, fibers with threshold smaller than the stress generated due to the external force break. The broken fibers have to redistribute the stress carried by them to the nearest intact fibers. We have implemented the following algorithm in this work:

(i) The system is exposed to a total load $N\sigma$ such that fiber at each site i carries a stress $\sigma(i) = \sigma$. If $\sigma_{th}(i) < \sigma(i)$ at any site i, the fiber breaks and its stress is equally distributed to its surviving neighbours (both directly and randomly connected). The breaking condition is examined for each site and is repeated until there is no more breaking for this particular external load.

(ii) The total load is now increased from $N\sigma$ to $N(\sigma + d\sigma)$. The new total load is then shared by the remaining unbroken fibers and once again the breaking condition is checked for each intact fiber.

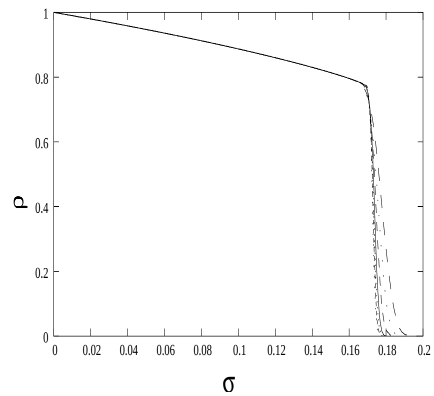

Figure 2: The variation of ρ with σ for different values of N. Here, $N = 2^{13}, 2^{14}, 2^{15}, 2^{16}, 10^5, 2^{17}, 2*10^5, 2^{18}$. The lower σ_c corresponds to a higher value of N.

(*iii*) It may so happen that during the failure dynamics a fiber encounters a situation where all its nearest neighbours are broken. This situation can be treated in two ways. One approach assumes that the stress carried by such a fiber is lost *i.e.*, the total load is not conserved [18]. On the other hand, we study a FBM with total external load being strictly conserved. To ensure this conservation, we propose the following: if a fiber encounters a situation in which one of its directly connected neighbours (in clockwise or anticlockwise direction) is broken, it gives its stress to the next nearest surviving neighbour in that direction. This rule however is not applicable to the randomly connected fiber *i.e.*, if the randomly connected neighbour of a broken fiber is already broken, the stress then is given only to its directly connected neighbours.
(*iv*) Once again, the load is increased and the process is repeated till the complete breakdown of the system.

It is clear from the above load sharing rule that even if the tips of fibers when placed on a graph as mentioned before resemble a Bethe lattice, this resemblance is valid only at time $t = 0$. The broken randomly connected neighbour of site i will reduce the co-ordination number of site i deviating the lattice from its initial geometry. We illustrate the third point of the algorithm with an example from Fig. 1. Let us assume that the fiber at the site 14 is the weakest among all the fibers connected to it. At any instant of time t, fiber at site 14 breaks. The additional load is shared equally among fibers at sites 11, 13 and 15. Consider a situation where at some higher load, fiber at site 11 also breaks; the load carried by 11 is equally shared only between the two directly connected fibers (at sites 10 and 12) if they are intact. Otherwise, the load is given to the next intact fiber along the direction of the broken directly connected fiber. Therefore effectively the long range random connection between 11 and 14 disappears as soon as 14 breaks.

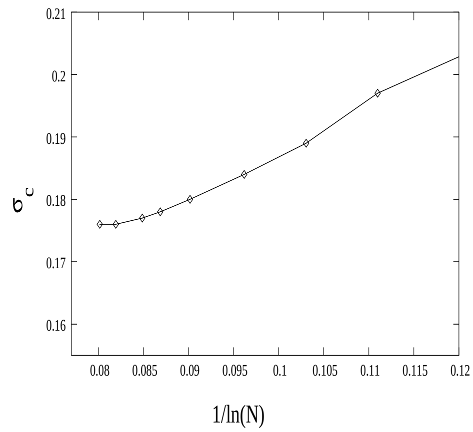

Figure 3: The dependence of critical stress σ_c on the system size N.

In our numerical simulations, $d\sigma$ is taken to be 0.0001 and the average is taken over 10^4 to 10^5 configurations. The variation of fraction of unbroken fibers ρ, with the applied stress is shown in Fig. 2. Also shown is the dependence of critical stress σ_c on N in Fig. 3. Clearly, there exist a non-zero value of σ_c for large N whereas in the case of pure local load sharing in a one-dimensional lattice, σ_c goes to zero in the thermodynamic limit[6]. Here, ρ behaves as the order parameter of the system which vanishes at the critical point.

To determine σ_c precisely, we use the standard finite size scaling method [19] with the scaling assumption

$$\rho(\sigma, N) = N^{-a} f((\sigma - \sigma_c) N^{\frac{1}{\nu}}) \tag{1}$$

where f(x) is the scaling function and the exponent ν describes the divergence of the correlation length ξ near the critical point, *i.e.*, $\xi \propto |\sigma - \sigma_c|^{-\nu}$. Demanding that the order parameter ρ scales as $\rho \propto (\sigma_c - \sigma)^{\beta}$, we arrive at the relation $a = \beta/\nu$. The plots of ρN^a with σ for different N crosses through a unique point $\sigma = \sigma_c$=0.173 with the exponent $a = 0.5$ (See Fig. 4).

With this value of σ_c and a, one can again use the scaling relation (1) to determine the exponent ν by making the data points to collapse to an almost single smooth curve near the critical point as displayed in Fig. (5). The value of the exponent ν obtained in this manner is equal to unity which automatically gives the value of the other exponent β=0.5. These matches with the mean field exponents derived analytically[5]. It should be noted that in Fig. (5), the collapse of data is better for higher system size. The results therefore establish the existence of a sharp thermodynamic transition with mean field (GLS) exponents eventhough the load sharing rule is local apart from random connections.

An important quantity associated with any breakdown process is the avalanche size distribution. An avalanche is defined as the number of fibers broken be-

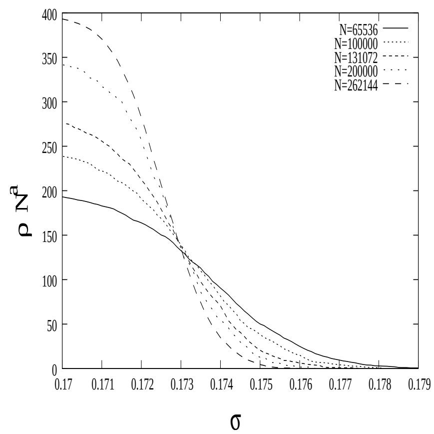

Figure 4: Precise determination of σ_c from the finite size scaling form (1). The σ_c at which the curves intersect is 0.173. $a = 0.5$.

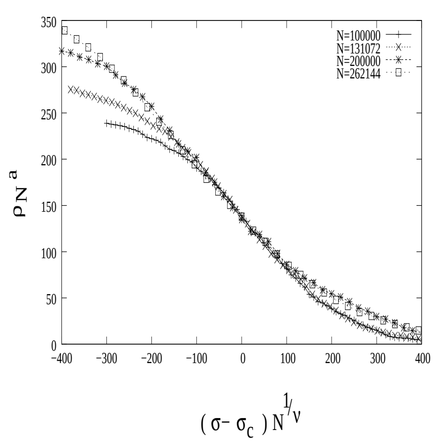

Figure 5: Data collapse with a=0.5, $\nu = 1$

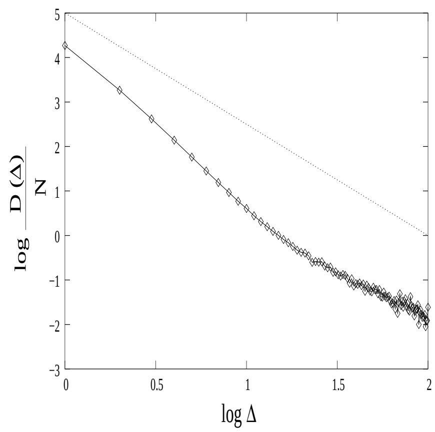

Figure 6: Avalanche size distribution of fibers placed on a random graph. Also drawn is a straight line with slope 5/2. Clearly, the exponent ξ takes the mean field value 5/2 for large Δ.

tween two successive external loadings. For RFBM with GLS, the distribution $D(\Delta)$ of an avalanche of size Δ follows a power law given as [20]

$$D(\Delta) \propto \Delta^{-\zeta}, \text{where } \zeta = 5/2. \tag{2}$$

when Δ is large.

To study the avalanche size distribution in the present model, we have applied the weakest fiber failure approach. In this approach, an external force is increased quasistatically such that only the weakest surviving fiber present in the bundle breaks. The failure of this weakest fiber causes an avalanche of failures of size Δ. The numerical result for the distribution of avalanche size is shown in Fig. 6. The exponent ζ takes the mean field value 5/2 for very large Δ.

3 Conclusion

We have studied a fiber bundle model where tips of fibers are placed on a graph with co-ordination number 3. A local load sharing mechanism is employed for the redistribution of load of the broken fiber to the intact fibers as described in Section II. The exact value of the critical stress and the critical exponents β and ν are obtained using finite size scaling method. These exponents correspond to the mean field (GLS) value as obtained in Ref.[5]. The mean field behaviour is due to a finite fraction of infinite range connection which survive in the process of dynamics.

The avalanche size distribution is studied using weakest fiber failure approach. We observe that the burst avalanche distribution picks up the mean field value $\zeta = 5/2$ only in the asymptotic limit (*i.e.*, for large Δ which occurs near the critical point). For smaller Δ, the behaviour is clearly different from the mean field limit. This suggests that the load sharing rule affects the behaviour of $D(\Delta)$ in the small Δ region. The small Δ-anomaly is not seen in fibers placed on a complex network[10].

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