LOCAL COHOMOLOGY OF MULTI-REES ALGEBRAS, JOINT REDUCTION NUMBERS AND PRODUCT OF COMPLETE IDEALS

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Abstract. We find conditions on the local cohomology modules of multi-Rees algebras of admissible filtrations which enable us to predict joint reduction numbers. As a consequence we are able to prove a generalisation of a result of Reid-Roberts-Vitulli in the setting of analytically unramified local rings for completeness of power products of complete ideals.

1. Introduction

The objective of this paper is to find suitable conditions on the local cohomology modules of multi-Rees algebras and associated graded rings of multigraded admissible filtrations of ideals in an analytically unramified local ring $(R, \mathfrak{m})$ and apply these to detect their joint reduction vectors and completeness of products of complete ideals.

Recall that if R is a commutative ring and I is an ideal of R then $a \in R$ is called integral over I, if a is a root of a monic polynomial $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ for some $a_i \in I^i$ for $i = 1, 2, \ldots, n$. The integral closure of I, denoted by $\overline{I}$, is the set of all elements of R which are integral over I. If $I = \overline{I}$, then I is called complete or integrally closed. O. Zariski [15] proved that product of complete ideals is complete in the polynomial ring $k[x, y]$ where k is an algebraically closed field of characteristic zero. This was generalised to two-dimensional regular local rings in Appendix 5 of [16]. This result is known as *Zariski's Product Theorem.* C. Huneke [6] showed that product of complete ideals is not complete in higher dimensional regular local rings. Since the appearance of this counterexample of Huneke, several results have appeared in the literature which identify classes of complete ideals in local rings of dimension at least 3 whose products are complete. The following result due to L. Reid, L. G. Roberts and M. A. Vitulli [13, Proposition 3.1] about complete monomial ideals is rather surprising:

Theorem 1.1. *Let $R = k[X_1, \ldots, X_d]$ be a polynomial ring of dimension $d \geq 1$ over a field k. Let I be a monomial ideal of R so that I^n is complete for all $1 \leq n \leq d-1$. Then I^n is complete for all $n \geq 1$.*

This can be thought of as a partial generalisation of the Zariski's Product Theorem for $d = 2$. This theorem was proved using tools from convex geometry. In this paper, we approach this result using

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vanishing of local cohomology modules of multi-Rees algebras and prove the following result about completeness of power products of $\mathfrak{m}$-primary monomial ideals.

Theorem 1.2. *Let $R = k[X_1, \ldots, X_d]$ where $d \geq 1$ and $\mathfrak{m} = (X_1, \ldots, X_d)$ be the maximal homogeneous ideal of R. Let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary monomial ideals of R. Suppose $\mathbf{I^n}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{n}| \leq d-1$. Then $\mathbf{I^n}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ with $|\mathbf{n}| \geq 1$.*

We prove the above result as a consequence of a more general result for complete ideals in analytically unramified local rings. In order to state this and other results proved in this paper, we recall certain definitions and set up notation.

Throughout this paper, $(R, \mathfrak{m})$ denotes a Noetherian local ring of dimension d with infinite residue field. Let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R and we denote the collection of these ideals $(I_1, \ldots, I_s)$ by $\mathbf{I}$. For $s \geq 1$, we put $\mathbf{e} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^s$ and for all $i = 1, \ldots, s$, $\mathbf{e_i} = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^s$ where 1 occurs at ith position. For $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, we write $\mathbf{I^n} = I_1^{n_1} \cdots I_s^{n_s}$ and $\mathbf{n}^+ = (n_1^+, \ldots, n_s^+)$ where $n_i^+ = \max\{0, n_i\}$. For $s \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, we put $|\alpha| = \alpha_1 + \cdots + \alpha_s$. We define $\mathbf{m} = (m_1, \ldots, m_s) \geq \mathbf{n} = (n_1, \ldots, n_s)$ if $m_i \geq n_i$ for all $i = 1, \ldots, s$. By the phrase "for all large $\mathbf{n}$", we mean $\mathbf{n} \in \mathbb{N}^s$ and $n_i \gg 0$ for all $i = 1, \ldots, s$.

Definition 1.3. *A set of ideals $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded* **I-filtration** *if for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^s$,* (i) $\mathbf{I^n} \subseteq \mathcal{F}(\mathbf{n})$, (ii) $\mathcal{F}(\mathbf{n})\mathcal{F}(\mathbf{m}) \subseteq \mathcal{F}(\mathbf{n}+\mathbf{m})$ *and* (iii) *if* $\mathbf{m} \geq \mathbf{n}$, $\mathcal{F}(\mathbf{m}) \subseteq \mathcal{F}(\mathbf{n})$.

Let $t_1, \ldots, t_s$ be indeterminates. For $\mathbf{n} \in \mathbb{Z}^s$, we put $\mathbf{t^n} = t_1^{n_1} \cdots t_s^{n_s}$ and denote the $\mathbb{N}^s$-graded **Rees ring of** $\mathcal{F}$ by $\mathcal{R}(\mathcal{F}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} \mathcal{F}(\mathbf{n})\mathbf{t^n}$ and the $\mathbb{Z}^s$-graded **extended Rees ring of** $\mathcal{F}$ by $\mathcal{R}'(\mathcal{F}) = \bigoplus_{\mathbf{n} \in \mathbb{Z}^s} \mathcal{F}(\mathbf{n})\mathbf{t^n}$. For an $\mathbb{N}^s$-graded ring $S = \bigoplus_{\mathbf{n} \geq \mathbf{0}} S_{\mathbf{n}}$, we denote the ideal $\bigoplus_{\mathbf{n} \geq \mathbf{e}} S_{\mathbf{n}}$ by S_{++}. Let $G(\mathcal{F}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} \mathcal{F}(\mathbf{n})/\mathcal{F}(\mathbf{n}+\mathbf{e})$ be the **associated multigraded ring of** $\mathcal{F}$ **with respect to** $\mathcal{F}(\mathbf{e})$. For $\mathcal{F} = \{\mathbf{I^n}\}_{\mathbf{n} \in \mathbb{Z}^s}$, we set $\mathcal{R}(\mathcal{F}) = \mathcal{R}(\mathbf{I})$, $\mathcal{R}'(\mathcal{F}) = \mathcal{R}'(\mathbf{I})$, $G(\mathcal{F}) = G(\mathbf{I})$ and $\mathcal{R}(\mathbf{I})_{++} = \mathcal{R}_{++}$.

Definition 1.4. *A $\mathbb{Z}^s$-graded $\mathbf{I}$-filtration $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ of ideals in R is called an* **I-admissible filtration** *if $\mathcal{F}(\mathbf{n}) = \mathcal{F}(\mathbf{n}^+)$ for all $\mathbf{n} \in \mathbb{Z}^s$ and $\mathcal{R}'(\mathcal{F})$ is a finite $\mathcal{R}'(\mathbf{I})$ -module.*

The principal examples of admissible filtrations with which we are concerned in this paper are (i) the $\mathbf{I}$-adic filtration $\{\mathbf{I^n}\}_{\mathbf{n} \in \mathbb{Z}^s}$ in a Noetherian local ring and (ii) the integral closure filtration $\{\overline{\mathbf{I^n}}\}_{\mathbf{n} \in \mathbb{Z}^s}$ in an analytically unramified local ring. It is proved in [10, Proposition 2.5] that if $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ is an $\mathbf{I}$-admissible filtration of ideals in R then $\mathcal{R}(\mathcal{F})$ is a finitely generated $\mathcal{R}(\mathbf{I})$-module.

Recall that an ideal J contained in an ideal I is called a reduction of I if $JI^n = I^{n+1}$ for all large n. The role that reductions of ideals play in the study of Hilbert-Samuel functions of $\mathfrak{m}$-primary ideals, is played by joint reductions, introduced by D. Rees in [12], of a sequence of $\mathfrak{m}$-primary ideals $I_1, \ldots, I_s$

to study the multigraded Hilbert-Samuel function $H(\mathbf{I},\mathbf{n})=\lambda(R/\mathbf{I}^{\mathbf{n}})$. Let $\mathbf{q}=(q_1,q_2,\ldots,q_s)\in\mathbb{N}^s$ and $|\mathbf{q}|=d\geq 1$. A set of elements $\{a_{ij}\in I_i \mid i=1,2,\ldots,s; j=1,2,\ldots,q_i\}$ is called a joint reduction of the set of ideals $(I_1,\ldots,I_s)$ of type $\mathbf{q}$ if there exists an $\mathbf{m}\in\mathbb{N}^s$ so that for all $\mathbf{n}\geq\mathbf{m}\in\mathbb{N}^s$,

$$\sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij} I_1^{n_1} I_2^{n_2}\ldots I_{i-1}^{n_{i-1}} I_i^{n_i-1} I_{i+1}^{n_{i+1}}\ldots I_s^{n_s}=I_1^{n_1}I_2^{n_2}\ldots I_s^{n_s}.$$

The vector $\mathbf{m}$ is called a joint reduction vector. We estimate joint reduction vectors using local cohomology modules of multi-Rees rings. In order to achieve this we need to work with joint reductions in a more general setting. D. Kirby and Rees [9] generalised it further in the setting of multigraded rings and modules which we recall next.

Definition 1.5. *Let $R=\bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}},\mathfrak{m})$ and $M=\bigoplus_{\mathbf{n}\in\mathbb{Z}^s} M_{\mathbf{n}}$ be a finite $\mathbb{Z}^s$-graded R-module. A* **joint reduction of type q of** *R* **with respect to** *M is a set of elements*

$$\mathcal{A}_{\mathbf{q}}(M)=\{a_{ij}\in R_{\mathbf{e_i}}: j=1,\ldots,q_i; i=1,\ldots,s\}$$

generating an ideal J of R irrelevant with respect to M, i.e. $(JM)_{\mathbf{n}}=M_{\mathbf{n}}$ for all large $\mathbf{n}$.

Kirby and Rees [9] proved existence of joint reduction of type $\mathbf{q}$ of R with respect to M if $|\mathbf{q}|\geq \dim\left(\dfrac{M}{\mathfrak{m}M}\right)+1$ and the residue field $R_{\mathbf{0}}/\mathfrak{m}$ is infinite. Here $\dim R$ is defined to be $\max(\operatorname{ht} P)$ where P ranges over the relevant prime ideals of R if R is not trivial and -1 if R is trivial. For M, $\dim M$ is defined to be $\dim\left(\dfrac{R}{\operatorname{Ann}_R M}\right)$.
Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d\geq 1$, $I_1,\ldots,I_s$ be $\mathfrak{m}$-primary ideals of R and $\mathcal{F}=\{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be a $\mathbb{Z}^s$-graded $\mathbf{I}$-admissible filtration of ideals in R. Let $\mathbf{q}=(q_1,\ldots,q_s)\in\mathbb{N}^s$ such that $|\mathbf{q}|=d$.

Definition 1.6. *A set of elements $\mathcal{A}_{\mathbf{q}}(\mathcal{F})=\{a_{ij}\in I_i: j=1,\ldots,q_i; i=1,\ldots,s\}$ is called a* **joint reduction of** *$\mathcal{F}$ of type $\mathbf{q}$ if the set $\{a_{ij}t_i\in\mathcal{R}(\mathbf{I})_{\mathbf{e_i}}: j=1,\ldots,q_i; i=1,\ldots,s\}$ is a joint reduction of type $\mathbf{q}$ of $\mathcal{R}(\mathbf{I})$ with respect to $\mathcal{R}(\mathcal{F})$, i.e. the following equality holds for all $\mathbf{n}\geq\mathbf{m}$ for some $\mathbf{m}\in\mathbb{N}^s$:*

$$\sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})=\mathcal{F}(\mathbf{n}).$$

The vector $\mathbf{m}$ is called a **joint reduction vector** *of $\mathcal{F}$ with respect to the joint reduction $\mathcal{A}_{\mathbf{q}}(\mathcal{F})$.*

Let I,J be $\mathfrak{m}$-primary ideals in a Noetherian local ring $(R,\mathfrak{m})$ of dimension $d\geq 2$, $\mu,\lambda\geq 1$ and $\mu+\lambda=d$. For the bigraded filtration $\mathcal{F}=\{I^rJ^s\}_{r,s\in\mathbb{Z}}$ and a joint reduction $\mathcal{A}(\mu,\lambda)=\{a_i,b_j,\mid a_i\in$

$I, b_j \in J, 1 \leq i \leq \mu, 1 \leq j \leq \lambda\}$, E. Hyry defined the joint reduction number of $\mathcal{F}$ with respect to $\mathcal{A}(\mu,\lambda)$ to be the smallest integer n satisfying

$$I^{n+1}J^{n+1} = (a_1,\ldots,a_\mu)I^nJ^{n+1} + (b_1,\ldots,b_\lambda)I^{n+1}J^n.$$

We adapt this definition to define joint reduction number for multigraded filtrations.

Definition 1.7. *Let* $\mathbf{q} = (q_1,\ldots,q_s) \in \mathbb{N}^s$ *such that* $|\mathbf{q}| = d \geq 1$. *The* **joint reduction number of $\mathcal{F}$ with respect to a joint reduction** $\mathcal{A}_{\mathbf{q}}(\mathcal{F}) = \{a_{ij} \in I_i : j = 1,\ldots,q_i; i = 1,\ldots,s\}$ *is the smallest integer* $n \in \mathbb{N}$, *denoted by* $\mathrm{jr}_{\mathcal{A}_{\mathbf{q}}}(\mathcal{F})$, *such that for all* $\mathbf{n} \in \mathbb{N}^s$,

$$\sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}\left(\sum_{k\in A}(n+1)\mathbf{e_k} + \mathbf{n} - \mathbf{e_i}\right) = \mathcal{F}\left(\sum_{k\in A}(n+1)\mathbf{e_k} + \mathbf{n}\right) \text{ where } A = \{i | q_i \neq 0\}.$$

We define the **joint reduction number of $\mathcal{F}$ of type $\mathbf{q}$** *to be*

$$\mathrm{jr}_{\mathbf{q}}(\mathcal{F}) = \min\{\mathrm{jr}_{\mathcal{A}_{\mathbf{q}}}(\mathcal{F}) \mid \mathcal{A}_{\mathbf{q}}(\mathcal{F}) \text{ is a joint reduction of } \mathcal{F} \text{ of type } \mathbf{q}\}.$$

A crucial step in our investigations is to establish a connection between joint reduction vectors and vanishing of multigraded components of local cohomology modules of multi-Rees algebras. The following result of Hyry plays a crucial role.

Lemma 1.8. [7, Lemma 2.3] *Let S be a Noetherian $\mathbb{Z}$-graded ring defined over a local ring $(R,\mathfrak{m})$. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal. Let M be a finitely generated $\mathbb{Z}$-graded S-module and $n_0 \in \mathbb{Z}$. Then $[H^i_{\mathcal{M}}(M)]_n = 0$ for all $n \geq n_0$ and $i \geq 0$ if and only if $[H^i_{(\mathfrak{a},S_+)}(M)]_n = 0$ for all $n \geq n_0$ and $i \geq 0$.*

For convenience, inspired by the above result, we introduce an invariant of local cohomology modules of multigraded modules over multigraded rings. Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}},\mathfrak{m})$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module.

Definition 1.9. *Let $\mathbf{m} \in \mathbb{Z}^s$. We say that the module M satisfies* **Hyry's condition** $\mathrm{H}_R(M,\mathbf{m})$ *if*

$$[H^i_{R_{++}}(M)]_{\mathbf{n}} = 0 \text{ for all } i \geq 0 \text{ and } \mathbf{n} \geq \mathbf{m}.$$

Suppose $R_{\mathbf{e_i}} \neq 0$ for all $i = 1,\ldots,s$. Let $\mathcal{M}$ be the maximal homogeneous ideal of R, for each $i = 1,\ldots,s$, $\mathcal{M}_i$ be the ideal of R generated by $R_{\mathbf{e_i}}$ and $R_{++} = \bigcap_i \mathcal{M}_i$.
Let I be any subset of $\{1,\ldots,s\}$ and J be a non-empty subset of $\{1,\ldots,s\}$. Then for disjoint I and J, we define $\mathcal{M}_{I,J} = (\bigcap_{i\in I}\mathcal{M}_i)\bigcap(\sum_{j\in J}\mathcal{M}_j)$. We prove a multigraded version of the above result due to Hyry.

Proposition 1.10. *Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$, $R_{\mathbf{e_i}} \neq 0$ for all $i = 1, \ldots, s$ and $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^s$-graded R-module. Let I be any subset of $\{1, \ldots, s\}$ and J be a non-empty subset of $\{1, \ldots, s\}$ such that I and J are disjoint. Suppose $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ and $[H^i_{\mathcal{M}}(M)]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $n_k > a_k$ for at least one $k \in \{1, \ldots, s\}$. Then $[H^i_{\mathcal{M}_{I,J}}(M)]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{a} + \mathbf{e}$. In particular, M satisfies Hyry's condition $\mathrm{H}_R(M, \mathbf{a} + \mathbf{e})$.*

In order to detect joint reduction vectors of multigraded admissible filtrations, we use the theory of filter-regular sequences for multigraded modules.

Definition 1.11. *A homogeneous element $a \in R$ is called an M-**filter-regular** if $(0 :_M a)_{\mathbf{n}} = 0$ for all large $\mathbf{n}$. Let $a_1, \ldots, a_r \in R$ be homogeneous elements. Then $a_1, \ldots, a_r$ is called an M-**filter-regular sequence** if a_i is $M/(a_1, \ldots, a_{i-1})M$-filter-regular for all $i = 1, \ldots, r$.*

Theorem 1.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R. Suppose $G(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{G(\mathbf{I})}(G(\mathcal{F}), \mathbf{m})$. Let $\mathbf{q} \in \mathbb{N}^s$ such that $|\mathbf{q}| = d$ and $\{a_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$ be a joint reduction of $\mathcal{F}$ of type $\mathbf{q}$ such that $a^*_{11}, \ldots, a^*_{1q_1}, \ldots, a^*_{s1}, \ldots, a^*_{sq_s}$ is a $G(\mathcal{F})$-filter-regular sequence where a^*_{ij} is the image of a_{ij} in $G(\mathbf{I})_{\mathbf{e_i}}$ for all $j = 1, \ldots, q_i$ and $i = 1, \ldots, s$. Then*

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n} - \mathbf{e_i}) \text{ for all } \mathbf{n} \geq \mathbf{m} + \mathbf{q}.$$

We can now state the main theorem of this paper which gives a generalisation of Reid-Roberts-Vitulli Theorem for zero-dimensional monomial ideals.

Theorem 1.13. *Let $(R, \mathfrak{m})$ be an analytically unramified Noetherian local ring of dimension $d \geq 2$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\overline{\mathcal{R}}(\mathbf{I}) = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} \overline{\mathbf{I}^{\mathbf{n}}}$ satisfy the condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\overline{\mathcal{R}}(\mathbf{I}), \mathbf{0})$. Suppose $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{n}| \leq d - 1$. Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ with $|\mathbf{n}| \geq 1$.*

We prove that if $\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay then it satisfies Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\overline{\mathcal{R}}(\mathbf{I}), \mathbf{0})$. By Hochster's Theorem [2, Theorem 6.3.5] about Cohen-Macaulayness of normal semigroup rings, $\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay if $\mathbf{I}$ consists of monomial ideals in a polynomial ring over a field.

2. Existence of joint reductions consisting of filter-regular sequences

Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over an Artinian local ring $(R_{\mathbf{0}}, \mathfrak{m})$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^s$-graded R-module. Let $\mathrm{Proj}^s(R)$ denote the set of all homogeneous prime ideals P in R such that $R_{++} \nsubseteq P$ and $M^{\Delta} = \bigoplus_{n \geq 0} M_{n\mathbf{e}}$. By [4, Theorem 4.1], there exists a numerical polynomial $P_M \in \mathbb{Q}[X_1, \ldots, X_s]$ of total degree $\dim M^{\Delta} - 1$ of the form

$$P_M(\mathbf{n}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha| \leq \dim M^{\Delta}-1}} (-1)^{\dim M^{\Delta}-1-|\alpha|} e_{\alpha}(M) \binom{n_1+\alpha_1-1}{\alpha_1} \cdots \binom{n_s+\alpha_s-1}{\alpha_s}$$

such that $e_{\alpha}(M) \in \mathbb{Z}$, $P_M(\mathbf{n}) = \lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}})$ for all large $\mathbf{n}$ and $e_{\alpha}(M) \geq 0$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| = \dim M^{\Delta} - 1$.

Proposition 2.1. *Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$ with infinite residue field $R_{\mathbf{0}}/\mathfrak{m}$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^s$-graded R-module and $\dim M^{\Delta} \geq 1$. Fix $i \in \{1, \ldots, s\}$. If $R_{\mathbf{e_i}} \neq 0$ then there exists $a \in R_{\mathbf{e_i}}$ such that a is M-filter-regular.*

Proof. Denote $M/H^0_{R_{++}}(M)$ by M'. Then $Ass(M') = Ass(M) \backslash V(R_{++})$. Let $Ass(M') = \{P_1, \ldots, P_k\}$. Let $\mathfrak{a}_i$ be the ideal of R generated by $R_{\mathbf{e_i}}$. Therefore for all $j = 1, \ldots, k$, $P_j \nsupseteq \mathfrak{a}_i$. Consider the $R_{\mathbf{0}}/\mathfrak{m}$-vector space $R_{\mathbf{e_i}}/\mathfrak{m}R_{\mathbf{e_i}}$. Then for each $j = 1, \ldots, k$,

$$(P_j \cap R_{\mathbf{e_i}} + \mathfrak{m}R_{\mathbf{e_i}})/\mathfrak{m}R_{\mathbf{e_i}} \neq R_{\mathbf{e_i}}/\mathfrak{m}R_{\mathbf{e_i}}.$$

Since $R_{\mathbf{0}}/\mathfrak{m}$ is infinite, there exists $a \in R_{\mathbf{e_i}} \setminus \bigcup_{j=1}^{k}(P_j \cap R_{\mathbf{e_i}} + \mathfrak{m}R_{\mathbf{e_i}})$.
By [10, Proposition 4.1], there exists $\mathbf{m}$ such that $[H^0_{R_{++}}(M)]_{\mathbf{n}} = 0$ for all $\mathbf{n} \geq \mathbf{m}$. Let $\mathbf{n} \geq \mathbf{m}$ and $x \in (0 :_M a)_{\mathbf{n}}$. Then $ax' = 0$ in M' where x' is the image of x in M'. Since a is a nonzerodivisor of M', $x \in [H^0_{R_{++}}(M)]_{\mathbf{n}} = 0$. □

Proposition 2.2. *Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over an Artinian local ring $(R_{\mathbf{0}}, \mathfrak{m})$. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^s$-graded R-module. Let $a_i \in R_{\mathbf{e_i}}$ be an M-filter-regular element. Then for all large $\mathbf{n}$,*

$$\lambda_{R_{\mathbf{0}}}\left(\frac{M_{\mathbf{n}}}{a_i M_{\mathbf{n}-\mathbf{e_i}}}\right) = \lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}}) - \lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}-\mathbf{e_i}})$$

and hence for all $\mathbf{n} \in \mathbb{Z}^s$, $P_{M/a_iM}(\mathbf{n}) = P_M(\mathbf{n}) - P_M(\mathbf{n} - \mathbf{e_i})$.

Proof. Consider the exact sequence of R-modules

$$0 \longrightarrow (0 :_M a_i)_{\mathbf{n}-\mathbf{e_i}} \longrightarrow M_{\mathbf{n}-\mathbf{e_i}} \xrightarrow{a_i} M_{\mathbf{n}} \longrightarrow \frac{M_{\mathbf{n}}}{a_i M_{\mathbf{n}-\mathbf{e_i}}} \longrightarrow 0.$$

Since a_i is M-filter-regular, for all large $\mathbf{n}$, we get

$$\lambda_{R_{\mathbf{0}}}\left(\frac{M_{\mathbf{n}}}{a_i M_{\mathbf{n}-\mathbf{e_i}}}\right) = \lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}}) - \lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}-\mathbf{e_i}})$$

and hence for all $\mathbf{n} \in \mathbb{Z}^s$, $P_{M/a_iM}(\mathbf{n}) = P_M(\mathbf{n}) - P_M(\mathbf{n} - \mathbf{e_i})$. □

Theorem 2.3. *Let* $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ *be a standard Noetherian* $\mathbb{N}^s$*-graded ring defined over an Artinian local ring* $(R_{\mathbf{0}}, \mathfrak{m})$ *with infinite residue field* $R_{\mathbf{0}}/\mathfrak{m}$ *and* $R_{\mathbf{e_i}} \neq 0$ *for all* $i = 1, \ldots, s$. *Let* $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ *be a finitely generated* $\mathbb{N}^s$*-graded* R*-module and* $\dim M^{\Delta} \geq 1$. *Let* $e_{\alpha}(M) > 0$ *for all* $\alpha \in \mathbb{N}^s$ *such that* $|\alpha| = \dim M^{\Delta} - 1$. *Then for any* $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ *such that* $|\mathbf{q}| = \dim M^{\Delta}$, *there exist* $a_{i1}, \ldots, a_{iq_i} \in R_{\mathbf{e_i}}$ *for all* $i = 1, \ldots, s$, *such that* $a_{11}, \ldots, a_{1q_1}, \ldots, a_{s1}, \ldots, a_{sq_s}$ *is an* M*-filter-regular sequence and for all large* $\mathbf{n}$, $M_{\mathbf{n}} = \sum_{i=1}^{s} \sum_{j=1}^{q_i} a_{ij} M_{\mathbf{n}-\mathbf{e_i}}$.

Proof. We use induction on $\dim M^{\Delta} = l$. Let $l = 1$. Then by Proposition 2.1, for each $i = 1, \ldots, s$, there exists $a_i \in R_{\mathbf{e_i}}$ such that a_i is M-filter-regular. Since $l = 1$, $P_M(\mathbf{n})$ is polynomial of total degree zero. Therefore by Proposition 2.2, $\lambda_{R_{\mathbf{0}}}(M_{\mathbf{n}}/a_i M_{\mathbf{n}-\mathbf{e_i}}) = 0$ for all large $\mathbf{n}$ and hence we get the required result. Suppose $l \geq 2$ and the result is true for all finitely generated $\mathbb{N}^s$-graded R-module T such that $1 \leq \dim T^{\Delta} \leq l-1$ and $e_{\alpha}(T) > 0$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| = \dim T^{\Delta} - 1$. Let M be finitely generated $\mathbb{N}^s$-graded R-module such that $\dim M^{\Delta} = l$ and $e_{\alpha}(M) > 0$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| = l - 1$. Fix $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ such that $|\mathbf{q}| = \dim M^{\Delta}$. Let $i = \min\{j \mid q_j \neq 0\}$. By Proposition 2.1, there exists $a_{i1} \in R_{\mathbf{e_i}}$ such that a_{i1} is an M-filter-regular element. Let $N = M/a_{i1}M$. Since $e_{\alpha}(M) > 0$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| = \dim M^{\Delta} - 1$, by Proposition 2.2, $P_N(\mathbf{n})$ is a polynomial of degree $l-2$ and hence $\dim N^{\Delta} = l - 1$. Let $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s$ such that $|\beta| = l - 2$ and $\alpha = \beta + \mathbf{e_i}$. Then $e_{\beta}(N) = e_{\alpha}(M) > 0$. Let $\mathbf{m} = \mathbf{q} - \mathbf{e_i} \in \mathbb{N}^s$, i.e. $m_i = q_i - 1$ and for all $j \neq i$, $m_j = q_j$. Since $|\mathbf{m}| = l - 1$, by induction hypothesis there exist $b_{j1}, \ldots, b_{jm_j} \in R_{\mathbf{e_j}}$ for all $j = 1, \ldots, s$ such that $b_{11}, \ldots, b_{1m_1}, \ldots, b_{s1}, \ldots, b_{sm_s}$ is an N-filter-regular sequence and for all large $\mathbf{n}$, $N_{\mathbf{n}} = \displaystyle\sum_{k=1}^{s} \sum_{j=1}^{m_k} b_{kj} N_{\mathbf{n}-\mathbf{e_k}}$. Let $a_{ik} = b_{i(k-1)}$ for all $k = 2, \ldots, , q_i$ and for all $j \neq i$, $a_{jk} = b_{jk}$ for all $k = 1, \ldots, , q_j$. Then for all large $\mathbf{n}$, $M_{\mathbf{n}} = \displaystyle\sum_{i=1}^{s} \sum_{j=1}^{q_i} a_{ij} M_{\mathbf{n}-\mathbf{e_i}}$. Since a_{i1} is M-filter-regular, $a_{11}, \ldots, a_{1q_1}, \ldots, a_{s1}, \ldots, a_{sq_s}$ is an M-filter-regular sequence. □

Theorem 2.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$ *and* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals in* R. *Let* $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ *be an* $\mathbf{I}$*-admissible filtration of ideals in* R *and* $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ *such that* $|\mathbf{q}| = d$. *Then there exists a joint reduction* $\{a_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$ *of* $\mathcal{F}$ *of type* $\mathbf{q}$ *such that* $a^*_{11}, \ldots, a^*_{1q_1}, \ldots, a^*_{s1}, \ldots, a^*_{sq_s}$ *is a* $G(\mathcal{F})$*-filter-regular sequence where* a^*_{ij} *is the image of* a_{ij} *in* $G(\mathbf{I})_{\mathbf{e_i}}$ *for all* $j = 1, \ldots, q_i$ *and* $i = 1, \ldots, s$.

Proof. Since $\mathcal{F}$ is an $\mathbf{I}$-admissible filtration, $G(\mathcal{F})$ is finitely generated $G(\mathbf{I})$-module. By [12, Theorem 2.4], there exists a polynomial

$$P_{\mathcal{F}}(\mathbf{n}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha|\leq d}} (-1)^{d-|\alpha|} e_{\alpha}(\mathcal{F}) \binom{n_1 + \alpha_1 - 1}{\alpha_1} \cdots \binom{n_s + \alpha_s - 1}{\alpha_s}$$

such that for all large $\mathbf{n}$, $P_{\mathcal{F}}(\mathbf{n}) = \lambda_R\left(R/\mathcal{F}(\mathbf{n})\right)$, $e_\alpha(\mathcal{F}) \in \mathbb{Z}$ and $e_\alpha(\mathcal{F}) > 0$ for all $\alpha \in \mathbb{N}^s$ where $|\alpha| = d$. Hence

$$\lambda\left(\frac{\mathcal{F}(\mathbf{n})}{\mathcal{F}(\mathbf{n}+\mathbf{e})}\right) = \lambda_R\left(\frac{R}{\mathcal{F}(\mathbf{n}+\mathbf{e})}\right) - \lambda_R\left(\frac{R}{\mathcal{F}(\mathbf{n})}\right)$$

is a numerical polynomial in $\mathbb{Q}[X_1, \ldots, X_s]$ of total degree $d-1$ for all large $\mathbf{n}$ and $e_\beta(G(\mathcal{F})) > 0$ for all $\beta \in \mathbb{N}^s$ where $|\beta| = d-1$. Therefore by Theorem 2.3, there exist $a_{i1}, \ldots, a_{iq_i} \in I_i$ for all $i = 1, \ldots, s$, such that $a_{11}^*, \ldots, a_{1q_1}^*, \ldots, a_{s1}^*, \ldots, a_{sq_s}^*$ is a $G(\mathcal{F})$-filter-regular sequence where $a_{ij}^* = a_{ij} + \mathbf{I}^{\mathbf{e}+\mathbf{e_i}} \in G(\mathbf{I})_{\mathbf{e_i}}$ for all $j = 1, \ldots, q_i$, $i = 1, \ldots, s$ and $G(\mathcal{F})_{\mathbf{n}} = \sum\limits_{i=1}^{s}\sum\limits_{j=1}^{q_i} a_{ij}^* G(\mathcal{F})_{\mathbf{n}-\mathbf{e_i}}$ for all large $\mathbf{n}$. Hence

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}) + \mathcal{F}(\mathbf{n}+\mathbf{e}) \text{ for all large } \mathbf{n}.$$

Since $\mathcal{F}$ is an $\mathbf{I}$-admissible filtration, by [12], for each $i = 1, \ldots, s$, there exist an integer r_i such that for all $\mathbf{n} \in \mathbb{Z}^s$, where $n_i \geq r_i$, $\mathcal{F}(\mathbf{n}+\mathbf{e_i}) = I_i\mathcal{F}(\mathbf{n})$, Hence for all large $\mathbf{n}$, we get

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}) + \mathcal{F}(\mathbf{n}+\mathbf{e}) \subseteq \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}) + I_1 \cdots I_s\mathcal{F}(\mathbf{n}).$$

Thus by Nakayama's Lemma, for all large $\mathbf{n}$, $\mathcal{F}(\mathbf{n}) = \sum\limits_{i=1}^{s}\sum\limits_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})$. $\square$

3. Vanishing of Local cohomology modules of Rees algebra of multigraded filtrations

Let $R = \bigoplus\limits_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$ and $R_{\mathbf{e_i}} \neq 0$ for all $i = 1, \ldots, s$. For a non-empty subset J of $\{1, \ldots, s\}$, we define $\mathcal{M}_J = \sum\limits_{j\in J} \mathcal{M}_j$.

Lemma 3.1. *Let $R = \bigoplus\limits_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$, $R_{\mathbf{e_i}} \neq 0$ for all $i = 1, \ldots, s$ and $M = \bigoplus\limits_{\mathbf{n}\in\mathbb{N}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^s$-graded R-module. Let $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ and J be any non-empty subset of $\{1, \ldots, s\}$. Suppose $[H^i_{\mathcal{M}}(M)]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $n_k > a_k$ for at least one $k \in J$. Then $[H^i_{\mathcal{M}_J}(M)]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $\sum\limits_{j\in J} n_j > \sum\limits_{j\in J} a_j$.*

Proof. Consider a group homomorphism $\phi : \mathbb{Z}^s \to \mathbb{Z}$ defined by $\phi(\mathbf{n}) = \sum\limits_{j\in J} n_j$. Then $R^\phi = \bigoplus\limits_{n\geq 0}(\bigoplus\limits_{\phi(\mathbf{n})=n} R_{\mathbf{n}})$. Let $S = (R^\phi)_0$ and $\mathcal{N}$ be the maximal homogeneous ideal of S. Therefore $(R^\phi)_{\mathcal{N}}$ is an $\mathbb{N}$-graded ring defined over the local ring $S_{\mathcal{N}}$ and $((\mathcal{M}_J)^\phi)_{\mathcal{N}}$ is the irrelevant ideal of $(R^\phi)_{\mathcal{N}}$. Then for all $i \geq 0$ and

$m > \sum_{j\in J} a_j$,

$$[H^i_{(\mathcal{M}^\phi)_\mathcal{N}}(M^\phi)_\mathcal{N}]_m = \Big([H^i_{\mathcal{M}^\phi}(M^\phi)]_m\Big)_\mathcal{N} = \left(\bigoplus_{\phi(\mathbf{n})=m} [H^i_\mathcal{M}(M)]_\mathbf{n}\right) \otimes_S S_\mathcal{N}.$$

Since $m > \sum_{j\in J} a_j$, $\phi(\mathbf{n}) = m$ implies $n_k > a_k$ for at least one $k \in J$. Hence $[H^i_{(\mathcal{M}^\phi)_\mathcal{N}}(M^\phi_\mathcal{N})]_m = 0$ for all $i \geq 0$ and $m > \sum_{j\in J} a_j$. Then by Lemma 1.8, taking $\mathfrak{a} = 0$ we get $[H^i_{((\mathcal{M}_J)^\phi)_\mathcal{N}}(M^\phi)_\mathcal{N}]_m = 0$ for all $i \geq 0$ and $m > \sum_{j\in J} a_j$. Thus $\left(\bigoplus_{\phi(\mathbf{n})=m} [H^i_{\mathcal{M}_J}(M)]_\mathbf{n}\right) \otimes_S S_\mathcal{N} = 0$ for all $i \geq 0$ and $m > \sum_{j\in J} a_j$. Therefore $[H^i_{\mathcal{M}_J}(M)]_\mathbf{n} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $\sum_{j\in J} n_j > \sum_{j\in J} a_j$. □

Proposition 3.2. *Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_\mathbf{n}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_\mathbf{0}, \mathfrak{m})$, $R_{\mathbf{e_i}} \neq 0$ for all $i = 1, \ldots, s$ and $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} M_\mathbf{n}$ be a finitely generated $\mathbb{N}^s$-graded R-module. Let I be any subset of $\{1, \ldots, s\}$ and J be a non-empty subset of $\{1, \ldots, s\}$ such that I and J are disjoint. Suppose $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ and $[H^i_\mathcal{M}(M)]_\mathbf{n} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $n_k > a_k$ for at least one $k \in \{1, \ldots, s\}$. Then $[H^i_{\mathcal{M}_{I,J}}(M)]_\mathbf{n} = 0$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{a} + \mathbf{e}$. In particular, M satisfies Hyry's condition $\mathrm{H}_R(M, \mathbf{a} + \mathbf{e})$.*

Proof. We follow the argument given in [3, Theorem 3.2.6] and use induction on $r = |I \cup J|$. Suppose $r = 1$. Since I, J are disjoint and $|J| \geq 1$, we have $I = \emptyset$ and the result follows from Lemma 3.1. Suppose $r \geq 2$ and the result is true upto $r - 1$. Let $I = \{i_1, \ldots, i_k\}$ and $J = \{i_{k+1}, \ldots, i_r\}$. we use induction on k. If $k = 0$ then again by Lemma 3.1, we get the result. Suppose $k \geq 1$ and the result is true upto $k - 1$. Let $\mathcal{I} = I \setminus \{i_k\}$ and $\mathcal{J} = J \cup \{i_k\}$. Then $\mathcal{M}_{\mathcal{I},J} + \mathcal{M}_{\mathcal{I},\{i_k\}} = \mathcal{M}_{\mathcal{I},\mathcal{J}}$ and $\mathcal{M}_{\mathcal{I},J} \cap \mathcal{M}_{\mathcal{I},\{i_k\}} = \mathcal{M}_{I,J}$. Consider the following Mayer-Vietoris sequence of local cohomology modules

$$\cdots \longrightarrow H^i_{\mathcal{M}_{\mathcal{I},J}}(M) \bigoplus H^i_{\mathcal{M}_{\mathcal{I},\{i_k\}}}(M) \longrightarrow H^i_{\mathcal{M}_{I,J}}(M) \longrightarrow H^{i+1}_{\mathcal{M}_{\mathcal{I},\mathcal{J}}}(M) \longrightarrow \cdots.$$

Using induction on k, we get $[H^{i+1}_{\mathcal{M}_{\mathcal{I},\mathcal{J}}}(M)]_\mathbf{n} = 0$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{a} + \mathbf{e}$ and using induction on r, we get $[H^i_{\mathcal{M}_{\mathcal{I},J}}(M)]_\mathbf{n} = 0 = [H^i_{\mathcal{M}_{\mathcal{I},\{i_k\}}}(M)]_\mathbf{n}$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{a} + \mathbf{e}$. For R_{++}, we take $I = \{1, \ldots, s-1\}$ and $J = \{s\}$. □

Let $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_\mathbf{n}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_\mathbf{0}, \mathfrak{m})$ and let $\mathcal{M}$ be the maximal homogeneous ideal of R. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^s} M_\mathbf{n}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. For all $i = 1, \ldots, s$, define [7] the a-invariants of M as

$$a^i(M) = \sup\{k \in \mathbb{Z} \mid [H^{\dim M}_\mathcal{M}(M)]_\mathbf{n} \neq 0 \text{ for some } \mathbf{n} \in \mathbb{Z}^s \text{ with } n_i = k\}.$$

Put $a(M) = (a^1(M), \ldots, a^s(M))$. We recall the following result.

Proposition 3.3. [11, Lemma 3.7] *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, $I_1,\dots,I_s$ be $\mathfrak{m}$-primary ideals of R and $\mathcal{F}=\{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R. Then $a(\mathcal{R}(\mathcal{F}))=-\mathbf{e}$.*

Remark 3.4. Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d and $I_1,\dots,I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F}=\{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R and $\mathcal{R}(\mathcal{F})$ be Cohen-Macaulay. Then by Propositions 3.3 and 3.2, $\mathcal{R}(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$.

The next theorem is a generalisation of a result due to Hyry [8, Theorem 6.1] for $\mathbb{Z}^s$-graded admissible filtration of ideals.

Theorem 3.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d and $I_1,\dots,I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F}=\{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R and let $\mathcal{R}(\mathcal{F})$ satisfy Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$. Then*

1. *$P_{\mathcal{F}}(\mathbf{n})=H_{\mathcal{F}}(\mathbf{n})$ for all $\mathbf{n}\in\mathbb{N}^s$,*
2. *for all $\alpha=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}^s$ such that $|\alpha|=d$,*

$$e_\alpha(\mathcal{F})=\sum_{n_1=0}^{\alpha_1}\cdots\sum_{n_s=0}^{\alpha_s}\binom{\alpha_1}{n_1}\dots\binom{\alpha_s}{n_s}(-1)^{d-n_1-\dots-n_s}\lambda\left(\frac{R}{\mathcal{F}(\mathbf{n})}\right)$$

where $\mathbf{n}=(n_1,\dots,n_s)$.

Proof. (1) Since $\mathcal{R}(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$, by [10, Theorem 4.3], we get $P_{\mathcal{F}}(\mathbf{n})=H_{\mathcal{F}}(\mathbf{n})$ for all $\mathbf{n}\in\mathbb{N}^s$.
(2) Consider the operators $(\Delta_i P_{\mathcal{F}})(\mathbf{n})=P_{\mathcal{F}}(\mathbf{n}+\mathbf{e_i})-P_{\mathcal{F}}(\mathbf{n})$ for all $i=1,\dots,s$. Then $(\Delta_s^{\alpha_s}\cdots\Delta_1^{\alpha_1}P_{\mathcal{F}})(\mathbf{0})=e_\alpha(\mathcal{F})$ for $\alpha=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}^s$, $|\alpha|=d$. By [14, Proposition 1.2],

$$(\Delta_s^{\alpha_s}\cdots\Delta_1^{\alpha_1}P_{\mathcal{F}})(\mathbf{0})=\sum_{n_1=0}^{\alpha_1}\cdots\sum_{n_s=0}^{\alpha_s}\binom{\alpha_1}{n_1}\dots\binom{\alpha_s}{n_s}(-1)^{d-n_1-\dots-n_s}P_{\mathcal{F}}(\mathbf{n})$$

where $\mathbf{n}=(n_1,\dots,n_s)$. Thus from part (1) we get the required result. □

Lemma 3.6. *Let $R=\bigoplus_{\mathbf{n}\in\mathbb{N}^s}R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}},\mathfrak{m})$ and $M=\bigoplus_{\mathbf{n}\in\mathbb{Z}^s}M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. Let $a\in R_{\mathbf{m}}$ be an M-filter-regular where $\mathbf{m}\neq\mathbf{0}$. Then for all $\mathbf{n}\in\mathbb{Z}^s$ and $i\geq 0$, the following sequence is exact*

$$[H^i_{R_{++}}(M)]_{\mathbf{n}}\longrightarrow\left[H^i_{R_{++}}\left(\frac{M}{aM}\right)\right]_{\mathbf{n}}\longrightarrow[H^{i+1}_{R_{++}}(M)]_{\mathbf{n}-\mathbf{m}}.$$

Proof. Consider the following short exact sequence of R-modules

$$0\longrightarrow(0:_M a)\longrightarrow M\longrightarrow\frac{M}{(0:_M a)}\longrightarrow 0.$$

Since a is M-filter-regular, $(0 :_M a)$ is R_{++}-torsion. Hence

$$H^i_{R_{++}}(M) \simeq H^i_{R_{++}}\left(\frac{M}{(0 :_M a)}\right) \text{ for all } i \geq 1.$$

Therefore the short exact sequence of R-modules

$$0 \longrightarrow \frac{M}{(0 :_M a)}(-\mathbf{m}) \xrightarrow{a} M \longrightarrow \frac{M}{aM} \longrightarrow 0$$

gives the desired exact sequence. □

Lemma 3.7. *Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$ and $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. Suppose M satisfies Hyry's condition $\mathrm{H}_R(M, \mathbf{m})$. Let $a_1, \ldots, a_l \in R_{\mathbf{e_j}}$ be an M-filter-regular sequence. Then $M/(a_1, \ldots, a_l)M$ satisfies Hyry's condition $\mathrm{H}_R(M/(a_1, \ldots, a_l)M, \mathbf{m} + l\mathbf{e_j})$.*

Proof. We use induction on l. Let $l = 1$. By Lemma 3.6, for all $i \geq 0$, we get the exact sequence

$$[H^i_{R_{++}}(M)]_{\mathbf{n}} \longrightarrow \left[H^i_{R_{++}}\left(\frac{M}{a_1 M}\right)\right]_{\mathbf{n}} \longrightarrow [H^{i+1}_{R_{++}}(M)]_{\mathbf{n}-\mathbf{e_j}}.$$

Since for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{m} + \mathbf{e_j}$, $[H^i_{R_{++}}(M)]_{\mathbf{n}} = 0$, we get $\left[H^i_{R_{++}}\left(\frac{M}{a_1 M}\right)\right]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{m} + \mathbf{e_j}$. Hence the result is true for $l = 1$.

Suppose $l \geq 2$ and the result is true upto $l - 1$. Let $N = M/(a_1, \ldots, a_{l-1})M$. Then

$$[H^i_{R_{++}}(N)]_{\mathbf{n}} = 0 \text{ for all } i \geq 0 \text{ and } \mathbf{n} \geq \mathbf{m} + (l-1)\mathbf{e_j}.$$

Since a_l is N-filter-regular, using $l = 1$ case, we get $\left[H^i_{R_{++}}\left(\frac{N}{a_l N}\right)\right]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \geq \mathbf{m} + l\mathbf{e_j}$. □

Proposition 3.8. *Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded Noetherian ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$ and $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. Suppose M satisfies Hyry's condition $\mathrm{H}_R(M, \mathbf{m})$. Let $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$, $a_{j1}, \ldots, a_{jq_j} \in R_{\mathbf{e_j}}$ for all $j = 1, \ldots, s$ such that $a_{11}, \ldots, a_{1q_1}, \ldots, a_{s1}, \ldots, a_{sq_s}$ be an M-filter-regular sequence. Then $M/(a_{11}, \ldots, a_{1q_1}, \ldots, a_{s1}, \ldots, a_{sq_s})M$ satisfies Hyry's condition $\mathrm{H}_R(M/(a_{11}, \ldots, a_{1q_1}, \ldots, a_{s1}, \ldots, a_{sq_s})M, \mathbf{m} + \mathbf{q})$.*

Proof. Define $N_0 = M$ and $N_j = N_{j-1}/(a_{j1}, \ldots, a_{jq_j})N_{j-1}$ for all $j = 1, \ldots, s$. Since $a_{j1}, \ldots, a_{jq_j} \in R_{\mathbf{e_j}}$ is an N_{j-1}-filter-regular sequence for all $j = 1, \ldots, s$, by Lemma 3.7, we get the required result. □

Theorem 3.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R. Suppose $G(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{G(\mathbf{I})}(G(\mathcal{F}), \mathbf{m})$. Let $\mathbf{q} \in \mathbb{N}^s$ such that $|\mathbf{q}| = d$ and $\{a_{ij} \in I_i : j = 1, \ldots, q_i; i =$*

$1,\ldots,s\}$ *be a joint reduction of* $\mathcal{F}$ *of type* $\mathbf{q}$ *such that* $a_{11}^*,\ldots,a_{1q_1}^*,\ldots,a_{s1}^*,\ldots,a_{sq_s}^*$ *is* $G(\mathcal{F})$*-filter-regular sequence where* a_{ij}^* *is the image of* a_{ij} *in* $G(\mathbf{I})_{\mathbf{e_i}}$ *for all* $j=1,\ldots,q_i$ *and* $i=1,\ldots,s$*. Then*

$$\mathcal{F}(\mathbf{n})=\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}) \text{ for all } \mathbf{n}\geq\mathbf{m}+\mathbf{q}.$$

Proof. By Proposition 3.8, we get

$$\left[H^i_{G(\mathbf{I})_{++}}\left(\frac{G(\mathcal{F})}{(a_{11}^*,\ldots,a_{1q_1}^*,\ldots,a_{s1}^*,\ldots,a_{sq_s}^*)G(\mathcal{F})}\right)\right]_{\mathbf{n}}=0$$

for all $i\geq 0$ and $\mathbf{n}\geq\mathbf{m}+\mathbf{q}$. Since $\{a_{ij}\in I_i: j=1,\ldots,q_i; i=1,\ldots,s\}$ is a joint reduction of $\mathcal{F}$, $G(\mathcal{F})/(a_{11}^*,\ldots,a_{1q_1}^*,\ldots,a_{s1}^*,\ldots,a_{sq_s}^*)G(\mathcal{F})$ is $G(\mathbf{I})_{++}$-torsion. Thus

$$H^0_{G(\mathbf{I})_{++}}\left(\frac{G(\mathcal{F})}{(a_{11}^*,\ldots,a_{1q_1}^*,\ldots,a_{s1}^*,\ldots,a_{sq_s}^*)G(\mathcal{F})}\right)=\frac{G(\mathcal{F})}{(a_{11}^*,\ldots,a_{1q_1}^*,\ldots,a_{s1}^*,\ldots,a_{sq_s}^*)G(\mathcal{F})}.$$

Hence for all $\mathbf{n}\geq\mathbf{m}+\mathbf{q}$, we have

$$\mathcal{F}(\mathbf{n})=\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+\mathbf{e}). \tag{3.9.1}$$

Now for all $k\geq 0$ and $\mathbf{n}\geq\mathbf{m}+\mathbf{q}$,

$$\mathcal{F}(\mathbf{n}+k\mathbf{e})=\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}+k\mathbf{e}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+(k+1)\mathbf{e})\subseteq\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+(k+1)\mathbf{e}).$$

Since $\mathcal{F}$ is an $\mathbf{I}$-admissible filtration, by [12], for each $i=1,\ldots,s$, there exists integer $r_i\in\mathbb{N}$ such that for all $\mathbf{n}\in\mathbb{N}^s$, where $n_i\geq r_i$, we have $\mathcal{F}(\mathbf{n}+\mathbf{e_i})=I_i\mathcal{F}(\mathbf{n})$. Let $r=\max\{r_i: i=1,\ldots,s\}$. Therefore

$$\begin{aligned}\mathcal{F}(\mathbf{n}+\mathbf{e}) &\subseteq \sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}+\mathbf{e}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+2\mathbf{e})\subseteq\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+2\mathbf{e})\\ &\subseteq \vdots\\ &\subseteq \sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})+\mathcal{F}(\mathbf{n}+(r+2)\mathbf{e})\subseteq\sum_{i=1}^{s}\sum_{j=1}^{q_i}a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i})+I_1\cdots I_s\mathcal{F}(\mathbf{n}).\end{aligned}$$

Hence by using Nakayama's Lemma, from the equality (3.9.1) we get the required result. □

Lemma 3.10. *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring of dimension* $d\geq 1$ *and* $I_1,\ldots,I_s$ *be* $\mathfrak{m}$*-primary ideals in* R*. Let* $\mathcal{F}=\{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ *be an* $\mathbf{I}$*-admissible filtration of ideals in* R*. Suppose* $\mathcal{R}(\mathcal{F})$ *satisfies Hyry's condition* $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{m})$ *where* $\mathbf{m}\in\mathbb{N}^s$*. Then* $G(\mathcal{F})$ *satisfies Hyry's condition* $\mathrm{H}_{G(\mathbf{I})}(G(\mathcal{F}),\mathbf{m})$*.*

Proof. Denote $\mathcal{R}'(\mathcal{F})/\mathcal{R}'(\mathcal{F})(\mathbf{e})$ by $G'(\mathcal{F})$. Consider the short exact sequence of $\mathcal{R}(\mathbf{I})$-modules

$$0 \longrightarrow \mathcal{R}'(\mathcal{F})(\mathbf{e}) \longrightarrow \mathcal{R}'(\mathcal{F}) \longrightarrow G'(\mathcal{F}) \longrightarrow 0. \tag{3.10.1}$$

This induces the long exact sequence of R-modules

$$\cdots \longrightarrow [H^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\mathbf{n}+\mathbf{e}} \longrightarrow [H^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\mathbf{n}} \longrightarrow [H^i_{\mathcal{R}_{++}}(G'(\mathcal{F}))]_{\mathbf{n}} \longrightarrow [H^{i+1}_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\mathbf{n}+\mathbf{e}} \longrightarrow \cdots .$$

Hence by [10, Proposition 4.2] and change of ring principle, we get the required result. □

Theorem 3.11. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $I_1,\ldots,I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R and $\mathcal{R}(\mathcal{F})$ satisfy Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$. Let $\mathbf{q} \in \mathbb{N}^s$ such that $|\mathbf{q}| = d$. Then there exists a joint reduction $\{a_{ij} \in I_i : j = 1,\ldots,q_i; i = 1,\ldots,s\}$ of $\mathcal{F}$ of type $\mathbf{q}$ such that*

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}) \text{ for all } \mathbf{n} \geq \mathbf{q} \qquad \text{and}$$

$$\mathrm{jr}_{\mathbf{q}}(\mathcal{F}) \leq \max\{q_i \mid i \in A\} - 1 \quad \text{where} \quad A = \{i \mid q_i \geq 1\}.$$

Proof. Since $\mathcal{R}(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$, by Lemma 3.10, $G(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{G(\mathbf{I})}(G(\mathcal{F}),\mathbf{0})$. Therefore by Theorem 2.4, there exists a joint reduction $\{a_{ij} \in I_i : j = 1,\ldots,q_i; i = 1,\ldots,s\}$ of $\mathcal{F}$ of type $\mathbf{q}$ such that such that $a^*_{11},\ldots,a^*_{1q_1},\ldots,a^*_{s1},\ldots,a^*_{sq_s}$ is a $G(\mathcal{F})$-filter-regular sequence where a^*_{ij} is the image of a_{ij} in $G(\mathbf{I})_{\mathbf{e_i}}$ for all $j = 1,\ldots,q_i$, $i = 1,\ldots,s$. Hence by Theorem 3.9, for all $\mathbf{n} \geq \mathbf{q}$,

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n}-\mathbf{e_i}).$$

□

Example 3.12. Let $R = k[|X,Y|]$. Then R is a regular local ring of dimension two. Let $I = (X,Y^2)$ and $J = (X^2,Y)$. Then I,J are complete parameter ideals in R. Consider the filtration $\mathcal{F} = \{I^rJ^s\}_{r,s\in\mathbb{Z}}$. Since I,J are complete ideals, by [16, Theorem 2′, Appendix 5], I^r, J^s and I^rJ^s are complete ideals for all $r,s \geq 1$. By [10, Proposition 3.2] and [10, Proposition 3.5], for all $r,s \in \mathbb{N}$, $H^1_{\mathcal{R}_{++}}(\mathcal{R}(I,J))_{(r,s)} = 0$. Note that (X^3+Y^3,XY) is a minimal reduction of IJ and

$$(X^3+Y^3,XY)IJ = I^2J^2.$$

Thus $r(IJ) \leq 1$ and hence $e_2(IJ) = 0$. Therefore using [10, Theorem 4.3] and [10, Lemma 2.11], we get $[H^2_{\mathcal{R}_{++}}(\mathcal{R}(I,J))]_{(r,s)} = 0$ for all $r,s \in \mathbb{N}$. Hence $\mathcal{R}(\mathcal{F})$ satisfies Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\mathcal{R}(\mathcal{F}),\mathbf{0})$. Note that $\mathcal{A} = \{X,Y\}$ is a joint reduction of (I,J) of type $\mathbf{e}$ and

$$XI^rJ^{s+1} + YI^{r+1}J^s = I^{r+1}J^{s+1} \text{ for all } r,s \in \mathbb{N}.$$

Thus $\operatorname{jr}_{\mathbf{e}}(\mathcal{F}) = \operatorname{jr}_{\mathcal{A}_{\mathbf{e}}}(\mathcal{F}) = 0$.

Theorem 3.13. *Let $(R, \mathfrak{m})$ be an analytically unramified Noetherian local ring of dimension $d \geq 2$ and let $I_1, \dots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\overline{\mathcal{R}}(\mathbf{I}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} \overline{\mathbf{I}^{\mathbf{n}}}$ satisfy Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\overline{\mathcal{R}}(\mathbf{I}), \mathbf{0})$. Suppose $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{n}| \leq d-1$. Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ with $|\mathbf{n}| \geq 1$.*

Proof. We use induction on $|\mathbf{n}|$. By given hypothesis the result is true upto $1 \leq |\mathbf{n}| \leq d-1$. Suppose $\mathbf{n} \in \mathbb{N}^s$ with $|\mathbf{n}| \geq d$ and the result is true for all $\mathbf{k} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{k}| < |\mathbf{n}|$. Let $\mathbf{m} = (m_1, \dots, m_s) \in \mathbb{N}^s$ such that $\mathbf{m} \leq \mathbf{n}$ and $|\mathbf{m}| = d$. Consider the filtration $\mathcal{F} = \{\overline{\mathbf{I}^{\mathbf{n}}}\}_{\mathbf{n} \in \mathbb{Z}}$. By [12], $\mathcal{F}$ is an $\mathbf{I}$-admissible filtration. Then by Theorems 2.4 and 3.9, there exists a joint reduction $\{a_{ij} \in I_i : j = 1, \dots, m_i; i = 1, \dots, s\}$ of $\mathcal{F}$ of type $\mathbf{m}$ such that

$$\overline{\mathbf{I}^{\mathbf{r}}} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \overline{\mathbf{I}^{\mathbf{r}-\mathbf{e_i}}} \text{ for all } \mathbf{r} \geq \mathbf{m}.$$

Thus $\overline{\mathbf{I}^{\mathbf{n}}} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \overline{\mathbf{I}^{\mathbf{n}-\mathbf{e_i}}}$. By induction hypothesis, $\mathbf{I}^{\mathbf{n}-\mathbf{e_i}}$ is complete for all $i \in A := \{i | n_i \geq 1\}$. Hence

$$\overline{\mathbf{I}^{\mathbf{n}}} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \overline{\mathbf{I}^{\mathbf{n}-\mathbf{e_i}}} = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} \mathbf{I}^{\mathbf{n}-\mathbf{e_i}} \subseteq \mathbf{I}^{\mathbf{n}}.$$

□

As a consequence of the above theorem we obtain a generalisation of a theorem of Reid, Roberts and Vitulli [13, Proposition 3.1] about complete monomial ideals.

Theorem 3.14. *Let $R = k[X_1, \dots, X_d]$ where $d \geq 1$ and $\mathfrak{m} = (X_1, \dots, X_d)$ be the maximal homogeneous ideal of R. Let $I_1, \dots, I_s$ be $\mathfrak{m}$-primary monomial ideals of R. Suppose $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{n}| \leq d-1$. Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ with $|\mathbf{n}| \geq 1$.*

Proof. If $d = 1$ then R is a PID and hence normal. Therefore every ideal is complete since principal ideals in normal domains are complete. Let $d \geq 2$. Since $I_1, \dots, I_s$ are monomial ideals, $\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay by [2, Theorem 6.3.5]. Let $W = R \setminus \mathfrak{m}$. Then $S = W^{-1}\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay. Since for any ideal I, $W^{-1}\overline{I} = \overline{W^{-1}I}$, we have

$$W^{-1}\overline{\mathcal{R}}(\mathbf{I}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} W^{-1}\overline{\mathbf{I}^{\mathbf{n}}} = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} (\overline{W^{-1}(\mathbf{I}^{\mathbf{n}})}) = \overline{\mathcal{R}}(W^{-1}I_1, \dots, W^{-1}I_s).$$

Therefore S satisfies Hyry's condition $\mathrm{H}_Q(S, \mathbf{0})$ where $Q = W^{-1}\mathcal{R}(\mathbf{I})$. Replace R by $W^{-1}R$. Therefore by Theorem 3.13, $W^{-1}(\mathbf{I}^{\mathbf{n}})$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $|\mathbf{n}| \geq 1$. Since $\mathfrak{m}$ is the maximal homogeneous ideal of R and $W^{-1}\left(\overline{\mathbf{I}^{\mathbf{n}}}/\mathbf{I}^{\mathbf{n}}\right) = 0$, we get the required result. □

We end the paper with three examples illustrating some of the results proved above.

Example 3.15. Let $S = \mathbb{Q}[[X,Y,Z]]$, $f = X^2+Y^2+Z^2$. Then $R = S/(f)$ is analytically unramified Cohen-Macaulay reduced local ring of dimension 2. Set $\mathfrak{m} = (X,Y,Z)/(f)$. Since $G_{\mathfrak{m}}(R) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq \mathbb{Q}[X,Y,Z]/(f)$ is reduced, $\mathfrak{m}^n$ is complete for all $n \geq 1$.
Consider the $\mathfrak{m}$-admissible filtration $\mathcal{F} = \{\overline{\mathfrak{m}^n}\}_{n\in\mathbb{Z}}$. The Hilbert polynomial of the filtration $\mathcal{F}$ is $P_{\mathcal{F}}(n) = 2\binom{n+1}{2} - n$. Set $\mathcal{R} = \mathcal{R}(\mathfrak{m})$. Since R is Cohen-Macaulay, $H^0_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F})) = 0$. By [1, Theorem 3.5], $[H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_n = \widetilde{\overline{\mathfrak{m}^n}}/\overline{\mathfrak{m}^n}$ for all $n \geq 0$ where $\{\widetilde{\overline{\mathfrak{m}^n}}\}_{n\in\mathbb{Z}}$ is the Ratliff-Rush closure filtration of $\mathcal{F}$. Therefore by [10, Proposition 3.2], $[H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_n = 0$ for all $n \geq 0$. By [1, Theorem 4.1], we get $[H^2_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_0 = 0$. Hence by [1, Lemma 4.7], $[H^2_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_n = 0$ for all $n \geq 0$. Hence $\mathcal{R}(\mathcal{F})$ satisfies the condition $\mathrm{H}_{\mathcal{R}(\mathfrak{m})}(\mathcal{R}(\mathcal{F}), 0)$.

The following examples show that Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathbf{I})}(\overline{\mathcal{R}}(\mathbf{I}), \mathbf{0})$ is sufficient but not necessary in Theorem 3.13.

Example 3.16. Let $S = \mathbb{Q}[[X,Y,Z]]$, $g = X^3+Y^3+Z^3$. Then $R = S/(g)$ is analytically unramified Cohen-Macaulay reduced local ring of dimension 2. Set $\mathfrak{m} = (X,Y,Z)/(g)$. Since $G_{\mathfrak{m}}(R) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq \mathbb{Q}[X,Y,Z]/(g)$ is reduced, $\mathfrak{m}^n$ is complete for all $n \geq 1$.
Consider the $\mathfrak{m}$-admissible filtration $\mathcal{F} = \{\overline{\mathfrak{m}^n}\}_{n\in\mathbb{Z}}$. The Hilbert polynomial of the filtration $\mathcal{F}$ is $P_{\mathcal{F}}(n) = 3\binom{n+1}{2} - 3n + 1$. Set $\mathcal{R} = \mathcal{R}(\mathfrak{m})$. Since R is Cohen-Macaulay, $H^0_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F})) = 0$. By [1, Theorem 3.5], $[H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_n = \widetilde{\overline{\mathfrak{m}^n}}/\overline{\mathfrak{m}^n}$ for all $n \geq 0$ (here $\{\widetilde{\overline{\mathfrak{m}^n}}\}_{n\in\mathbb{Z}}$ is the Ratliff-Rush closure filtration of $\mathcal{F}$). Therefore $[H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))]_0 = 0$ for all $n \geq 0$. By [1, Theorem 4.1], we get $\lambda\left(H^2_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))\right)_0 = 1$. Hence $\mathcal{R}(\mathcal{F})$ does not satisfy Hyry's condition $\mathrm{H}_{\mathcal{R}(\mathfrak{m})}(\mathcal{R}(\mathcal{F}), 0)$.

Example 3.17. Let $R = k[x,y,z]$ where k is a field of characteristic not equal to 3 and $I = (x^4, x(y^3+z^3), y(y^3+z^3), z(y^3+z^3)) + (x,y,z)^5$. In [5, Theorem 3.12], S. Huckaba and Huneke showed that $\mathrm{ht}(I) = 3$, I is normal ideal, i.e. $\overline{I^n} = I^n$ for all $n \geq 1$ and $H^2(X, \mathscr{O}_X) \neq 0$ where $X = \operatorname{Proj} \mathcal{R}(I)$. Hence $H^3_{\mathcal{R}_{++}}(\mathcal{R}(I))_0 \neq 0$. Thus $\mathcal{R}(I)$ does not satisfy Hyry's condition $\mathrm{H}_{\mathcal{R}(I)}(\mathcal{R}(I), 0)$.

References

[1] C. Blancafort, *On Hilbert functions and cohomology*, J. Algebra **192** (1997), 439-459. 15

[2] W. Bruns and J. Herzog, *Cohen-Macaulay normal rings*, Cambridge studies in Advanced Mathematics, **39**, Cambridge University Press, 1993, 13H10 (13-02). 5, 14

[3] Huy Tài Hà, *Multigraded regularity, a^*-invariant and the minimal free resolution*, J. Algebra **310** (2007), 156-179. 9

[4] M. Herrmann, E. Hyry, J. Ribbe and Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra **197** (1997), 311-341. 6

[5] S. Huckaba and C. Huneke, *Normal ideals in regular rings*, J. Reine Angew. Math. **510**, (1999), 63-82. 15

[6] C. Huneke, *The primary components and integral closures of ideals in 3-dimensional regular local rings*, Math. Annalen, **275** (1986), 617-635. 1

[7] E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. **351** (1999), 2213-2232. 4, 9

[8] E. Hyry, *Cohen-Macaulay multi-Rees algebras*, Compositio Math. **130** (2002), 319-343. 10

[9] D. Kirby and D. Rees, *Multiplicities in graded rings I : The general theory*, Commutative algebra: syzygies, multiplicities, and birational algebra (W. J. Heinzer, C. L. Huneke and J. D. Sally, eds.), Contemp. Math.,**159** (1994), 209-267. 3

[10] S. K. Masuti, P. Sarkar and J. K. Verma, *Hilbert polynomials of multigraded filtrations of ideals*, J. Algebra, **444** (2015), 527-566. 2, 6, 10, 13, 15

[11] S. K. Masuti, T. J. Puthenpurakal and J. K. Verma, *Local Cohomology of Multi-Rees algebras with applications to joint reductions and complete ideals*, Acta Math. Vietnamica, **40** (2015), 479-510. 10

[12] D. Rees, *Generalisations of reductions and mixed multiplicities*, J. London Math. Soc. (2) **29** (1984), 397-414. 2, 7, 8, 12, 14

[13] L. Reid, L. G. Roberts and M. A. Vitulli, *Some results on normal homogeneous ideals,* Comm. Algebra **31** (2003), 4485-4506. 1, 14

[14] Snapper, E., *Polynomials associated with divisors*, J. Math. Mech. **9** (1960), 123-129. 10

[15] O. Zariski, *Polynomial ideals defined by infinitely near base points*, American Journal of Mathematics **60** (1938), 151-204. 1

[16] O. Zariski and P. Samuel, Commutative algebra, Vol 2, van Nostrand, 1960. 1, 13

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