

Non-equilibrium quantum relaxation across a localization-delocalization transition

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We consider the one-dimensional XX -model in a quasi-periodic transverse-field described by the Harper potential, which is equivalent to a tight-binding model of spinless fermions with a quasi-periodic chemical potential. For weak transverse field (chemical potential), $h < h_c$, the excitations (fermions) are delocalized, but become localized for $h > h_c$. We study the non-equilibrium relaxation of the system by applying two protocols: a sudden change of h (quench dynamics) and a slow change of h in time (adiabatic dynamics). For a quench into the delocalized (localized) phase, the entanglement entropy grows linearly (saturates) and the order parameter decreases exponentially (has a finite limiting value). For a critical quench the entropy increases algebraically with time, whereas the order parameter decreases with a stretched-exponential. The density of defects after an adiabatic field change through the critical point is shown to scale with a power of the rate of field change and a scaling relation for the exponent is derived.

I. INTRODUCTION

Non-equilibrium relaxation in a closed quantum system following a change of some parameter(s) in the Hamiltonian (such as the amplitude of the transverse field, $h(t)$) is of recent interest, both experimentally and theoretically. Considering the speed of variation of the parameter, we generally discriminate between two limiting processes. For the *quench dynamics*, the parameter is modified instantaneously, which experimentally can be realized in ultra cold atomic gases^{1–11} using the phenomenon of Feshbach resonance. In this process the evolution of different observables after the quench is of interest, as well as the possible existence and properties of the stationary state, in particular in integrable and non-integrable systems^{12–62}. In the other limiting relaxation process, in the so called *adiabatic dynamics* the parameter is varied very slowly, usually linearly in time, such as $h(t) = t/\tau$ across a phase-transition point. In this case one is interested in the density of defects, which are produced when the system falls out of equilibrium close to the critical point^{63–80}.

Most of the results for non-equilibrium quantum relaxation are obtained for homogeneous systems, for which the eigenstates are generally extended. As a consequence after a quench the general (time- and space-dependent) correlation functions decay exponentially, which can be explained (even quantitatively) within a semi-classical theory^{18,37,42,43}. In the stationary state thermalization is expected to hold for non-integrable models^{16–26} whereas for integrable models it was a general belief that the stationary state is described by a so called generalized Gibbs ensemble (GGE). Very recent studies^{58–62} show, however, that the GGE is not generally correct. When it does not work it is due to the fact that the generalized eigenstate thermalization hypothesis fails and it strongly appears to be linked with the presence of bound states in the spectrum.

Concerning adiabatic dynamics, variants of the Kibble-Zurek scaling theory^{63–65} are found to hold: the density of defects scales as $\tau^{-\kappa}$ and κ is related to the static critical exponents z and ν , as well as to the dimension of the system.

Among inhomogeneous quantum systems, random quantum spin chains have most frequently been studied in the context of non-equilibrium relaxation^{81–85}. In these disordered one-dimensional systems, the eigenstates are localized even in the presence of interactions, which prevents thermalization after a quench. Consequently an unusual relaxation can be observed: after a (non-critical) quench both the average entanglement entropy and the magnetization approach a non-vanishing stationary value. After a critical quench (i.e. a quench to the critical point), the dynamics is ultra-slow: the entanglement entropy grows in time as $\ln \ln t$ ^{82–85}, whereas the magnetization behaves as $[\ln(t)]^{-A}$ with a disorder dependent exponent, A ⁸⁶. For the adiabatic dynamics the defect density is found to scale as⁶⁷ $1/\ln^2(\tau)$, which is a consequence of the equilibrium dynamical scaling relation⁸⁷: $\xi \sim \ln^2(\tau)$, ξ being the correlation length.

Localization of eigenstates can exist in non-disordered systems, too, as for instance in quasi-periodic systems. A well known example is the Aubry-André model⁸⁸, which is a one-dimensional hopping model with a specific quasi-periodic potential denoted as Harper's potential⁸⁹. This model could be experimentally realized by ultra cold atomic gases in optical lattices having two periodic optical waves with different incommensurate wavelengths⁹⁰. For weak quasi-periodic potential the eigenstates are extended, but they become localized for a sufficiently strong potential. A similar scenario has been predicted for interacting particles: sufficiently strong quasi-periodic potential leads to many-body localization^{91,92}. The quench dynamics in the Aubry-André model for hard-core bosons has been studied recently⁹³, where the GGE scenario was shown to be valid in the extended phase, but fails in the

localized phase.

In the present paper we revisit the non-equilibrium relaxation properties of the Aubry-André model. New features of our study are the following. We consider a magnetic model, the $S = 1/2$ XX-chain in a quasi-periodic transverse field, which - after a Jordan-Wigner transformation - is equivalent to a tight-binding model of spinless fermions in a quasi-periodic chemical potential. We study the non-equilibrium dynamics after a sudden change of the amplitude of the transverse field and compute the dynamical evolution of the entanglement entropy, as well as the relaxation of the magnetization. We investigate separately, when the quench is performed to the extended or to the localized phase, as well as to the transition point. We also study adiabatic dynamics, which has not been considered before, and calculate the density of defects which are created during the process, when the amplitude of the transverse field is passed linearly through the localization-delocalization transition point.

The paper is organized as follows: The model and the observables of interest are introduced in Sec.II. Results for the quench dynamics and the adiabatic dynamics are shown in Secs.III and IV, respectively. Our paper is closed by a discussion in the last section.

II. MODEL AND OBSERVABLES

A. Quasi-periodic XX-chain

We consider the spin-1/2 XX-chain in the presence of a position dependent transverse field, which is defined by the Hamiltonian:

$$\mathcal{H} = -\frac{J}{4} \sum_{n=1}^L (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) - \sum_{n=1}^L h_n \sigma_n^z, \quad (1)$$

in terms of the $\sigma_n^{x,y,z}$ Pauli-matrices at site n . In the calculation we apply either periodic boundary conditions, thus $\sigma_{L+1}^x \equiv \sigma_1^x$ and $\sigma_{L+1}^y \equiv \sigma_1^y$, or free boundary conditions, when the first sum in Eq.(1) runs up to $L - 1$. In the following we fix $J = 1$ and use a quasi-periodic potential:

$$h_n = h \cos(2\pi\beta n) \quad (2)$$

where β is an irrational number: typically we use $\beta = \frac{\sqrt{5}-1}{2}$ the inverse of the golden mean, which is the ‘‘most’’ irrational number. Using the Jordan-Wigner transformation the Hamiltonian is expressed in terms of fermion creation (c_n^\dagger) and annihilation (c_n) operators⁹⁴:

$$\mathcal{H} = -\frac{1}{2} \sum_{n=1}^{L-1} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - h \sum_{n=1}^L \cos(2\pi\beta n) c_n^\dagger c_n, \quad (3)$$

thus in Eq.(3) we have a tight-binding model of spinless fermions in a quasi-periodic chemical potential. (For periodic boundary conditions there is an extra term in

Eq.(3): $(c_L^\dagger c_1 + c_1^\dagger c_L) \exp(i\pi\mathcal{N})/2$, where $\mathcal{N} = \sum_{n=1}^L c_n^\dagger c_n$ is the number of fermions.)

This type of potential appears first in Harper’s paper⁸⁹, in which he showed that Hamiltonian in Eq.(3) for $h = 1$ describes an electron on a square lattice in a perpendicular magnetic field. Introducing a new set of fermion operators η_q through the canonical transformation:

$$\eta_q = \sum_{n=1}^L \phi_{q,n} c_n, \quad (4)$$

with $\sum_{q=1}^L \phi_{q,n} \phi_{q,n'} = \delta_{n,n'}$ the Hamiltonian in Eq.(3) is transformed to a diagonal form:

$$\mathcal{H} = \sum_q \epsilon_q (\eta_q^\dagger \eta_q - 1/2). \quad (5)$$

Here the energy of modes, ϵ_q , and the components of vectors, $\phi_{q,n}$ satisfy the almost Mathieu equation⁹⁵:

$$\frac{1}{2} \phi_{q,n-1} + h_n \phi_{q,n} + \frac{1}{2} \phi_{q,n+1} = -\epsilon_q \phi_{q,n}. \quad (6)$$

There is a vast literature about properties of the almost Mathieu equation, as well as on the properties quasi-periodic Hamiltonians both in mathematical⁹⁶ and physical⁹⁷ points of view.

B. Aubry-André duality

Following Aubry and André⁸⁸ a new set of fermion operators are introduced:

$$c_{\bar{k}} = \frac{1}{\sqrt{L}} \sum_n \exp(i2\pi\bar{k}\beta n) c_n \quad (7)$$

which are eigenstates of the momentum operator with eigenvalue: $k = \bar{k} F_{n-1} \text{mod} F_n$, where F_n is the n -th Fibonacci number and $L = F_n$. In terms of these the Hamiltonian is given by:

$$\mathcal{H} = -\frac{h}{2} \left[\sum_{\bar{k}=1}^L (c_{\bar{k}}^\dagger c_{\bar{k}+1} + c_{\bar{k}+1}^\dagger c_{\bar{k}}) - \frac{2}{h} \sum_{\bar{k}=1}^L \cos(2\pi\beta\bar{k}) c_{\bar{k}}^\dagger c_{\bar{k}} \right]. \quad (8)$$

Note that Eq.(8) is in the same form as that in Eq.(3), thus the Hamiltonian satisfies the duality relation:

$$\mathcal{H}(h) = h\mathcal{H}(1/h). \quad (9)$$

Through Eq.(9) the small h regime of the Hamiltonian, in which the eigenstates are extended in the real space are connected with the large h regime, in which the eigenstates have extended properties in the Fourier space, thus these are in the real space localized. The localization transition takes place at the self-duality point, thus the critical amplitude of the field is $h_c = 1$. For $h > 1$ the

localized states have a finite correlation length, ξ , which is given by⁸⁸:

$$\xi = \frac{1}{\ln(h)}, \quad h > 1, \quad (10)$$

for all eigenstates of \mathcal{H} . Similar conclusion holds for the eigenvectors, $\phi_{q,n}$ in Eq.(6) which are used to diagonalize the Hamiltonian in Eq.(5). The $\phi_{q,n}$ -s are localized in the $h > 1$ regime with the same correlation length given in Eq.(10) and for large $|h|$ these are given by:

$$\phi_{q,n} = \delta_{n,n_q}, \quad \epsilon_q = -h \cos(2\pi\beta n_q), \quad |h| \gg 1. \quad (11)$$

C. Observables in the quench dynamics

In the quench process the amplitude of the transverse field is suddenly changed from a value of h_0 for $t < 0$ to another value, say h for $t > 0$ and the Hamiltonians are denoted by \mathcal{H}_0 and \mathcal{H} , respectively. For $t < 0$ the system is in the ground state of the initial Hamiltonian, $|\Psi_0^{(0)}\rangle$, while for $t > 0$ its time-evolution involves the new Hamiltonian, \mathcal{H} , and given by $|\Psi_0(t)\rangle = \exp(-i\mathcal{H}t)|\Psi_0^{(0)}\rangle$, thus generally $|\Psi_0(t)\rangle$ is not an eigenstate of \mathcal{H} . We set \hbar to unity through out this paper. The expectation value $A(t)$ of an observable, \hat{A} , is given by $\langle \Psi_0^{(0)} | \hat{A}_H(t) | \Psi_0^{(0)} \rangle$, where $\hat{A}_H(t) = \exp(i\mathcal{H}t)\hat{A}\exp(-i\mathcal{H}t)$ is \hat{A} in the Heisenberg picture. One can calculate time-dependent correlation functions in similar way.

In the actual problem we calculate the entanglement entropy $\mathcal{S}_\ell(t)$ of the first ℓ spins of the chain and the rest of the system, which is defined as: $\mathcal{S}_\ell(t) = -\text{Tr}_\ell[\rho_\ell(t) \ln \rho_\ell(t)]$. Here $\rho_\ell(t) = \text{Tr}_{n>\ell} |\Psi_0(t)\rangle \langle \Psi_0(t)|$ is the reduced density matrix with $|\Psi_0(t)\rangle$ being the state of the complete system at time t obtained after solving the Schrödinger equation. In a *homogeneous chain* for $L \rightarrow \infty$ and $\ell \gg 1$ the entanglement entropy has two different regions³⁷. For $t < \ell/v_{\max}$, where v_{\max} is some maximal velocity of quasi-particles, the entanglement entropy increases linearly: $\mathcal{S}_\ell(t) \sim t$; while for $t > \ell/v_{\max}$, it saturates as $\mathcal{S}_\ell(t) \sim \ell$. For *random quantum spin chains*, due to localized excitations the entanglement entropy saturates at a finite value, except at the critical point, where there is an ultra-slow increase of the form⁸²: $\mathcal{S}_\ell(t) \sim \ln \ln t$. In the one-dimensional Fibonacci *quasi-crystal*, where the spectrum of excitations is singular continuous⁹⁶, the entropy grows in a power-law form: $\mathcal{S}_\ell(t) \sim t^\sigma$, with $0 < \sigma < 1$ being a function of the quench parameters⁹⁸.

Another observable we calculate is the local order-parameter (magnetization), $m_l(t)$, at a position l in an open chain. Here we follow the method of Yang⁹⁹ and define $m_l(t)$ for large L by the off-diagonal matrix-element: $m_l(t) = \langle \Psi_0^{(0)} | \sigma_l^x(t) | \Psi_1^{(0)} \rangle$, where $|\Psi_1^{(0)}\rangle$ is the first excited state of \mathcal{H}_0 . In a *homogeneous chain* of infinite length ($L \rightarrow \infty$), the magnetization for a bulk site $l \gg 1$ has an exponential decay^{42,48},

both in time: $m_l(t) \sim \exp(-t/\tilde{\tau})$ for $t < l/v_{\max}$ and in space: $m_l(t) \sim \exp(-l/\tilde{\xi})$ for $t \gg l/v_{\max}$. Here the non-equilibrium relaxation time, $\tilde{\tau}$, and the non-equilibrium correlation length, $\tilde{\xi}$ are given functions of the quench parameters, h_0 and h . For *random quantum spin chains* the local magnetization relaxes to a finite limiting value, except at the critical point, where the decay is logarithmically slow⁸⁶: $m_b(t) \sim [\ln t]^{-A}$ and A depends on the form of the disorder. In the one-dimensional Fibonacci *quasi-crystal* the relaxation of the bulk magnetization is given in a stretched-exponential form⁹⁸: $m_b(t) \sim \exp(-C/t^\mu)$. Here the exponent μ and the exponent of the entanglement entropy, σ , are found to be close to each other, at least in the so called non-oscillatory phase.

D. Density of defects in the adiabatic dynamics

In adiabatic dynamics, the amplitude of the transverse field in Eq.(2) is varied linearly: $h = h(t) = t/\tau$ and we are interested in the density of defects created during this process. At the starting point, at $t = -\infty$ the ground state of the system, denoted by $\Psi_0(-\infty)$, is a classical product state, since the spins follow the direction of the local field. It is $\sigma_n^z = 1$ ($c_n^\dagger c_n = 1$) for $\cos(2\pi\beta n) > 0$ and $\sigma_n^z = -1$ ($c_n^\dagger c_n = 0$) for $\cos(2\pi\beta n) < 0$.

In the following we consider the length of the chain an *even* number, so that in that state $\Psi_0(-\infty)$ the total magnetization is zero and it is half-filled in terms of fermions. As time goes on the system evolves according to the time-dependent Schrödinger equation: its state at time t satisfies the relation: $d\Psi/dt = -i\mathcal{H}(t)\Psi(t)$, with the boundary condition: $\Psi(-\infty) = \Psi_0(-\infty)$. Solving the eigenvalue problem of the Hamiltonian at time t results in a ground state $\Psi_0(t)$, which generally differs from $\Psi(t)$, obtained through dynamic evolution. Our goal is to determine how far is $\Psi(t)$ from the true ground state as a function of the parameter τ . This is quantified by the total excitation probability, P , which can be calculated in the fermionic description in the following way. First, we notice that the Heisenberg equation of motion for the operators $c_{n,H}(t)$ are linear⁶⁷, since the Hamiltonian in Eq.(3) is quadratic. From this follows that the evolution of vectors, $\tilde{\phi}_{q,n}(t)$, which enter in the diagonalization of the Hamiltonian in Eq.(6) satisfy the differential equation:

$$i \frac{d\tilde{\phi}_{q,n}}{dt} = \frac{1}{2}\tilde{\phi}_{q,n-1} + h_n\tilde{\phi}_{q,n} + \frac{1}{2}\tilde{\phi}_{q,n+1}, \quad (12)$$

with the boundary condition: $\tilde{\phi}_{q,n}(-\infty) = \phi_{q,n}(-\infty)$, where the latter are given in Eq.(11). Note, that $\phi_{q,n}(t)$, which denotes the equilibrium value of the vector evaluated with the potential at time t through Eq.(6) is generally different from its dynamically evolved value: $\tilde{\phi}_{q,n}(t)$ and from this can we calculate the excitation probability.

To do so we note that at the starting state at $t = -\infty$ half of the fermionic states in Eq.(8) are occupied,

these are denoted by Q^- , whereas the other half of the fermionic states, the excited ones, denoted by Q^+ , are empty. By strictly adiabatic time evolution the excited states would stay empty. The amount of excitations than can be measured through the excitation probability:

$$P_t = \frac{2}{L} \sum_{q \in Q^+} \sum_{q' \in Q^-} p_{q,q'} , \quad (13)$$

in terms of the partial excitation probabilities:

$$p_{q,q'} = \left| \sum_n \tilde{\phi}_{q,n}(t) \phi_{q',n}(t) \right|^2 . \quad (14)$$

Note that P_t is normalized in the sense that $0 \leq P_t \leq 1$.

In the actual calculation we have taken two limiting final states: i) $t = 0$, when the quench is performed at the middle of the extended phase and ii) $t = \infty$, when the quench goes across the extended phase and ends at the other limiting side of the localized phase. In the first case the localization-delocalization transition point is crossed once at $h = -1$, while in the second protocol it is crossed twice, at $h = \pm 1$.

III. QUENCH DYNAMICS

In the (sudden) quench dynamics we have used $\beta = (\sqrt{5}-1)/2$, the inverse golden-mean ratio for the parameter of the Harper potential and the length of the finite chains were fixed to a Fibonacci number F_n . We have calculated the entanglement entropy and the local magnetization up to $L = F_{17} = 1597$.

A. Entanglement entropy

The entanglement entropy, \mathcal{S}_ℓ is calculated between a block of length, $\ell = F_{n-2}$ and its environment of length F_{n-1} with periodic boundary conditions. (For details of the calculation of the entanglement entropy in the free-fermion basis see the Appendix of Ref.[100].) We used the ground state corresponding to the initial field $h_0 = 0$ as the initial state and then made quenches to the extended ($0 < h < 1$) and to the localized phases ($h > 1$), as well as to the critical point ($h = 1$). Numerical results for $\mathcal{S}_\ell(t)$ are shown in Figure 1.

The dynamics of the entanglement entropy has two different regimes (as for homogeneous chain): for short times it is an increasing function of time and for long times it saturates to some value. For quenches to the extended phase the time-dependence in the initial period is linear, $\mathcal{S}_\ell(t) \approx \alpha(h)t$ and the saturation value is $\tilde{\mathcal{S}}_\ell \sim \ell$. This behavior is qualitatively similar to homogeneous system. Estimates of the prefactor of the linear term, α are shown in Fig. 2. Starting from $h = h_0 = 0$ α is first increasing, has a maximum around $h = 0.5$ and then decreasing to 0 at $h = 1$.

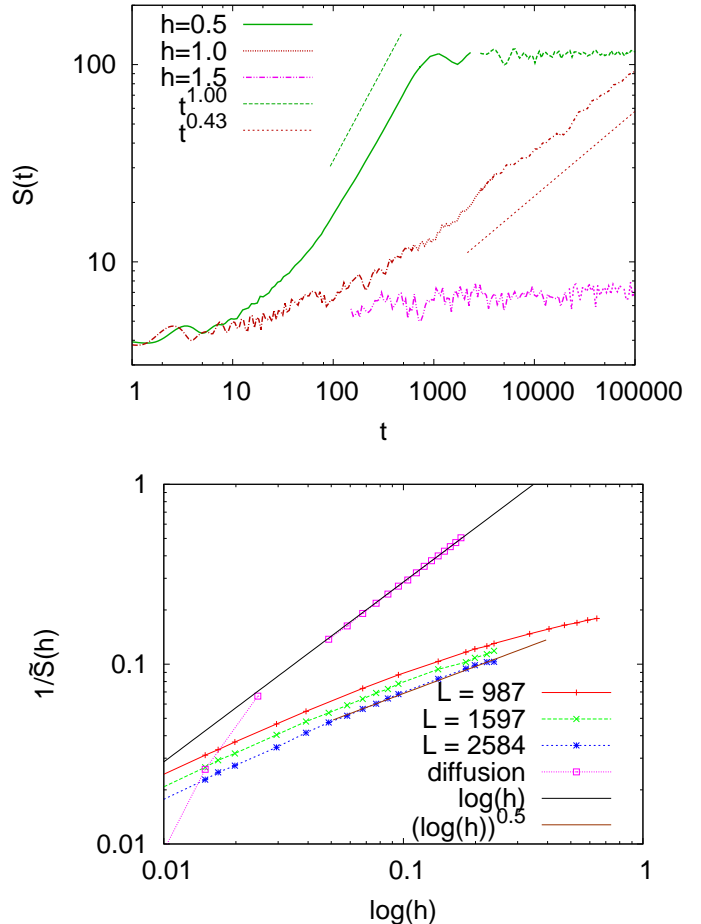


FIG. 1: (Color online) Dynamical entanglement entropy after a quench from $h_0 = 0$ to different values of h (upper panel). Saturation values of the entanglement entropy and the limiting value of the width of the wave packet (diffusion) in the localized phase show a power-law divergence close to the transition point (lower panel).

After a quench into the localized phase the entropy saturates quickly to an ℓ independent value: $\tilde{\mathcal{S}}_\ell = \tilde{\mathcal{S}}(h)$, $h > 1$. We have checked that close to the transition point $\tilde{\mathcal{S}}(h)$ diverges:

$$\tilde{\mathcal{S}}(h) \sim |\ln(h)|^{-\sigma'} , \quad (15)$$

with an exponent: $\sigma' = 0.50(4)$, see in the lower panel of Fig.1.

Finally, if the quench is performed to the transition point the growth of the entropy is given in a power-law form:

$$S(t) \sim t^\sigma , \quad (16)$$

with an exponent $\sigma = 0.43(5)$. Using phenomenological scaling theory a relation between the exponents σ' and σ can be derived in the following way. Under uniform scaling transformation, when lengths are rescaled by a factor $b > 1$ the entanglement entropy behaves as:

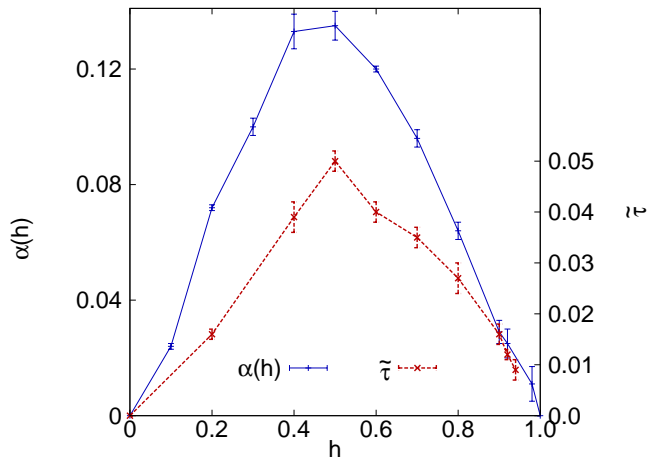


FIG. 2: (Color online) Prefactor of the linear part of the dynamical entanglement entropy (left axis) and the relaxation time (right axis) after a quench from $h_0 = 0$ to different values of h .

$\tilde{\mathcal{S}}(\ln h, t) = b^s \tilde{\mathcal{S}}(b/\ln h, t/b^z)$ for $h \geq 1$, where we have used the form of the correlation length in Eq.(10) and $z = 1$ is the dynamical exponent. Now taking the scale factor $b = t^{1/z}$ we obtain $\tilde{\mathcal{S}}(\ln h, t) = t^{s/z} \hat{\mathcal{S}}(t^{1/z} \ln h)$. At the critical point, $h = 1$, the scaling function has the limiting value $\lim_{u \rightarrow \infty} \hat{\mathcal{S}}(u) = \text{cst}$, thus $\sigma = s/z = s$. Similarly, taking $b = 1/\ln(h)$ we can show that $\sigma' = s$, thus $\sigma = \sigma'$ in agreement with the numerical results.

The properties of the dynamical entropy can be explained in terms of anomalously diffusing quasiparticles, see in Sec. III C.

B. Local magnetization

The local magnetization, $m_l(t)$ is measured in a free chain of length $L = F_n$ at a position $l = F_{n-2}$, for technical details see the Appendix of Ref.[98]. In this region of the chain the local magnetization is practically independent of l and we consider it as the bulk magnetization and will be denoted by $m_b(t)$. The numerically calculated time-dependent bulk magnetizations after a quench from $h_0 = 0$ to different values of h are shown in Fig.3. If the quench is performed to the extended phase ($0 < h < 1$) the decay of magnetization is exponential: $m_b(t) \sim \exp(-t/\tilde{\tau})$, as in the homogeneous system. Estimates for the characteristic time, $\tilde{\tau}(h)$ are given in Fig.2: with varying h it has similar characteristic as the prefactor of the linear part of the entanglement entropy. If the quench is performed to the localized phase $h > 1$ the magnetization approaches a finite limiting value. Finally, for the critical quench ($h = 1$) the decay is stretched exponential:

$$m_b(t) \sim A(t) \exp(-Ct^\mu), \quad (17)$$

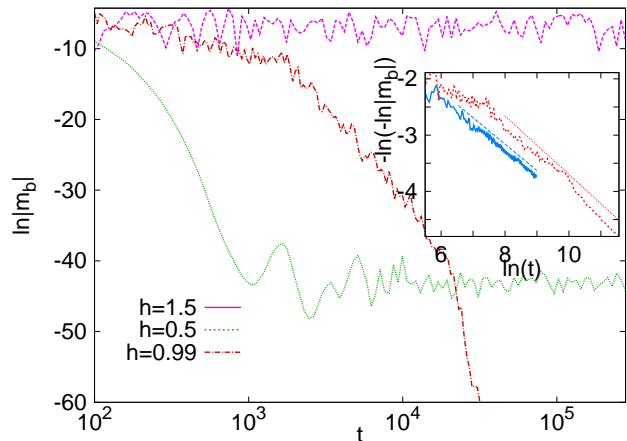


FIG. 3: (Color online) Bulk magnetization after a quench from $h_0 = 0$ to different values of h . In the inset quench to the critical region is shown in agreement with the stretched-exponential form in Eq.(17) (the straight lines have a slope $\mu = 0.47$).

where $A(t)$ is some oscillatory function and $\mu = 0.47(5)$. This is illustrated in the inset of Fig.3. This behavior is interpreted in terms of quasiparticles in the following section.

C. Quasiparticle interpretation

Non-equilibrium quench dynamics is well described within the framework of a semiclassical theory^{18,37,42,43}. It is based on the concept of quasiparticle that are produced uniformly in the system during the quench and which move classically after production. We regard these quasiparticles as wave packets, which are localized at some site at $t = 0$ and which perform afterwards a diffusive motion. Following previous studies in quasicrystals^{98,101} we construct the wave packet connecting sites n and n' at time t in the form:

$$W_{n,n'}(t) = \sum_q \cos(\epsilon_q t) \phi_{q,n} \phi_{q,n'}, \quad (18)$$

in terms of the eigenvectors and eigenvalues of Eq.(6) calculated with the amplitude h , i.e. after the quench. Due to normalization of the eigenvectors $W_{n,n'}(0) = \delta_{n,n'}$. The width of the wave-packet created at site n after time t is given by:

$$d(n, t) = \left[\sum_{n'} (n - n')^2 |W_{n,n'}(t)|^2 \right]^{1/2}, \quad (19)$$

which is then averaged over the starting positions, thus $d(t) = \overline{d(n, t)}$.

We have calculated $d(t)$ for different values of the amplitude of the transverse field and these are shown in

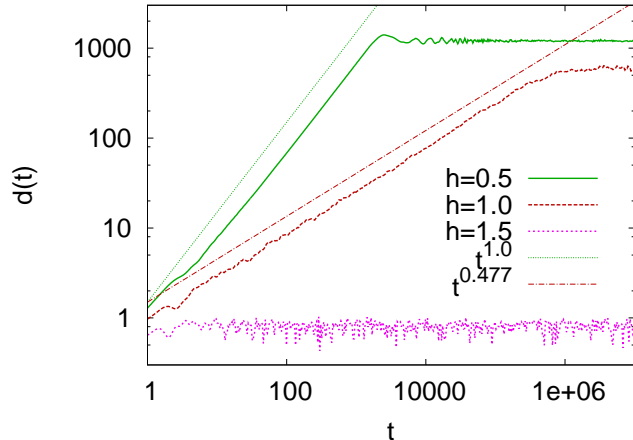


FIG. 4: Time-dependent width of the wave packet at different amplitudes of the transverse field.(Color online)

Fig.4. In agreement with previous studies¹⁰² $d(t)$ grows linearly in the extended phase ($0 < h < 1$) thus the quasiparticles move ballistically. From this follows - repeating the arguments of the semiclassical theory⁴³ - thus the dynamical entropy grows linearly and the bulk magnetization has an exponential decay. In the localized phase ($h > 1$) the width of the wave-packet stays finite, $d(t) \rightarrow \tilde{d}$. We have checked that close to the transition point this limiting value scales as the localization length in the system: $\tilde{d} \sim \xi$, see in the lower panel of Fig.1.

Finally, at the transition point ($h = 1$) the width of the wave packet grows algebraically with time: $d(t) \sim t^D$, where the diffusion exponent is estimated as $D = 0.477(10)$. In the semiclassical theory the anomalous diffusion of quasiparticles manifests itself in the modified form of the dynamical entanglement entropy in Eq.(16) and of the bulk magnetization in Eq.(17). The corresponding exponents, σ , μ and D should be equal, which is indeed satisfied within the error bars of the numerical estimates.

IV. ADIABATIC DYNAMICS

The adiabatic dynamics is calculated numerically in systems of finite size $L = 2F_n$ with $\beta = F_{n-1}/F_n$ as an approximant of the inverse golden mean ratio. (In the fermionic representation in Eq.(3) for simplicity we used the so called c -cyclic boundary condition, see in Ref.⁹⁴.) We set $|h_{\max}| = 10$ for the largest amplitude of the transverse field and checked that the numerical results are stable: they do not change if we used instead $|h_{\max}| = 20$. The differential equation in Eq.(12) is integrated numerically using a Runge-Kutta method with adaptive stepsize in time to keep the relative error less than 10^{-6} .

Numerical results of the excitation probability as a function of the time-scale, τ is shown in Fig.5 for the two types of final states, with $t = 0$ and $t = \infty$, respectively.

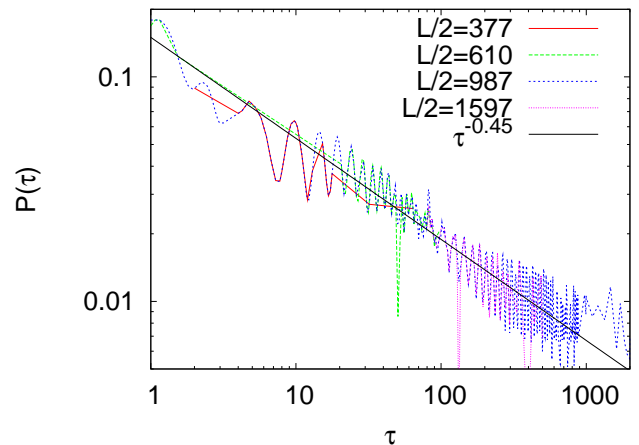
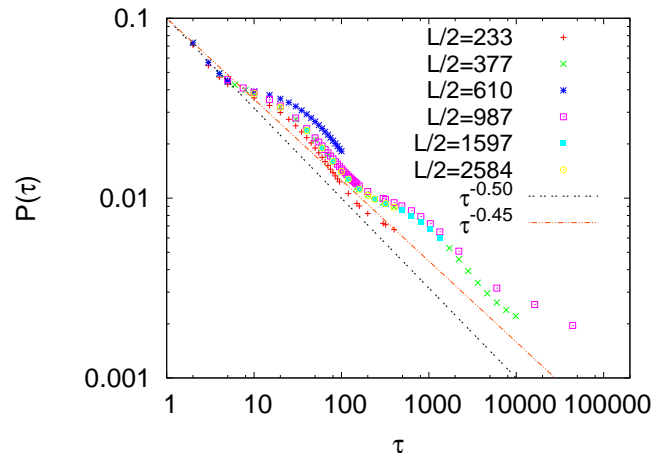


FIG. 5: Excitation probability as a function of the time-scale, τ , after an adiabatic process from $h = -\infty$ to $h = 0$ (upper panel) and to $h = \infty$ (lower panel) calculated in finite systems of sizes $L = 2F_n$ with $n = 13, 14, \dots, 18$.(Color online)

In the first case, $t = 0$ the largest Fibonacci parameter in the calculation was $n = 18$, while for $t = \infty$ it was $n = 17$. In both cases the excitation probability has an asymptotic power-law dependence:

$$P_t(\tau) \sim A_t(\tau)\tau^{-\kappa}, \quad (20)$$

but the prefactors, $A_t(\tau)$ have different functional forms. In the first case with $t = 0$ when the localization-delocalization transition is crossed once (at $h = -1$) the prefactor has a weak, approximately log-periodic oscillating form: $A_0(\tau) \sim \sin^2(\log(\tau/\tau_0))$. This type of log-periodic oscillations are due to discrete scale invariance and these are often present in quasi-periodic and aperiodic systems¹⁰³. Due to this correction the decay exponent, κ can only be estimated with some uncertainty:

$$\kappa = 0.45(5). \quad (21)$$

In the second protocol with $t = \infty$ when the localization-delocalization transition is crossed twice (at $h = -1$ and $h = 1$) the prefactor has oscillations in τ , $A_\infty(\tau) \sim$

$\sin^2(\tau/\tau_\infty + cst.)$ with $\tau_\infty \approx 0.15$, to which also a log-periodic correction is supplemented. This oscillatory phase is analogous to the Stückelberg oscillations^{104,105} of a periodically driven two-level system which arises due to the interference of probability amplitude between the ground and the excited state, when the region of avoided level crossing is passed twice. In the second case due to the oscillations the estimate of κ is somewhat less accurate. We checked, however, that the numerical data in Fig.5 are compatible with the estimate for κ in Eq.(21).

In the following we explain the numerical value of the decay exponent in Eq.(21) and relate it to the combination of other exponents. First, let us recapitulate the reasoning of traditional scaling theory⁶⁵. The amplitude of the transverse field at time \tilde{t} is given by $h(\tilde{t}) = 1 + \tilde{t}/\tau$, and therefore the distance from the critical point $\delta(\tilde{t}) = \tilde{t}/\tau$. This implies that the equilibrium relaxation time of the system at time \tilde{t} is $\tilde{t}' \sim \xi^z \sim \delta^{-\nu z} = (\tilde{t}/\tau)^{-\nu z}$, where ξ is the equilibrium correlation length. When the relaxation time \tilde{t}' is of the same order as the time \tilde{t} the system falls out of equilibrium, i.e. the ground state cannot follow adiabatically the field change any more. The condition $\tilde{t} = \tilde{t}'$ implies

$$\tilde{t} \sim \tau^{\frac{\nu z}{\nu z + 1}}. \quad (22)$$

For $|t| < \tilde{t}$ defects are produced and transitions to excited states occur. The typical distance between neighbouring defects is then given by ξ ($\sim \tilde{t}^{1/z}$), thus the phase-space of excitations in a d -dimensional system is $\Omega \sim \xi^{-d} \sim \tau^{-\frac{d\nu}{\nu z + 1}}$. Then, it is usually expected that the elementary transition probabilities, such as $p_{q,q'}$ in Eq.(14) are independent of the scale thus $P(\tau) \sim \Omega$ and we arrive at the scaling relation:

$$P_{sc}(\tau) \sim \tau^{-\frac{d\nu}{\nu z + 1}}. \quad (23)$$

For the Aubry-André model with $d = 1$ and $\nu = z = 1$ the prediction of traditional scaling theory is $\kappa_{sc} = 0.5$, which is somewhat larger than (although at the border of) the numerical estimate in Eq.(21). However, the assumptions used in the derivation of $P_{sc}(\tau)$ are not valid for the Aubry-André model since the ground state of the Hamiltonian in Eq.(5) is not a continuous function of the amplitude of the transverse field at $h = \pm 1$. Therefore we study numerically the scaling behavior of the elementary transition probabilities, $p_{q,q'}$, calculated at $t = 0$, i.e. for the first protocol. First we notice that $p_{q,q'} = p_{q',q}$ and arrange the $p_{q,q'}$ -s in decreasing order. Then in Eq.(13) we sum up the contribution of the largest N terms:

$$P(N, L, \tau) = \frac{2}{L} \sum_{q \in Q^+ q' \in Q^-}^{N'} p_{q,q'}, \quad (24)$$

which is denoted by the prime at the summation and this quantity is called the partial excitation probability. For large- N we can rearrange the parameters q (and also q'),

such that in Eq.(13) by restricting the summations to $q, q' \leq \sqrt{N}$ we get (asymptotically) $P(N, L, \tau)$. Generally, for $q_1 < q_2$ ($q'_1 < q'_2$) the free-fermionic energies in Eq.(6) satisfy $\epsilon_{q_2} < \epsilon_{q_1} < 0$ ($0 < \epsilon_{q'_1} < \epsilon_{q'_2}$).

We have calculated the partial excitation probability, $P(N, L, \tau)$, normalized with its limiting value $P_0(\tau)$ for different sizes and for different decay parameters. For large N and L the partial excitation probability is found to be a function N/L^2 , thus $P(N, L, \tau) = \tilde{P}(N/L^2, \tau)$, as illustrated in the upper panel of Fig.6 for different values of L at a fixed value of τ . The τ -dependence of $\tilde{P}(N/L^2, \tau)$ is shown in the lower panel of Fig.6 at a fixed (large) L and for different values of $\tau < L$. With increasing τ the scaling functions appear to approach the same limiting curve, thus $P(N, L, \tau)$ is factorized as

$$P(N, L, \tau) = \pi(N/L^2)P_0(\tau), \quad (25)$$

for large enough τ .

As seen in the lower panel of Fig.6 in the log-log plot $\pi(N/L^2)$ has a linear section over several decades and then it saturates for large arguments, say for $N > N_{\text{eff}}$. Thus we can approximate

$$\frac{P(N, L, \tau)}{P_0(\tau)} \approx \begin{cases} \frac{P(N, L, \tau)}{P(N_{\text{eff}}, L, \tau)} \sim (N/N_{\text{eff}})^\omega & N \leq N_{\text{eff}} \\ 1, & N > N_{\text{eff}} \end{cases} \quad (26)$$

From the data in the lower panel of Fig.6 we estimate $\omega = 0.90(2)$. Now let us consider the scaling behavior of $P(N, L, \tau) = \tilde{P}(N/L^2, \tau)$, when lengths are rescaled by a factor $b > 1$. Keeping in mind that $N_{\text{eff}}/L^2 \sim \Omega$ is the phase-space of excitations given by $\Omega \sim \xi^{-1}$ we obtain:

$$\tilde{P}(N/L^2, \tau) = b^{-\omega} \tilde{P}(bN/L^2, b^{-\frac{\nu z + 1}{\nu}} \tau). \quad (27)$$

Here the prefactor, $b^{-\omega}$ follows from Eq.(26) and the scaling dimension of τ can be read from Eq.(22). Now taking $b = \tau^{\frac{\nu}{\nu z + 1}}$ we get:

$$\tilde{P}(N/L^2, \tau) = \tau^{-\frac{\omega\nu}{\nu z + 1}} \pi(\tau^{\frac{\nu}{\nu z + 1}} N/L^2), \quad (28)$$

thus at $N \approx N_{\text{eff}}$:

$$P_0(\tau) \sim \tau^{-\frac{\omega\nu}{\nu z + 1}}, \quad (29)$$

and $\kappa = \kappa_{sc}\omega$ (also $N_{\text{eff}}/L^2 \sim \tau^{-\frac{\nu}{\nu z + 1}}$). With the measured value of ω we get $\kappa \simeq 0.45$ in agreement with the direct estimate in Eq.(21).

V. DISCUSSION

In this paper we have studied the non-equilibrium dynamics of the Aubry-André model for the $S = 1/2$ -spin XX -chain in the presence of a quasi-periodically modulated transverse field, which is equivalent to a tight-binding model of spinless fermions in a quasi-periodic chemical potential. In this model there is a localization-delocalization quantum phase-transition separating the

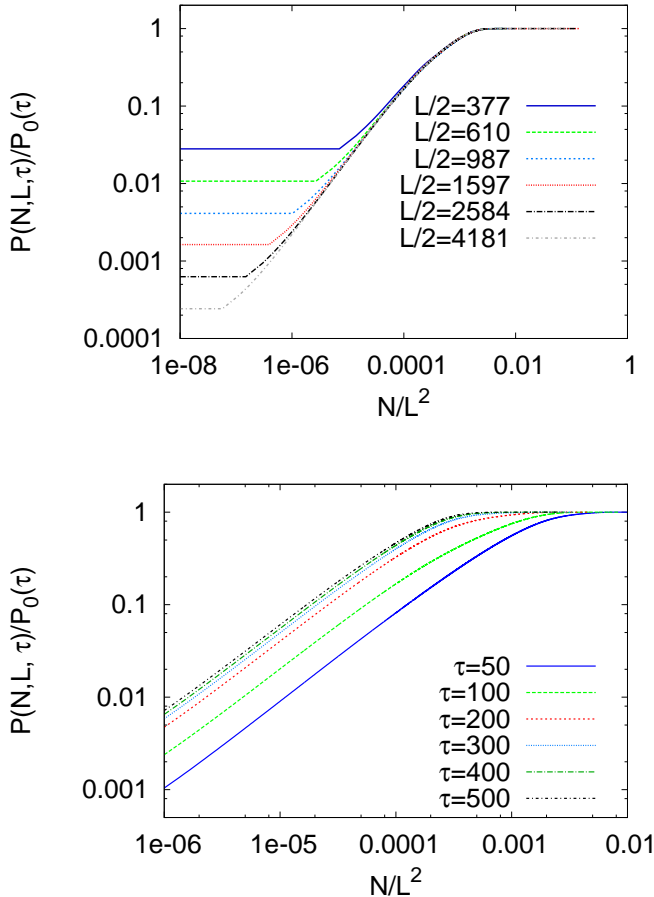


FIG. 6: Normalized partial excitation probabilities as a function of $(N/L)^2$ for different sizes: $L = 2F_n$ with $n = 13, 14, \dots, 18$ at $\tau = 100$ (upper panel) the same at $L = 2F_{18}$ for different values of τ (lower panel) both in log-log scale. (Color online)

extended and the localized phases. By varying the amplitude of the transverse field in time, $h(t)$, we have studied the properties of non-equilibrium quantum relaxation at zero temperature. We considered in details two limiting cases of the dynamics.

First we studied quench dynamics, in which $h(t)$ is changed suddenly at $t = 0$ and focused on the dynamics of the entanglement entropy, as well as on the relaxation of the local order-parameter. For quenches to the extended phase the non-equilibrium dynamics turns out to be qualitatively similar as in the homogeneous model: the entanglement entropy increases linearly, while the local order-parameter decays exponentially. The characteristic parameters, the prefactor of the linear part of the entanglement entropy, as well as the relaxation time are found to depend on the details of the quench process. This type of non-equilibrium behavior is consistent with the GGE scenario. In contrast to this, after a quench into the localized phase there is no thermalization in the stationary state: both the entanglement entropy and the

local order-parameter approaches a finite limiting value. Finally, for a critical quench the entanglement entropy increases as a power law, whereas the local order-parameter decays with a stretched exponential. This type of behavior is related to the singular continuous form of the spectrum of the critical Hamiltonian, as already noticed in the quench dynamics of quantum Fibonacci quasicrystals⁹⁸. The properties of the critical quench have been explained in the frame of a semi-classical theory in terms of anomalously diffusing quasi-particles, which are created uniformly in space during the quench.

In the second type of non-equilibrium process we have varied $h(t)$ linearly in time with a rate $1/\tau$ and studied the density of defects in the ground state created during this process. If the localization-delocalization transition point is passed once the density of defects follows a power-law dependence, $\sim \tau^{-\kappa}$, while if two symmetrically placed transition points are passed then the density of defects has a multiplicative oscillating correction, similar to the Stückelberg phase of periodically driven two-level systems. Using scaling arguments we have related κ to another critical exponents as given in Eq.(29). In this expression also the scaling dimension ω of the excitation probability enters. For homogeneous systems it is generally expected that $\omega = 1$. In our case, when the spectrum of the Hamiltonian is not continuous at the transition point, as well as the spectrum of the critical Hamiltonian is singular continuous we have $\omega < 1$. It is expected that $\omega \neq 1$ is a general rule for quasi-periodic and aperiodic Hamiltonians.

Finally, we discuss the question of the non-equilibrium dynamics of the Hamiltonian in Eq.(1) for different values of the quasi-periodicity parameter β in Eq.(2). If β is a rational number of the form $\beta = 1/(2q)$ with q being an integer, then in the adiabatic process the decay exponent is given by⁸⁰ $\kappa = q/(q + 1)$. The same result holds for $\beta = p/(2q)$, when p is an odd integer and p and q are relative primes, at least for not too large values of q . Thus these results can not be analytically continued to the case, when β is an irrational number. If β is an irrational number and different from the inverse of the golden mean ratio studied in this paper, than the critical exponents of the non-equilibrium dynamics are expected to be β dependent. Some hint in favor of this assumption can be found in the diffusion properties of the quasiparticles, see in Sec.III C. Indeed the diffusion exponent, D , is measured to be β dependent¹⁰² and the same is expected to hold for the non-equilibrium exponents σ and μ .

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