ON HIGGS BUNDLES ON ELLIPTIC SURFACES

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Abstract. Let $\pi : X \to C$ be a relatively minimal non-isotrivial elliptic surface over the field of complex numbers, where $g(C) \geq 2$. In this article, we demonstrate an equivalence between the category of semistable parabolic Higgs bundles on C, and the category of semistable Higgs bundles on X with vanishing second Chern class, and determinant a vertical divisor.

Contents

1. Introduction

Motivation and statement of results. Consider a relatively minimal elliptic surface $\pi : X \to C$ over $\mathbb{C}$, with $\chi(X) > 0$. Let $c_1, \ldots, c_n$ be the set of points on C where the fibration π has a multiple fiber of multiplicity m_i respectively. The data $(C, \mathbf{c}, \mathbf{m}) := (C, c_1, \ldots, c_n, m_1, \ldots, m_n)$ defines a 2-orbifold. It is well-known then that we have a natural isomorphism of groups induced by π (see [7, Theorem 24, p 189])

$$\pi_1(X, *) \cong \pi_1^{orb}(C, *) \tag{**}$$

The orbifold fundamental group $\pi_1^{orb}(C, *)$ is defined as follows: Recall the fundamental group $\pi_1(C - \{c_1, \ldots, c_n\}, *)$ has $2g + n$ generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n$ subject to the relation

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1.$$

The orbifold fundamental group $\pi_1^{orb}(C, *)$ is then defined to be the quotient of $\pi_1(C - \{c_1, \ldots, c_n\}, *)$ by the smallest normal subgroup containing the elements $\gamma_i^{m_i}$. Thus, $\pi_1^{orb}(C, *)$ is freely generated by the elements $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n$ subject to the relations

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1, \;\; and \;\; \gamma_i^{m_i} = 1.$$

A natural class of elliptic surfaces with positive Euler characteristic are the so called non-isotrivial elliptic fibrations. An elliptic surface $\pi : X \to C$ is called isotrivial, if after passing to a finite cover $B \to C$, the surface $X \times_C B$ is birational to a product $B \times E$, where E is a complex elliptic curve. An elliptic surface is called non-isotrivial if it is not isotrivial.

Now the space of Jordan equivalence classes of representations of $\pi_1(X, *)$ in $GL(n, \mathbb{C})$ can be identified with the moduli space of S-equivalence classes of semistable rank n Higgs bundles on X, with vanishing Chern classes from the work of Simpson (see [17],[18]).
Similarly, the space of Jordan equivalence classes of representations of $\pi_1^{orb}(C, *)$, correspond to S-equivalence classes of parabolic rank n Higgs bundles on C. Here, by Jordan equivalence we mean the equivalence relation on the space of representations given by identifying representations which have isomorphic Jordan-Holder filtrations. The isomorphism (**) suggests a natural correspondence between these moduli spaces. If we restrict our representations to unitary representations, then the corresponding moduli spaces are that of S-equivalence classes of semistable vector bundles on X with vanishing Chern classes, and S-equivalence classes of parabolic vector bundles on C with parabolic degree 0 (see [13, 16, 12, 6]).
An algebraic geometric isomorphism between these moduli spaces was exhibited by Stefan Bauer in [2] (see [3, 4, 9] for related questions).
In this paper, our aim is to establish a similar correspondence in the case of Higgs bundles on non-isotrivial relatively minimal elliptic surfaces $\pi : X \to C$ with $g(C) \geq 2$.
Recall, a Higgs bundle on a variety Y, is a pair (V, θ) where V is a vector bundle on Y and $\theta \in Hom(V, V \otimes \Omega^1_Y)$, which satisfies

$$\theta \wedge \theta = 0.$$

Coming to our situation, fix a polarization H on X and consider the following categories:
$\mathcal{C}_X^{vHiggs} :=$ The category of H-semistable Higgs bundles (V, θ) on X with vanishing second Chern class and $det(V)$ a vertical divisor.
$\mathcal{C}_{(C,\mathbf{c},\mathbf{m})}^{ParHiggs} :=$ The category of semistable parabolic Higgs bundles on C with parabolic structures above the points c_i (with weights at c_i belonging to $\frac{1}{m_i}\mathbb{Z} \cap [0, 1)$).
Our main result is the following

Theorem 1. *There is a natural equivalence of categories $\mathcal{C}_X^{vHiggs}$ and $\mathcal{C}_{(C,\mathbf{c},\mathbf{m})}^{ParHiggs}$.*

Strategy. Our strategy, is the outcome of an attempt at adapting the study in [2], to the situation of Higgs bundles.
The category of parabolic bundles (Higgs) on a curve C, with genus atleast 2 is equivalent to the category of bundles (Higgs) on ramified Galois covers, equivariant for the action of the Galois group. Keeping this in mind, we consider an elliptic fibration $\tilde{\pi} : \widetilde{X} \to \widetilde{C}$ with natural morphisms $q : \widetilde{X} \to X$ and $p : \widetilde{C} \to C$, such that

$$\pi \circ q = p \circ \tilde{\pi}.$$

We have further, $q : \widetilde{X} \to X$ is etale Galois. We also have $p : \widetilde{C} \to C$ is Galois with the same Galois group as that of q.
Moreover, the fibration $\tilde{\pi} : \widetilde{X} \to \widetilde{C}$ is a non-isotrivial, relatively minimal elliptic surface with no multiple fibers. We show that to prove Theorem 1 it is enough to construct an equivalence of categories of semistable equivariant (for the Galois group) Higgs bundles on $\widetilde{X}$ and equivariant Higgs bundles on $\widetilde{C}$.
Subsequently, we argue that the equivariant situation as above can be derived from Theorem 1 applied to the fibration $\tilde{\pi} : \widetilde{X} \to \widetilde{C}$.

So we are reduced to proving Theorem 1 in the case of fibrations with no multiple fibers.

Now in the case when the fibration has no multiple fibers, we show that every semi-stable Higgs bundle (V, θ) on X with vanishing second Chern class and determinant vertical is the pull back of a semistable Higgs bundle on C. To that end, we first observe that it is enough to show that the bundle V is the pull-back of a bundle from the curve C. To see this let W be a vector bundle on C and consider the bundle $U := \pi^*(W)$ on X. We use Lemma 2 and projection formula to conclude

$$H^0(X, \mathcal{E}nd(U, U) \otimes \Omega^1_X)) = H^0(C, \mathcal{E}nd(W, W) \otimes K_C).$$

Hence, every Higgs field on the bundle $\pi^*(W)$ is the pull-back of a Higgs field on W.
To show V is the pull-back of a bundle on C, it is enough to show its restriction to every fiber is trivial. We reduce this further to showing that, the restriction of V to the generic fiber of π is trivial. The generic fiber (possibly after a base extension) of π is an elliptic curve. Now we can use the beautiful classification results on vector bundles over elliptic curves due to Atiyah (see [1]), to study the generic restriction of V. This theme of understanding the global picture by studying the generic fiber is something which we use repeatedly in this article.

Related work and further comments. The assumption $g(C) \geq 2$ has been used only to ensure the existence of Galois covers with prescribed ramification points and ramification indices. Hence, the results of section 2 and subsection 3.1 are valid without this assumption. In particular, our proofs are valid in the case of a fibration with no multiple fibers without any assumption on the genus of C.

In [15, section 5], there is a discussion on the correspondence relating semistable Higgs bundles on elliptic surfaces with vanishing Chern classes and semistable parabolic Higgs bundles (parabolic degree 0) on the curve. But the discussion is limited to the case when the vector bundle underlying the Higgs bundle has a two step Harder-Narasimhan filtration.
In [10, Theorem 14.5], the authors consider the $1-1$ correspondence between the moduli space of semistable Higgs bundles on X with vanishing Chern classes and the moduli space of semistable parabolic Higgs bundles on the curve C of parabolic degree 0, arising as a consequence of the work of Simpson[17, 18]. This correspondence is at the level of topological spaces. The question of establishing an algebraic geometric correspondence between these moduli spaces is proposed in [10], as a remark following the above mentioned theorem. We are able to show such an isomorphism (see corollary 2) in the case of non-isotrivial elliptic surfaces.

Further, as in [2] our study is not restricted to the situation of vanishing Chern classes. We feel that results of this article must be true for isotrivial elliptic surfaces with positive Euler characteristic as well, but we do not know how to prove it.

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2. Preliminaries

All varieties considered are over the field of complex numbers unless mentioned otherwise.

2.1. **Elliptic surfaces.** An Elliptic surface is a fibered surface $\pi : X \to C$ where the general fibers are genus 1 curves and X, C are a smooth projective surface and a smooth projective curve over $\mathbb{C}$ respectively. We call an elliptic surface as above relatively minimal if there are no exceptional curves (curves with self intersection number -1) on the fibers.
Just to be consistent with the definition of vertical bundles (Definition 1) defined in the next subsection, we call a divisor D *vertical*, if D is linearly equivalent to a divisor of the form $\Sigma_i r_i F_i$ where $r_i \in \mathbb{Q}$ and F_i for every i is a divisor corresponding to a fiber at some point of C. We call a divisor D *vertically supported* if $Supp(D)$ maps to a proper closed subset of C under π. A *vertically supported* divisor D always satisfies $D^2 \leq 0$ and is *vertical* precisely when $D^2 = 0$. For two divisors D_1 and D_2, we write $D_1 \equiv D_2$ if D_1 is numerically equivalent to D_2. The vertical divisors corresponding to various fibers are all numerically equivalent. Hence as far as intersection theory is concerned, we may work with a fixed fiber, say we denote by F. The sheaf $R^1\pi_*(\mathcal{O}_X)$ is a line bundle on C and we have in the case of relatively minimal elliptic surfaces

$$\chi(X) = 12deg(L)$$

where L denotes the dual of the line bundle $R^1\pi_*(\mathcal{O}_X)$ on C. We have the canonical bundle formula due to Kodaira (see [7, Theorem 15, p 176])

$$K_X \cong \pi^*(K_C \otimes L) \otimes \mathcal{O}_X(\Sigma_i(m_i - 1)F_i).$$

where F_i are effective divisors on X whose G.C.D of the coefficients of the components are 1 and the multiple fibers of π are precisely of the form $m_i F_i$.
Hence, on a relatively minimal elliptic surface, we have the canonical divisors of X are vertical divisors. If $Y \to C$ is an elliptic surface which is not relatively minimal, then assume after blowing down the exceptional curves $\{E_1, \dots, E_r\}$ we get a relatively minimal model say X. We then have

$$K_Y \cong K_X \otimes \mathcal{O}_Y(E_1 + \ldots + E_r).$$

Hence, K_Y can be represented by vertically supported divisor. In particular, we have for any elliptic surface $X \to C$,

$$K_X.F = 0.$$

We will need the following characterization of vertical divisors in the subsequent sections

Lemma 1. *Assume $\chi(X) > 0$. Then for a divisor D,*

$$D \text{ vertical} \iff D.F = 0 \text{ and } D^2 = 0$$

Proof. D vertical clearly implies

$$D.F = 0 = D^2.$$

Conversely, assume $D.F = 0$ and $D^2 = 0$. Let H be an ample divisor on X. Choose a pair of integers m, n such that $(mD+nF).H = 0$. Since now $(mD+nF)^2 = 0$, we get from Hodge index theorem on surfaces that $mD + nF \equiv 0$ and $D \equiv rF$ where $r \in \mathbb{Q}$. If D is vertically supported then $D = aF$, with $a \in \mathbb{Q}$. This is the case if D is effective. Hence, to conclude the proof it is enough to show $D_l = D + lF$ is effective where $l \in \mathbb{N}$ as D_l satisfies the hypothesis of the Lemma and the preceding discussion applied to D_l says D_l is vertical and hence so do $D = D_l - lF$. To see this choose $l >> 0$, so that $(D_l).H > (K_X).H$. Then

$$H^2(X, \mathcal{O}_X(D_l)) = H^0(X, Hom(D_l, K_X))^* = 0.$$

Applying Riemann-Roch theorem, we see that

$$H^0(X, \mathcal{O}_X(D_l)) = H^1(X, \mathcal{O}_X(D_l)) + \chi(\mathcal{O}_X) > 0$$

and hence we have D_l is effective. □

The following Lemma is necessary for our study

Lemma 2. *(see [14]) Let X be a non-isotrivial elliptic surface with no multiple fibers. Then the natural map*

$$K_C \to \pi_*(\Omega^1_X)$$

is an isomorphism.

Proof. Consider the short exact sequence

$$0 \to \pi^*(K_C) \to \Omega^1_X \to \Omega^1_{X/C} \to 0.$$

Applying π_*, we get the following long exact sequence

$$0 \to K_C \to \pi_*(\Omega^1_X) \to \pi_*(\Omega^1_{X/C}) \xrightarrow{\sigma} K_C \otimes R^1\pi_*(\mathcal{O}_X)$$

The sheaf $\pi_*(\Omega^1_{X/C})$ is a rank 1 sheaf on C, while $K_C \otimes R^1\pi_*(\mathcal{O}_X)$ is a rank 1 locally free sheaf. The map σ restricted to the generic fiber is the kodaira-spencer map which is non-zero if X is assumed non-isotrivial. Hence the kernel of σ is precisely $\pi_*(\Omega^1_{X/C})_{Tor} = \pi_*((\Omega^1_{X/X})_{Tor})$. So we have

$$0 \to K_C \to \pi_*(\Omega^1_X) \to \pi_*((\Omega^1_{X/C})_{Tor}) \to 0.$$

Now from [11, Proposition 1] we get $H^0(C, \pi_*((\Omega^1_{X/C})_{Tor})) = 0$. But as $\pi_*((\Omega^1_{X/C})_{Tor})$ is a torsion sheaf on C, it has to be the 0-sheaf since a non-zero torsion sheaf on a curve always has sections. Hence we have

$$K_C \xrightarrow{\cong} \pi_*(\Omega^1_X).$$

□

2.2. **Vertical Bundles.** We keep the assumption that $\pi : X \to C$ is a *non-isotrivial* elliptic fibration. Denote by K, the function field $k(C)$ of C. For a vector bundle V on X, we denote by V_K the bundle on $X_K := X \times_C spec(K)$ given by pull back of V to X_K through the natural map $X_K \to X$. Similarly, for an extension L/K of fields we denote by $X_L := X \times_C spec(L)$ and V_L the bundle on X_L given by the pull back of V to X_L through the morphism $X_L \to X$.
Let us recall the definition of *Vertical Bundles* as defined in [2, Definition 1.3,p 512]

Definition 1. *A rank n vector bundle V on X is called vertical, if V has a filtration*

$$(0) = V_0 \subset V_1 \subset \ldots \subset V_n = V.$$

by sub-bundles V_i, with $V_i/V_{i-1} \cong \mathcal{O}_X(D_i)$ where D_i are vertical divisors.

The main result of this subsection is the following proposition which relates vertical bundles V on X and V_K.

Proposition 1. *Let V be a vector bundle with $c_2(V) = 0$ and $D = det(V)$ is a vertical divisor. Then, V is vertical if and only if V_K is semistable.*

Proof. If V is vertical, then clearly V_K is semistable. Now for the converse, let $\bar{K}/K$ be an algebraic closure of K. Consider the elliptic curve $X_{\bar{K}}$ and vector bundle $V_{\bar{K}}$ on $X_{\bar{K}}$. From assumption we have $V_{\bar{K}}$ is semistable with trivial determinant. From Atiyah's classification results on vector bundles on elliptic curves (see [1]), we have $V_{\bar{K}} \cong \oplus_i L_i I_{m_i}$ where L_i are degree 0 line bundles on $X_{\bar{K}}$ and I_m denotes the unique indecomposable bundle on $X_{\bar{K}}$ of rank m and trivial determinant. Let L/K denote a finite Galois extension so that for every index i $L_i \in Pic^0(X_L)$. We then have a decomposition of V_L as $\oplus_i L_i I_{m_i}$. Let $f : \tilde{C} \to C$ be the finite Galois cover of C corresponding to L/K. Choose a minimal resolution $\tilde{X}$ of $X \times_C \tilde{C}$. Since X was non-isotrivial, the same holds true for $\tilde{X}$ and hence $\chi(\tilde{X}) > 0$. Denote by $\tilde{V}$, the pull back of V to $\tilde{X}$. The bundle $\tilde{V}$ also satisfies $c_2(\tilde{V}) = 0$ and $\tilde{D} = det(\tilde{V})$ is a vertical divisor with $\tilde{D}^2 = 0$. We have $\tilde{V_L} = V_L$ and hence has a filtration by the line bundles L_i. We can extend this filtration on $\tilde{V_L}$ to a filtration by torsion free subsheaves on $\tilde{V}$,

$$(0) = \tilde{V_0} \subset \dots \tilde{V}_{n-1} \subset \tilde{V_n}.$$

such that $\tilde{V_i}/\tilde{V}_{i-1} \cong \mathcal{O}_X(D_i) \otimes I_{Z_i}$. Using the additivity of Chern classes, we get

$$\Sigma_i D_i = \tilde{D}, \tag{2.1}$$

$$\Sigma_{i<j} D_i D_j + lt(Z_i) = 0. \tag{2.2}$$

Squaring equation(2.1) and using the fact that $\tilde{D}^2 = 0$, we get

$$\Sigma_{i<j} D_i D_j = -\frac{1}{2}\Sigma_i D_i^2. \tag{2.3}$$

Substituting equation (2.3) in equation (2.2), we get

$$\Sigma_i lt(Z_i) = \frac{1}{2}\Sigma_i D_i^2. \tag{2.4}$$

Since by assumption $D_i F = 0$, we have

$$D_i^2 \leq 0, \forall i.$$

On the other hand we have

$$lt(Z_i) \geq 0, \ \ \forall i.$$

Hence from equation (2.4) we get the only possibility is

$$lt(Z_i) = 0 \ \ and \ \ D_i^2 = 0.$$

Now from Lemma 1 we can conclude D_i are vertical divisors for all i. In particular

$$L_i \cong \mathcal{O}_{X_L} \ \ \forall i.$$

Now consider the short exact sequence

$$0 \to L_0 \cong \mathcal{O}_{X_L} \xrightarrow{t_0} V_L \to V_L/L_0 \to 0.$$

Let $G := Gal(L/K)$ be the Galois group. We have G acts on V_L and hence on the sections $H^0(X_L, V_L)$. Now replace t_0 by $Tr(t_0) = \Sigma_{g \in G} g(t_0)$ and we can assume t_0 is G-invariant and hence L_0 is a G-invariant trivial sub-bundle of V_L. Hence by Galois descent we have a section $s_0 : \mathcal{O}_{X_K} \to V_K$ with t_0 being the induced section of V_L via base change and $V_L/L_0 \cong (V_K/s_0(\mathcal{O}_{X_K}))_L$. Replacing now V_L by V_L/L_0 and t_0 by $t_1 : L_1 \cong \mathcal{O}_{X_L} \to V_L/L_0$ and repeating the argument we see that V_K has a filtration by sub-bundles with sub-quotients all trivial line bundles. Extend this filtration to a filtration of V and as in the case of $\tilde{V}$, we see that V is vertical. Thus we have proved the proposition. □

Let us recall now the definition of Higgs bundles on a projective variety Y. A Higgs bundle on Y is a pair (V, θ), where V is a vector bundle on Y and $\theta : V \to V \otimes \Omega^1_Y$ is a homomorphism with

$$\theta \wedge \theta = 0.$$

We fix a polarization H on Y and let r be $\dim(Y)$. We say Higgs bundle (V, θ) on Y is semistable if for every subsheaf $W \subset V$ preserved by θ (i.e $\theta(W) \subset W \otimes \Omega^1_Y$), we have

$$c_1(W).H^{r-1}/rank(W) \leq c_1(V).H^{r-1}/rank(V).$$

Now consider the case when Y is a surface. For a vector bundle V on Y, Denote by $\Delta(V)$ the number $(r-1)c_1(V)^2 - 2rc_2(V)$ which is called the *Bogomolov Number* of V. If a vector bundle V admits a Higgs field ϕ so that (V, ϕ) is a semistable Higgs bundle on Y (with respect to H), then we have the Bogomolov inequality

$$\Delta(V) \leq 0.$$

Further if $\Delta(V) = 0$, then the pair (V, ϕ) is semi-stable with respect to any other polarization on Y [5, Theorem 1.3]. Hence now as a corollary of Proposition 1 we have the following generalization of [2, Lemma 1.4, p 512].

Corollary 1. *If (V, θ) is a semistable Higgs bundle with $c_2(V) = 0$ and $D = det(V)$ vertical, then V is a vertical bundle.*

Proof. From Proposition 1 it is enough to show V_K is semistable. If V_K is not semistable, Then since X_K is a genus 1 curve, The H-N filtration of V_K induces a decomposition $V_K = \oplus_{i=1}^{j} W_i$ where W_i is the destabilizing subsheaf of V_K/W_{i-1} if we set $W_0 = (0)$. In particular each W_i is semistable and

$$deg(W_0) > \ldots > deg(W_j).$$

Now $(\Omega^1_X)_K$ is a rank 2 vector bundle on X_K which is an extension of $\mathcal{O}_{X_K}$ by itself. In particular $(\Omega^1_X)_K$ is semistable of degree 0 and so the bundles $W_i \otimes (\Omega^1_X)_K$ are semistable with $deg(W_i \otimes (\Omega^1_X)_K) = 2deg(W_i)$ and $rk(W_i \otimes (\Omega^1_X)_K) = 2rk(W_i)$. We have then

$$\mu(W_0) = deg(W_0)/rk(W_0) > \mu(W_i \otimes (\Omega^1_X)_K) = deg(W_i)/rk(W_i), \;\; i \geq 2.$$

So

$$H^0(X_K, Hom(W_0, W_i \otimes (\Omega^1_X)_K)) = (0), \forall i \geq 2.$$

Hence, we have $\theta_K(W_0) \subseteq W_0 \otimes (\Omega^1_X)_K$. Now extend W_0 to a torsion free sub-sheaf $W \subset V$ with V/W torsionfree as well. Since θ_K preserves W_0, the Higgs field θ preserves the subsheaf W. On the other hand, as we have $c_1(W).F = deg(W_0) > 0$, for a suitable $m >> 0$ and the polarisation $H + mF$, the slope W exceeds that of V. But since $\Delta(V) = 0$, this will contradict the stability of (V, θ) with respect to H. □

Before we end this section we would like to address two natural questions regarding vertical bundles. The first one is about when a sub-sheaf of a vertical bundle V is itself vertical. Clearly such a sub-sheaf $N \subset V$ satisfies $c_1(N).F = 0$. We will see below [Lemma 3] that this condition is in fact sufficient. The other question is specific to the case when π has no multiple fibers. In this situation there is a natural class of vertical bundles, which are the pull backs of bundles on C to X. If V is such a bundle then we have $V_K = \mathcal{O}_{X_K}^{\oplus r}$ where $r = rk(V)$. Once again this condition turns out to be sufficient [Lemma 4].

Lemma 3. *Let V be a vertical bundle and $N \subset V$ a sub-sheaf with torsion free quotient V/N. Then, N is vertical precisely when $c_1(N).F = 0$.*

Proof. Since N and V/N are torsion free, we have N_K and $(V/N)_K$ are locally free on X_K. Further by assumption both are of degree 0. On the other hand as V is vertical, $V_K \cong \oplus_i I_{k-i}$, where I_{k_i} is the unique indecomposable bundle of rank k_i and trivial determinant. Hence both N_K and $(V/N)_K$ also admit filtrations where the successive quotients are trivial line bundles. Any such filtration on N_K and $(V/N)_K$ can be extended to a filtration on N and V/N with successive quotients all of rank 1 and of the form $\mathcal{O}_X(D_i) \otimes I_{Z_i}$ where D_i is a vertically supported divisor and Z_i is a closed set of points on X. But this filtration is also a filtration on V. Now an argument involving Chern classes as in the proof of Proposition 1 gives us $Z_i = \emptyset$ for every i and D_i are vertical divisors. Hence both N and V/N are vertical.

□

Lemma 4. *Assume π has no multiple fibers. Then a vertical bundle V is isomorphic to $\pi^*(W)$ where W is a bundle on C if and only if $V_K = \mathcal{O}_{X_K}^{\oplus r}$ where $r = rk(V)$.*

Proof. Let V be a vertical bundle with $V_K = \mathcal{O}_{X_K}^{\oplus r}$. Since by assumption π has no multiple fibers, a line bundle corresponding to a vertical divisor restricts to the trivial line bundle on any fiber of π. Hence V restricted to any fiber is an iterated extension of trivial line bundles. In particular if for $c \in C$, we denote by X_c by the fiber (scheme theoretic) of π above c and V_c the restriction $V \mid_{X_c}$, then as $h^0(X_c, \mathcal{O}_{X_c}) = 1$, we have

$$h^0(X_c, V_c) \leq r.$$

The equality occurs precisely at the points $c \in C$ where V_c is the trivial rank r bundle on X_c. Now from semi-continuity principle the set $Z = \{c \in C \mid h^0(X_c, V_c) = r\}$ is a non-empty closed subset of C. But on the other hand we have if $\zeta \in C$, the generic point of C, then $h^0(X_\zeta, V_\zeta) = h^0(X_K, V_K) = r$. Hence $\zeta \in Z$ and thus $Z = C$. Thus V restricts to the trivial rank r bundle on every fiber and consequently $V \cong \pi^*(\pi_*(V))$. □

3. Main Theorem

Let $\pi : X \to C$ denote a non-isotrivial elliptic fibration possibly with multiple fibers and $\chi(X) > 0$. Fix a polarization H on X. Denote by $\mathcal{C}_X^{vHiggs}$, the category whose objects are (H-)semistable Higgs bundles (V, θ) on X with $c_2(V) = 0$, $det(V)$ a vertical divisor, and morphisms being Higgs bundle morphisms. Let $\mathbf{c} := \{c_1, \dots, c_l\}$ be the points on C where the fibers of π are multiple. Let the multiplicities of these fibers be $\mathbf{m} := \{m_1, \dots, m_l\}$ respectively. Recall the notion of a parabolic vector bundle on C. A parabolic vector bundle on C with a parabolic structure at a point $c \in C$, consists of a vector bundle V, together with a Flag

$$F^\bullet(V_c) := (0) \subset F^1(V_c) \subset F^2(V_c) \subset \dots F^r(V_c) = V_c.$$

and weights $\alpha_i \in \mathbb{R}$ assigned to each subspace $F^i(V_c)$ such that

$$0 < \alpha_1 < \dots < \alpha_r \leq 1.$$

To such a parabolic vector bundle $(V, F^\bullet(V_c), \{\alpha_i\})$ we can associate a real number called the parabolic degree given by

$$Pardeg(V) := deg(V) + \Sigma_i \alpha_i dim(F^i(V_c)/F^{i-1}(V_c)).$$

In general, if there are parabolic structures on more than one point, then the definition of parabolic degree has to be appropriately modified. There is a natural induced parabolic structure on every sub-bundle of V and we have an obvious notion of semistability (stability) using the parabolic degree instead of the usual degree. Now we also can define a parabolic Higgs bundle. Since in literature there are two different notions of a Higgs field, we want to specify what we mean by a parabolic Higgs field. For a parabolic vector bundle $(V, F^\bullet(V_c), \{\alpha_i\})$ as defined above, a parabolic Higgs field is a morphism

$$\phi : V \to V \otimes K_C(c)$$

such that we have

$$\phi(F^i(V_c)) \subseteq F^{i-1}(V_c) \otimes K_C(c).$$

Now we can define semistability for a parabolic Higgs bundle $(V, F^\bullet(V_c), \phi)$ as in the usual case by restricting the slope condition to sub-bundles preserved by the

Higgs field ϕ. The definition of parabolic Higgs bundles in the case of parabolic structures at more than one point is the same as above except we have to replace $K_C(c)$ by $K_C(c_1+\ldots+c_l)$, where c_i are the parabolic points. Assume from now on that $g(C)\geq 2$. Further, assume the weights associated with the filtration $F^\bullet(V_{c_j})$ at c_j all are rational and lie in $\frac{1}{m_j}\mathbb{Z}\cap[0,1]$. Let $\mathcal{C}^{ParHiggs}_{(C,\mathbf{c},\mathbf{m})}$ denote the category of parabolic semistable Higgs bundles with weights as described above. The following theorem is the main result of this article

Theorem 1. *There is a natural equivalence of categories* $\mathcal{C}^{vHiggs}_X$ *and* $\mathcal{C}^{ParHiggs}_{(C,\mathbf{c},\mathbf{m})}$.

Since we have assumed $g(C)\geq 2$, there exists a Galois cover $p:\widetilde{C}\to C$ with Galois group denoted by Γ and the local ramification groups above c_j being the cyclic group $\frac{\mathbb{Z}}{m_j\mathbb{Z}}$ for every j. Let $\mathcal{C}^{\Gamma-Higgs}_{\widetilde{C}}$ denote the category of Γ-equivariant Higgs bundles on $\tilde{C}$. We then have a natural equivalence of categories

$$p^\Gamma_*:\mathcal{C}^{\Gamma-Higgs}_{\widetilde{C}}\overset{\cong}{\to}\mathcal{C}^{ParHiggs}_{(C,\mathbf{c},\mathbf{m})}$$

Now consider the commutative diagram

$$\begin{array}{ccc}\widetilde{X} & \xrightarrow{q} & X\\ \Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\pi}\\ \widetilde{C} & \xrightarrow{p} & C\end{array}$$

where $\widetilde{X}$ is a minimal desingularization of $X\times_C\widetilde{C}$ (see [8, Section 1.6,p 95-108]). We have $\tilde{\pi}:\widetilde{X}\to\widetilde{C}$ is a relatively minimal non-isotrivial elliptic surface with no multiple fibers. Further, we have q is an etale Galois cover with Galois group Γ.

Denote by $\mathcal{C}^{\Gamma-vHiggs}_{\widetilde{X}}$, the category of Γ-semistable Γ-equivariant Higgs bundles on $\widetilde{X}$ with $c_2=0$ and c_1 vertical. From Galois descent, we have an equivalence of categories

$$q^*:\mathcal{C}^{vHiggs}_X\overset{\cong}{\to}\mathcal{C}^{\Gamma-vHiggs}_{\widetilde{X}}.$$

Hence to prove the theorem it suffices to construct a natural equivalence between the categories $\mathcal{C}^{\Gamma-vHiggs}_{\widetilde{X}}$ and $\mathcal{C}^{\Gamma-Higgs}_{C}$. To that end we will first prove the theorem in the case when the elliptic fibration has no multiple fibers.

3.1. **Proof of Theorem 1 in the case of no multiple fibers.** Denote by $\mathcal{C}^{Higgs}_C$, the category of semistable Higgs bundles on C. we have a natural map

$$d\pi:\pi^*(K_C)\to\Omega^1_X$$

Now for a Higgs bundle (W,ϕ) on C, let $V=\pi^*(W)$. Then we denote by $d\pi(\phi)\in Hom(V,V\otimes\Omega^1_X)$ the composition $(Id_V\otimes d\pi)\circ(\pi^*(\phi))$. Clearly $d\pi(\phi):V\to V\otimes\Omega^1_X$ is a Higgs field on V. Now we have the following Lemma

Lemma 5. *If (W,ϕ) is a semistable Higgs bundle on C, then for any chosen polarisation on X, the Higgs bundle $(V,d\pi(\phi))$ is semistable on X*

Proof. Since $\Delta(V)=0$, it is enough to prove that there exists a polarization with respect to which $(V,d\pi(\phi))$ is semistable. Assume the contrary and let H be a polarization for which the pair $(V,d\pi(\phi))$ is unstable. Since the bundle V_K is trivial and hence semistable, for any sub-sheaf of $N\subset V$, we have $c_1(N).F\leq 0$. Hence, changing the polarization from H to $H+mF$ for $m>>0$, either turns $(V,d\phi(\phi))$ into a semistable Higgs bundle in which case we are done or else the maximal destabilizing sub-sheaf V_{max} satisfies $c_1(V_{max}).F=0$. But as V_K is trivial and $(V_{max})_K$ is a degree 0 sub-bundle of V_K, the only possibility is $(V_{max})_K$ is itself trivial. Hence so do $(V/V_{max})_K$. Now from Lemma 3 and Lemma 4 we have $V_{max} \cong \pi^*(\pi_*(V_{max}))$. Hence, $\pi_*(V_{max})$ is a sub-bundle of W of rank same as that of V_{max}. Further, we have

$$\mu(\pi_*(V_{max})) = \frac{c_1(V_{max}).H/F.H}{rk(V_{max})} > \mu(W) = \frac{c_1(W).H/F.H}{rk(W)}.$$

and $\pi_*(V_{max})$ is invariant under ϕ, which contradicts semistability of (W, ϕ). Hence, $(V, d\pi(\phi))$ is semistable for the polarization H. □

Thus, we have a well defined functor

$$\pi^* : \mathcal{C}_C^{Higgs} \to \mathcal{C}_X^{vHiggs}$$

given by

$$(W, \phi) \mapsto (\pi^*(W), d\pi(\phi)).$$

From Lemma 2 we have the natural map $K_C \to \pi_*(\Omega_X^1)$ is an isomorphism. Hence, if $V = \pi^*(W)$ for W a bundle on C, then from projection formula every Higgs field θ on V is of the form $d\pi(\phi)$ for ϕ a Higgs field on W. Hence, the functor π^* is full and faithful. We will see below that π^* is essentially surjective as well and hence is an equivalence of categories, which proves Theorem 1 when π has no multiple fibers.

Remark 1. *The statement for line bundles (even without the assumption of non-isotriviality) is a consequence of Hodge theory for complex surfaces. Recall we have under the assumption of* $\chi(X) > 0$,

$$g(C) = h^{1,0} = dim_{\mathbb{C}}(H^1(X, \mathcal{O}_X)) = dim_{\mathbb{C}}(H^0(X, \Omega_X^1)) = h^{0,1}.$$

On the other hand, the dimension of the subspace $H^0(X, \pi^*(K_C)) \subseteq H^0(X, \Omega_X^1)$ *is* $g(C)$ *as well. Hence, we have the equality*

$$H^0(X, \pi^*(K_C)) = H^0(X, \Omega_X^1).$$

In particular, every 1*-form on* X *is the pull back of a form on* C*. Now a rank* 1 *Higgs bundle of the form in the theorem above is a pair* (L, θ) *where* L *is isomorphic to a line bundle of the form* $\mathcal{O}_X(D)$ *with* D *vertical (hence in the case of no multiple fibers,* D *is the pull back of a divisor on* C*) and* θ *is a* 1*-form. So the statement holds true for rank* 1 *Higgs bundles as in the theorem.*

Since we have assumed X to be non-isotrivial, we have from Lemma 2, $\pi_*(\Omega_X^1)$ is the line bundle K_C on C. Hence from semicontinuity principle, we have

$$dim_K(H^0(X_K, (\Omega_X^1)_K)) = 1.$$

Consider now the restriction of the short exact sequence

$$0 \to \pi^*(K_C) \to \Omega_X^1 \to \Omega_{X/C}^1 \to 0.$$

to X_K. Since $(\pi^*(K_C))_K \cong \mathcal{O}_{X_K} \cong (\Omega_{X/C}^1)_K$, we see that $(\Omega_X^1)_K$ is an extension of $\mathcal{O}_{X_K}$ by $\mathcal{O}_{X_K}$. Up to isomorphism, there are only 2 such bundles on X_K, the one being the trivial rank 2 bundle and the other the indecomposable bundle I_2 (see [1]). Since we have seen already that $dim_K(H^0(X_K, (\Omega_X)_K)) = 1$, the bundle $(\Omega_X^1)_K$ cannot be the trivial bundle and hence it is isomorphic to I_2. So the pair (V_K, ϕ_K) is a I_2-valued Higgs pair on X_K. Such an I_2 valued Higgs pair is equivalent to a morphism

$$I_2^* \to End(V, V)$$

such that fiber wise the image lands inside a family of commuting matrices. The following Lemma about I_2-valued Higgs pairs is what we need for our purposes

Lemma 6. *Let E be an elliptic curve over a field k, and $\phi : V \to V \otimes I_2$ be an I_2-valued Higgs field. We then have, for any section $\alpha \in H^0(E, End(I_2, \mathcal{O}_E))$, the induced element $\beta = \alpha \circ \phi \in H^0(E, End(V, V))$ is Nilpotent.*

Proof. The bundle I_2 is an extension of $\mathcal{O}_E$ by $\mathcal{O}_E$ and hence we have a short exact sequence

$$0 \to \mathcal{O}_E \xrightarrow{s} I_2 \xrightarrow{t} \mathcal{O}_E \to 0.$$

Further

$$H^0(E, I_2) = k < s >, \;\; H^0(E, End(I_2, \mathcal{O}_E)) = k < t > .$$

In particular, for $a \in H^0(E, I_2)$ and $b \in H^0(E, End(I_2, \mathcal{O}_E))$, we always have

$$ba = 0 \in H^0(E, \mathcal{O}_E).$$

We also have

$$I_2 \cong I_2^*$$

Fix an isomorphism as above and then we have

$$H^0(E, I_2^*) = k < t^* >, \;\; H^0(E, End(I_2^*, \mathcal{O}_E)) = k < s^* > .$$

Consider now the morphism (which we denote by θ as well) induced by the Higgs field

$$\theta : I_2^* \to End(V, V).$$

We have a trace map $Tr_V : End(V, V) \to \mathcal{O}_X$ and $Tr_V \circ \theta \in H^0(E, End(I_2^*, \mathcal{O}_E))$. Let $Tr_V \circ \theta = \lambda s^*$ and $\alpha = \gamma t^*$. Then

$$Tr(\beta) = Tr_V \circ \theta \circ \alpha = \lambda\gamma s^* t^* = 0.$$

Let L/k be a finite extension so that we have a decomposition of V_L as direct sum of generalized eigenspaces of β,

$$V_L = \bigoplus_{\delta_j \in L} V_L^{\delta_j}$$

Since θ point wise lands in a family of commuting endomorphisms, we have $V_L^{\delta_j}$ are preserved by θ. Hence, we have induced maps

$$\theta^{\delta_j} : I_2^* \to End(V_L^{\delta_j}, V_L^{\delta_j}).$$

and $\beta^{\delta_j} = \theta^{\delta_j} \circ \alpha$. In particular

$$\beta = \oplus \beta^{\delta_j}.$$

Now as in the case of V, we get

$$rank(V_L^{\delta_j})\delta_j = Tr(\beta^{\delta_j}) = 0.$$

Hence, either $\beta = 0$ or all the eigenvalues are 0 and hence β is nilpotent. $\square$

As a consequence of the above Lemma we have the following

Lemma 7. *Let (V, θ) be an I_2-valued Higgs pair, with V a semistable rank r degree 0-bundle on E. Then either ,*
(a) $V = L \otimes \mathcal{O}_E^{\oplus r}$ with $deg(L) = 0$, and $\theta \circ (id_V \otimes t) = 0$, or
(b) $\exists W \subset E$ with $deg(W) = 0$ and $\theta(W) \subset W \otimes I_2$.

Proof. Consider the endomorphism

$$T = \theta \circ t : V \to V$$

We have from Lemma 6 that T is a nilpotent endomorphism. Let $W := Ker(T)$. Now as V is semistable of degree 0, we have $deg(W) \leq 0$. On the other hand by same reasoning $deg(Im(T)) \leq 0$. Hence, $deg(W)$ is forced to be 0. So if $\phi \neq 0$, then W is a proper degree 0 sub-bundle invariant under θ and we are done. If $\theta = 0$, then θ factors through $s : \mathcal{O}_E \to I_2$, i.e we have an endomorphism $\phi : V \to V$ such that

$$\theta = (id_V \otimes s) \circ (\phi).$$

Using Atiyah's classification results on bundles on elliptic curves [1], it is easy to see that unless $V = L \otimes \mathcal{O}_E^{\oplus r}$ where $deg(L) = 0$, ϕ always leaves invariant a proper degree 0 sub-bundle of V. $\square$

We have now all the ingredients to prove Theorem 1 when the fibration has no multiple fibers. We provide below 2 different arguments, the first one though works only in the case when the Higgs bundle has no sub-Higgs sheaves.

3.1.1. *Higgs bundles with no sub-Higgs sheaves.*

Proof. Consider the spectral cover $Y \subset T^*X$ associated to a Higgs bundle (V, θ). Let rank of V be r. The fact that (V, θ) has no sub-Higgs sheaves is equivalent to Y being irreducible and the natural map $q : Y \to X$ is a finite map, which restricted to the smooth locus Y^{sm} of Y is a ramified r-sheeted cover of $q(Y^{sm})$. Further, we have $V = q_*(L)$ where L is a rank 1 torsion free sheaf on Y. Now think of the Higgs field θ as a morphism

$$\theta : TX \to End(V, V).$$

For $x \in X$, the image of the induced morphism of vector spaces

$$\theta(x) : T_xX \to End(V_x, V_x)$$

by integrability condition on θ lies inside a commuting family of endomorphism. Hence, the matrices in the image of $\theta(x)$ can be simultaneously triangularized and the eigenvalues correspond to linear maps $T_xX \to \mathbb{C}$ or equivalently elements of T_x^*X which is precisely the set $q^{-1}(x) \subset Y$. Though there might not exist global sections of T^*X which restrict to the eigenvalues point wise, we can find sections of suitable symmetric powers of T^*X which correspond to the co-efficients of the characteristic polynomials. The discriminants of the point wise characteristic polynomials can also be extended to a section of a suitable symmetric power of T^*X. Let us call it $\Delta(\theta)$. Now as we have seen already T^*X restricts to the unique indecomposable rank 2 bundle of trivial determinant when restricted to the smooth fibers. Further it has a unique section which if non-zero is nowhere vanishing. Assume now $x \in X$ with fiber of π over $y = \pi(x)$ smooth and $\Delta(\theta)(x) = 0$. Then $\Delta(\theta)$ vanishes on the entire fiber $\pi^{-1}(y)$. In particular, as the vanishing locus of $\Delta(\theta)$ is a closed set, it has to be nowhere vanishing on an open set $\pi^{-1}(U)$ where $U \subset C$ is open. In particular we see that q is unramified on $q^{-1}(U)$ and the ramification locus is a vertically supported Divisor on X. Denote the scheme theoretic fiber of f over X_K by Y_K which is a disjoint union of elliptic curves over K. On the other hand the torsion free sheaf L restricts to a line bundle L_K on Y_K and $V_K = (q_K)_*(L_K)$. If we denote by G the Galois group (note here we do not assume Y_K to be connected, but the Galois group makes sense), then we have

$$q_K^*(V_K) = \oplus_{\sigma \in G} \sigma(L_K)$$

Hence $\#(G)(deg(L_K) = \#(G)deg(V_K) = 0$. On the other hand $H^0(Y_K, L_K) = H^0(X_K, V_K) \neq 0$ and hence the only possibility is $L_K \cong \mathcal{O}_{Y_K}$. But then since q_K is unramified $(q_K)_*(\mathcal{O}_{Y_K}) = \oplus_{i=1}^m (\oplus_{j=1}^{n_i} K_i^j)$ where K_i are torsion line bundles on X_K defining a connected subcover $q_K^i : Y_K^i \subset Y_K \to X_K$. But as V_K is already an extension by trivial line bundles, the only possibility is $K_i = \mathcal{O}_{X_K}$ for every i and hence Y_K is a disjoint union of copies of X_K and $V_K = \oplus_{j=1}^r \mathcal{O}_{X_K}$. $\square$

3.1.2. *The general case.*

Proof. Consider the I_2 Higgs pair (V_K, θ_K) on X_K. We have V_K is an iterated extension of trivial line bundles. Now from Lemma 7 we have either V_K is trivial or has a degree 0 (hence semistable) sub-bundle $W_K \subset V_K$ preserved by θ_K. Clearly, W_K is also an iterated extension by trivial line bundles as V_K is so. We can extend W_K to a sub-sheaf W of V with torsion free quotient V/W and $\theta(W) \subseteq W \otimes \Omega^1_X$. Further $det(W).F = 0$. Every sub-sheaf $Q \subset V$ preserved by θ satisfies $det(Q).F \leq 0$ as V_K is semistable of degree 0. Now changing polarization from H to $H + mF$ for a suitable $m \in \mathbb{N}$, we can assume the subsheaf $V_{max} \subset V$, which has maximum slope among the sub-sheaves preserved by θ satisfies $det(V_{max}).F = 0$. In particular $(V_{max})_K$ is also an iterated extension by trivial line bundles and so do the quotient $(V/V_{max})_K$. Chose a filtration by trivial line bundles on $(V_{max})_K$ and $(V/V_{max})_K$ and extend them to X as filtration on V_{max} and (V/V_{max}) where the sub-quotients are rank 1 torsion free sheaves of type $\mathcal{O}_X(D_i) \otimes I_{Z_i}$ with D_i being vertically supported divisors for every index i. Now observe this filtration inturn gives a filtration on V and as in the proof of Proposition 1 we can see that infact $Z_i = \emptyset$ and D_i are vertical divisors. Hence both V_{max} and V/V_{max} are vertical bundles. Denote the induced Higgs fields on V_{max} and V/V_{max} by θ_0 and θ_1 respectively. From the assumption both of them are semistable Higgs bundles on X as well of rank smaller than that of V. Hence by induction we have semistable Higgs bundles (W_0, ϕ_0) and (W_1, ϕ_1) on C such that

$$(V_{max}, \theta_0) \cong (\pi^*(W_0), \pi^*(\phi_0)), \;\; (V/V_{max}, \theta_1) \cong (\pi^*(W_1), \pi^*(\phi_1)).$$

Note that we have

$$deg(W_0) = det(V_{max}).H/F.H \leq deg(W_1) = det(V/V_{max}).H/F.H.$$

Now consider the short exact sequence (infact a short exact sequence of Higgs bundles on X)

$$0 \to V_{max} \to V \to V/V_{max} \to 0. \tag{3.1}$$

Applying π_* to equation 3.1, we get a long exact sequence

$$0 \to W_0 \to \pi_*(V) \to W_1 \xrightarrow{\eta} W_1 \otimes L^{-1}$$

where $L = R^1\pi_*(\mathcal{O}_X)^{-1}$. Now recall since X is relatively minimal and $\chi(X) > 0$, we have $deg(L) > 0$. The map η is compatible with the Higgs fields ϕ_0 and ϕ_1 on W_0 and W_1 respectively. But

$$deg(W_0) > deg(W_1 \otimes L^{-1})$$

and hence as (W_0, ϕ_0) and (W_1, ϕ_1) are semistable as Higgs bundles on C, the morphism $\eta = 0$. Hence

$$rk(\pi_*(V)) = rk(V) \implies V_K \cong \mathcal{O}_{X_K}^{\oplus rk(V)}.$$

□

3.2. **Proof of Theorem 1 in the case of multiple fibers.** Recall the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widetilde{C} & \xrightarrow{p} & C \end{array}$$

where $\tilde{\pi} : \widetilde{X} \to \widetilde{C}$ is a non-isotrivial relatively minimal elliptic surface with no multiple fibers. We have $\widetilde{C} \to C$ is Galois with Galois group Γ. Further $\widetilde{X} \to X$

is etale Galois with Galois group also Γ. From the previous subsection, we have an equivalence of categories

$$\tilde{\pi}^* : \mathcal{C}^{Higgs}_{\widetilde{C}} \to \mathcal{C}^{vHiggs}_{\widetilde{X}}.$$

Since $\tilde{\pi}$ is Γ equivariant, the functor $\tilde{\pi}^*$ induces a functor

$$\tilde{\pi}^*_{\Gamma} : \mathcal{C}^{\Gamma-Higgs}_{\widetilde{C}} \to \mathcal{C}^{\Gamma-vHiggs}_{\widetilde{X}}$$

We claim now the functor $\tilde{\pi}^*_{\Gamma}$ is an equivalence of categories. To that end note that every Γ-semistable Higgs bundle on $\widetilde{X}$ is semistable in the usual sense. Hence every object $(V, \theta) \in Ob(\mathcal{C}^{\Gamma-Higgs}_{\widetilde{X}})$ is isomorphic to a Higgs bundle of the form $(\tilde{\pi}^*(W), d\tilde{\pi}(\phi))$ as $\tilde{\pi}^*$ is an equivalence of categories. The only thing to verify is if this isomorphism can be obtained in the category of Γ-equivariant Higgs bundles on $\widetilde{X}$. The idea is to show that for any chosen isomorphism of Higgs bundles λ in $Isom((\tilde{\pi}^*(W), d\tilde{\pi}(\phi)), (V, \theta))$ we can provide (W, ϕ) with a natural Γ structure such that the isomorphism λ is Γ invariant.
To see this denote by $\tau^{\widetilde{X}}_g$ and $\tau^{\widetilde{C}}_g$ the respective automorphisms of $\widetilde{X}$ and $\widetilde{C}$ corresponding to $g \in \Gamma$. Recall from the definition of Γ-equivariance, we have isomorphisms

$$\alpha_g \in Isom((V, \theta), ((\tau^{\widetilde{X}}_g)^*(V), (\tau^{\widetilde{X}}_g)^*(\theta))).$$

satisfying

$$(\tau^{\widetilde{X}}_h)^*(\alpha_g)\alpha_h = \alpha_{gh}.$$

Now we have

$$\tilde{\pi} \circ \tau^{\widetilde{X}}_g = \tau^{\widetilde{C}}_g \circ \tilde{\pi}, \ \ \forall g \in \Gamma.$$

Fix an isomorphism

$$\lambda \in Isom((\tilde{\pi}^*(W), d\tilde{\pi}(\phi)), (V, \theta))$$

Set

$$\beta_g = (\tau^{\widetilde{X}}_g)^*(\lambda) \circ \alpha_g \circ \lambda.$$

We have then

$$\begin{aligned}\beta_g &\in Isom((\tilde{\pi}^*(W), d\tilde{\pi}(\phi)), (\tilde{\pi}^*(\tau^{\widetilde{C}}_g)^*W), d\tilde{\pi}(\tau^{\widetilde{C}}_g)^*\phi))) \\ &= Isom((W, \phi), ((\tau^{\widetilde{C}}_g)^*(W), (\tau^{\widetilde{C}}_g)^*(\phi))).\end{aligned}$$

and clearly

$$(\tau^{\widetilde{C}}_h)^*(\beta_g)\beta_h = \beta_{gh}.$$

Hence β_g induce a Γ-structure on (W, ϕ) such that the isomorphisms λ are Γ-equivariant.

Let Δ be a vertical divisor and $d = \frac{\Delta.H}{F.H}$. Denote by $M^{Higgs}_X(r, \Delta, 0)$ the moduli space of S-equivalence classes of rank r H-semistable Higgs bundles on X with vanishing second Chern class and determinant numerically equivalent to Δ.
Denote by $M^{ParHiggs}_{(C,\mathbf{c},\mathbf{m})}(r, d)$ the moduli space of S-equivalence classes of parabolic Higgs bundles on C with parabolic structures above the points c_i (and weights above c_i lying in $\frac{\mathbb{Z}}{m_i\mathbb{Z}}$) and parabolic degree d.
We have the following Corollary of Theorem 1

Corollary 2. *The moduli spaces* $M^{Higgs}_X(r, \Delta, 0)$ *and* $M^{ParHiggs}_{(C,\boldsymbol{c},\boldsymbol{m})}(r, d)$ *are isomorphic as algebraic varieties.*

Proof. As in the proof of Theorem 1 it is enough to consider the case of a fibration with no multiple fibers. We denote by $\mathcal{M}^{Higgs}_X(r, \Delta, 0)$ ($\mathcal{M}^{Higgs}_C(r, d)$) the moduli functors whose corresponding coarse moduli spaces are $M^{Higgs}_X(r, \Delta, 0)$ ($M^{Higgs}_C(r, d)$ respectively).

Recall these moduli functors are from category of finite-type schemes over $\mathbb{C}$ to category of sets. For a given finite type scheme T, $\mathcal{M}_X^{Higgs}(r,\Delta,0)(T)$ ($\mathcal{M}_C^{Higgs}(r,d)$) is the set of equivalence classes of flat families of semistable Higgs bundles on X of rank r, vanishing second Chern class and determinant numerically equivalent to Δ (semistable Higgs bundles on C of rank r and degree d respectively) parametrised by T. Recall a family parametrised by T corresponding to $\mathcal{M}_X^{Higgs}(r,\Delta,0)$, is a pair $(\mathcal{V},\psi)$ where $\mathcal{V}$ is a sheaf on $X\times T$, flat over T and $\psi \in Hom(\mathcal{V},\mathcal{V}\otimes_{\mathcal{O}_{X\times T}} pr_1^*(\Omega_X^1))$ where pr_1 denotes the projection map from $X\times T$ to X. Further for every closed point $t\in T$, we have for the natural closed embedding $t: X\hookrightarrow X\times T$, The pair

$$(V_t,\psi_t) := (t^*\mathcal{V}, t^*\psi)$$

is an object in $\mathcal{C}_X^{vHiggs}$ with $det(V_t)\equiv\Delta$. Let π_T denote the morphism

$$\pi_T := \pi\times id_T : X\times T \to C\times T.$$

From Theorem 1, we get the pair $(\mathcal{W},\phi) := ((\pi_T)_*(\mathcal{V}),(\pi_T)_*(\psi))$ is a flat family of objects in $\mathcal{C}_C^{Higgs}$ with $deg(\mathcal{W}_t) = d$ for every t. Thus we get a Natural transformation of functors

$$\pi_* : \mathcal{M}_X^{Higgs}(r,\Delta,0)\to\mathcal{M}_C^{Higgs}(r,d)$$

Conversely starting from a flat family $(\mathcal{W},\phi)$ of objects in $\mathcal{C}_C^{Higgs}$, with $deg(\mathcal{W}_t)=d$ for every t, parametrised by T, we have $((\pi)_T^*(\mathcal{W}),(\pi_T)^*(\phi))$ is a flat family of objects in $\mathcal{C}_X^{vHiggs}$ parametrised by T as considered above.
Thus we get a natural transformation

$$\pi^* : \mathcal{M}_C^{Higgs}(r,d)\to\mathcal{M}_X^{Higgs}(r,\Delta,0)$$

The composition $\pi_*\circ\pi^*$ and $\pi^*\circ\pi_*$ are clearly the identity transformations of the corresponding functors.
Hence the moduli functors $\mathcal{M}_X^{Higgs}(r,\Delta,0)$ and $\mathcal{M}_C^{Higgs}(r,d)$ are naturally equivalent and so the corresponding coarse moduli spaces $M_X^{Higgs}(r,\Delta,0)$ and $M_C^{Higgs}(r,d)$ are isomorphic as varieties. □

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