

# On the structure of the second eigenfunctions of the $p$ -Laplacian on a ball

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## Abstract

In this paper, we prove that the second eigenfunctions of the  $p$ -Laplacian,  $p > 1$ , are not radial on the unit ball in  $\mathbb{R}^N$ , for any  $N \geq 2$ . Our proof relies on the variational characterization of the second eigenvalue and a variant of the deformation lemma. We also construct an infinite sequence of eigenpairs  $\{\tau_n, \Psi_n\}$  such that  $\Psi_n$  is nonradial and has exactly  $2n$  nodal domains. A few related open problems are also stated.

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## 1 Introduction

Let  $B_1 \subset \mathbb{R}^N$  be the open unit ball centred at the origin. We consider the following eigenvalue problem:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } B_1, \\ u &= 0 && \text{on } \partial B_1, \end{aligned} \tag{1.1}$$

where  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator with  $p > 1$  and  $\lambda$  is the spectral parameter. A real number  $\lambda$  for which (1.1) admits a non-zero weak solution in  $W_0^{1,p}(B_1)$  is called an eigenvalue of (1.1) and corresponding solutions are called the eigenfunctions associated with  $\lambda$ .

For  $p = 2$ , it is well known that the set of all eigenvalues of (1.1) can be arranged in a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \dots \rightarrow \infty$$

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and the corresponding normalized eigenfunctions form an orthonormal basis for the Sobolev space  $W_0^{1,2}(B_1)$ . Further, using the Courant-Weinstein variational principle (Theorem 7.8.14 of [4]), these eigenvalues can be expressed as follows:

$$\lambda_k := \inf_{\{u \perp \{u_1, \dots, u_{k-1}\}, \|u\|_2=1\}} \int_{B_1} |\nabla u|^2 dx, \quad k = 1, 2, 3, \dots,$$

where  $u_i$  is an eigenfunction corresponding to  $\lambda_i$ . For  $p \neq 2$ , using Ljusternik-Schnirelman theorem, an infinite sequence  $\{\mu_n\}$  of eigenvalues of (1.1) is provided in [7]. Possibly a different sequence  $\{\lambda_n\}$  of variational eigenvalues of (1.1) is provided in [5]. We stress that a complete description of the set of all eigenvalues of (1.1) for  $p \neq 2$  is a challenging open problem. Nevertheless, a complete description of the set of all radial eigenvalues  $\{\gamma_n\}$  (eigenvalue with a radial eigenfunction) of (1.1) is given in [3]. The authors of [3] showed that  $\lambda$  is a radial eigenvalue of (1.1) if and only if the following ODE has a non-zero solution:

$$\begin{aligned} -(r^{N-1} |u'(r)|^{p-2} u'(r))' &= \lambda r^{N-1} |u(r)|^{p-2} u(r) \quad \text{in } (0, 1), \\ u'(0) &= 0, \quad u(1) = 0. \end{aligned} \tag{1.2}$$

Regardless of the methods by which the eigenvalues are obtained, one can uniquely identify the first two eigenvalues of (1.1) as below:

$$\begin{aligned} \lambda_1 &= \min\{\lambda : \lambda \text{ is an eigenvalue of (1.1)}\}, \\ \lambda_2 &= \min\{\lambda > \lambda_1 : \lambda \text{ is an eigenvalue of (1.1)}\}. \end{aligned}$$

It is well known that the eigenfunctions corresponding to  $\lambda_1$  are radial and keep the same sign on  $B_1$ . All other eigenfunctions change its sign on  $B_1$ . The structure of the second eigenfunctions are not well understood, except for  $p = 2$ . In this case, the Fourier method for the Laplacian in the polar coordinates gives the precise form of the second eigenfunctions. In particular, it is evident that the second eigenfunctions are not radial. One anticipates the same results also for  $p \neq 2$ .

In [12], Parini proved that the second eigenfunctions are not radial in a special case, where  $B_1$  is the disc ( $B_1 \subset \mathbb{R}^2$ ) and  $p$  is close to 1. In [1], this result is extended for every  $p \in (1, \infty)$  using a computer aided proof. Indeed, these methods are not readily extendable to dimensions greater than 2. Here, we give a simple analytic proof for their result which works in all dimensions ( $N \geq 2$ ) and for every  $p \in (1, \infty)$ . Our proof relies on the variational characterization of  $\lambda_2$  given in [5] and a variation of the deformation lemma given in [8]. We also use a result from [2] that states that for a fixed  $r \in (0, 1)$ ,

$$\lambda_1(B_1 \setminus \overline{B_r(x)}) \leq \lambda_1(B_1 \setminus \overline{B_r(0)}),$$

where  $B_r(x) \subset B_1$  is the ball with centre  $x$  and radius  $r$ . Now we state our main result:

**Theorem 1.1.** *Let  $B_1$  be the unit ball centred at the origin in  $\mathbb{R}^N$  with  $N \geq 2$  and let  $1 < p < \infty$ . Let  $\lambda_2$  be the second eigenvalue of (1.1). Then the eigenfunctions corresponding to  $\lambda_2$  are not radial.*

In this paper we also construct a sequence  $\{\tau_n, \Psi_n\}$  of eigenpairs of (1.1) such that the eigenfunction  $\Psi_n$  is nonradial and has exactly  $2n$  nodal domains. Furthermore, the sequence  $\{\tau_n\}$  is strictly increasing and unbounded. In fact the nodal domains can be specified using the spherical coordinate system for  $\mathbb{R}^N$  which consists of a radial coordinate  $r$  and angular coordinates  $\theta_1, \dots, \theta_{N-1}$  where  $\theta_1, \dots, \theta_{N-2} \in [0, \pi]$  and  $\theta_{N-1} \in [0, 2\pi)$ . By a sector of the ball  $B_1$  we mean the set  $S$  given by  $S = \{x \in B_1 : 0 < \theta_* < \theta_{N-1} < \theta^* < 2\pi\}$ . We prove the following theorem.

**Theorem 1.2.** *Let  $B_1 \subset \mathbb{R}^N$ . Then for each  $n \in \mathbb{N}$  there exists an eigenpair  $\{\tau_n, \Psi_n\}$  of (1.1) such that  $\Psi_n$  has exactly  $2n$  nodal domains where each nodal domain is a sector with measure  $\frac{|B_1|}{2n}$ .*

The rest of this paper is organized as follows. In Section 2, we consider Dirichlet eigenvalue for  $p$ -Laplacian on a general domain and discuss the existence and the regularity properties of the eigenfunctions. We also discuss the variational characterizations of eigenvalues and state a version of the deformation lemma. In Section 3, we give a proof for Theorem 1.1. The last section consists of a proof of Theorem 1.2 and some important open problems related to eigenvalues of  $p$ -Laplacian.

## 2 Preliminary

In this section we consider the eigenvalue problem on a bounded domain  $\Omega$  in  $\mathbb{R}^N$  :

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We discuss the existence and regularity properties of the eigenfunctions of (2.1). If  $\lambda$  is an eigenvalue of (2.1) and  $u \in W_0^{1,p}(\Omega)$  is an associated eigenfunction, then we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} |u|^{p-2} uv \, dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.2}$$

Now we consider the following two functionals on  $W_0^{1,p}(\Omega)$  :

$$J(u) = \int_{\Omega} |\nabla u|^p \, dx, \quad G(u) = \int_{\Omega} |u|^p \, dx.$$

Using the Lagrange multiplier theorem, it can be easily verified that the critical values and critical points of  $J$  on the manifold  $\mathcal{S} = G^{-1}(1)$  satisfy (2.2). Indeed, the eigenvalues of (2.1) and the critical values of  $J$  on  $\mathcal{S}$  are one and the same. The least critical value of  $J$  on  $\mathcal{S}$  is given by

$$\lambda_1 = \inf_{u \in \mathcal{S}} J(u).$$

In the next proposition, we list some of the important properties of  $\lambda_1$  and the corresponding eigenfunctions.

**Proposition 2.1.** *Let  $\lambda_1$  be the first eigenvalue of (2.1). Then*

- (i)  $\lambda_1$  is simple
- (ii) any eigenfunction corresponding to  $\lambda_1$  keeps the same sign on  $\Omega$ ,
- (iii) any eigenfunction corresponding to an eigenvalue  $\lambda > \lambda_1$  changes its sign on  $\Omega$ ,
- (iv) if  $\Omega = B_r(0)$ , then the eigenfunctions corresponding to  $\lambda_1$  are radial.

*Proof.* For a proof of (i) and (ii) see [11], (iii) follows from Theorem 1.1 of [9]. Finally (iv) is evident from (i) and (iii) by noting the existence of a radial positive eigenfunction for (2.1) when  $\Omega = B_r(0)$ .  $\square$

An infinite set of critical values of  $J$  on  $\mathcal{S}$  are obtained in [7] using the variational methods. Their approach relies on the notion of Krasnoselskii genus of a symmetric closed set. For a symmetric closed subset  $\mathcal{A} \subset \mathcal{S}$ , Krasnoselskii genus of  $\mathcal{A}$  is defined as

$$\gamma(\mathcal{A}) := \inf \{n \in \mathbb{N} : \exists \text{ a continuous odd map from } \mathcal{A} \text{ into } \mathbb{R}^n \setminus \{0\}\}$$

with the convention  $\inf\{\emptyset\} = \infty$ . For each  $n \in \mathbb{N}$ , let

$$\begin{aligned} \mathcal{E}_n &:= \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = \overline{\mathcal{A}}, \mathcal{A} = -\mathcal{A} \text{ and } \gamma(\mathcal{A}) \geq n\}, \\ \mu_n &:= \inf_{\mathcal{A} \in \mathcal{E}_n} \sup_{u \in \mathcal{A}} J(u). \end{aligned}$$

Then  $\mu_n$  is a critical value of  $J$  on  $\mathcal{S}$  (see Proposition 5.4 of [7]). Possibly another set of critical values are obtained in [5] by considering a special collection of sets with genus  $n$  in  $\mathcal{S}$ . Note that, the unit sphere  $\mathcal{S}^{n-1}$  in  $\mathbb{R}^n$  has genus  $n$  and hence its image under an odd continuous map has the same genus. For each  $n \in \mathbb{N}$ , let

$$\begin{aligned} \mathcal{F}_n &:= \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = h(\mathcal{S}^{n-1}), h \text{ is an odd continuous map from } \mathcal{S}^{n-1} \rightarrow \mathcal{S}\}, \\ \mu_n^* &:= \inf_{\mathcal{A} \in \mathcal{F}_n} \sup_{u \in \mathcal{A}} J(u). \end{aligned}$$

Then  $\mu_n^*$  is a critical value of  $J$  on  $\mathcal{S}$  (see Theorem 5 of [5]). Since  $\mathcal{F}_n \subset \mathcal{E}_n$ , we always have  $\mu_n \leq \mu_n^*$ . It is known that  $\lambda_i = \mu_i = \mu_i^*$  for  $i = 1, 2$ . This result for  $i = 1$  follows as the set  $\{u, -u\}$  lies in both  $\mathcal{E}_1$  and  $\mathcal{F}_1$  for  $u \in \mathcal{S}$ . Let  $u$  be an eigenfunction corresponding to  $\lambda_2$ . Then by (ii) of Proposition 2.1 both  $u^+$  and  $u^-$  are nonzero. Thus the set  $\mathcal{A} := \{au^+ + bu^- : |a|^p \|u^+\|_p^p + |b|^p \|u^-\|_p^p = 1\}$  lies in both  $\mathcal{E}_2$  and  $\mathcal{F}_2$ . Now as  $J(au^+ + bu^-) = \lambda_2$ , we get  $\mu_2 \leq \lambda_2$  and  $\mu_2^* \leq \lambda_2$ . Since there is no eigenvalue between  $\lambda_1$  and  $\lambda_2$ , it follows that  $\lambda_2 = \mu_2 = \mu_2^*$ . In particular, we have the following variational characterization of  $\lambda_2$  that we use later:

$$\lambda_2 = \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} J(u). \quad (2.3)$$

The next proposition is a consequence of the deformation lemma (see Lemma 3.7 of [8], see also Theorem 2.1 and Remark 2.3 of [6]). Note that  $J \in \mathcal{C}^1(W_0^{1,p}(\Omega); \mathbb{R})$  and  $\mathcal{S}$  is a  $\mathcal{C}^1$  manifold. Further,  $J(u) = J(-u)$  and  $\mathcal{S} = -\mathcal{S}$ .

**Proposition 2.2.** *Let  $\mathcal{S}, J$  be as before. Let  $\mathcal{K}$  be a compact subset of  $S$ . If  $\|J'(u)\|_* \geq \varepsilon > 0$  for all  $u \in \mathcal{K}$ , then there exists a continuous one parameter family of homeomorphisms  $\Psi : S \times [0, 1] \rightarrow S$  such that*

- (i)  $J(\Psi(u, t)) \leq J(u) - \varepsilon t$ , for every  $u \in \mathcal{K}$ ,  $t \in [0, 1]$ ,
- (ii)  $\Psi(-u, t) = -\Psi(u, t)$ , for all  $u \in \mathcal{S}$ ,  $t \in [0, 1]$ .

*In particular, if  $\mathcal{K} \in \mathcal{F}_n$  and  $J$  has no critical point on  $\mathcal{K}$ , then the set  $\tilde{\mathcal{K}} = \{\Psi(u, 1) : u \in \mathcal{K}\}$  is in  $\mathcal{F}_n$  and*

$$\sup_{u \in \tilde{\mathcal{K}}} J(u) < \sup_{u \in \mathcal{K}} J(u). \quad (2.4)$$

We also need the following result on the regularity of the eigenfunctions of (2.1) which is a consequence of Theorem 1 of [10].

**Proposition 2.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary. Let  $\phi$  be an eigenfunction of (2.1). Then there exists  $\alpha \in (0, 1)$  such that  $\phi \in C^{1,\alpha}(\bar{\Omega})$ .*

### 3 Radial asymmetry of the second eigenfunctions

In this section we prove our main result. First we state a lemma that follows from Proposition 4.1 of [3].

**Lemma 3.1.** *Let  $\gamma_2$  be the second radial eigenvalue of (1.2). Then any radial eigenfunction corresponding to  $\gamma_2$  has exactly two nodal domains - a ball and an annulus with centre at the origin. In particular, there exist  $r \in (\frac{1}{2}, 1)$  such that  $\lambda_1(B_r(0)) = \gamma_2 = \lambda_1(B_1 \setminus \overline{B_r(0)})$ .*

Now using the 'r' given by the above lemma, we construct a special collection of sets in  $\mathcal{F}_2$ . Let  $r$  be as in Lemma 3.1. Then for each  $n \in \mathbb{N} \cup \{0\}$ , we construct a special set  $\mathcal{A}_n \in \mathcal{F}_2$  such that  $\sup_{u \in \mathcal{A}_n} J(u) = \gamma_2$ . Let  $\{t_n\}$  be a sequence in  $[0, 1 - r)$  such that  $t_0 = 0$  and  $t_n \rightarrow 1 - r$ . For each  $n \in \mathbb{N} \cup \{0\}$ , let  $B_n = B_r(t_n e_1)$  and  $\Omega_n = B_1 \setminus \overline{B_n}$  where  $e_1$  is the unit vector in the direction of the first coordinate axis. Let  $u_n, v_n$  be the respective first eigenfunctions on  $B_n$  and  $\Omega_n$  satisfying  $u_n > 0$  on  $B_n$ ,  $v_n > 0$  on  $\Omega_n$  and  $\|u_n\|_p = \|v_n\|_p = 1$ . By translation invariance of the  $p$ -Laplacian, we have  $\lambda_1(B_n) = \gamma_2$ . Further, from Theorem 1 of [2], we also have  $\lambda_1(\Omega_n) \leq \gamma_2$ . Let  $\tilde{u}_n$  and  $\tilde{v}_n$  be the zero extensions to the entire  $B_1$ . For each  $n \in \mathbb{N} \cup \{0\}$ , we consider

$$\mathcal{A}_n := \{a\tilde{u}_n + b\tilde{v}_n : |a|^p + |b|^p = 1\}.$$

One can easily verify that  $\mathcal{A}_n \in \mathcal{F}_2$  and  $\sup_{u \in \mathcal{A}_n} J(u) = \gamma_2, \forall n \in \mathbb{N} \cup \{0\}$ .

Now we ask the question whether  $\mathcal{A}_n$  contains a critical point of  $J$  on  $S$  or not. This leads to the following two alternatives:

- (i) for every  $n \in \mathbb{N}$ ,  $\mathcal{A}_n$  contains at least one critical point of  $J$  on  $S$ ,
- (ii) there exists  $n_0 \in \mathbb{N}$  such that  $\mathcal{A}_{n_0}$  does not contain any critical point of  $J$  on  $S$ .

In the next lemma we show that alternative (i) does not hold.

**Lemma 3.2.** *Let  $\mathcal{A}_n$  be as above. Then alternative (i) does not hold.*

*Proof.* Let  $u_n$  and  $\tilde{u}_n$  be as above. Then  $u_n(x) = u_0(x - t_n e_1)$  and hence the sequence  $\{\tilde{u}_n(x)\}$  converges to  $u^*(x) = \tilde{u}_0(x - (1-r)e_1)$  both pointwise and in  $W_0^{1,p}(B_1)$ . On the other hand, the sequence  $\{\tilde{v}_n\}$  is bounded by  $\gamma_2$  in  $W_0^{1,p}(B_1)$ . Thus up to a subsequence,  $\tilde{v}_n$  converges to some  $v^*$  weakly in  $W_0^{1,p}(B_1)$  and a.e. in  $B_1$ . If alternative (i) holds, then we get a sequence  $\{\phi_n = a_n \tilde{u}_n + b_n \tilde{v}_n : |a_n|^p + |b_n|^p = 1\}$  of eigenfunctions of (1.1) with eigenvalues  $J(\phi_n)$ . By Proposition 2.3, the eigenfunctions are in  $C^1(\overline{B_1})$  and hence we must have  $a_n b_n < 0$ . Now we may assume that  $a_n > 0$  and  $b_n < 0$  for each  $n$ . Further, the sequences  $\{J(\phi_n)\}, \{a_n\}$  and  $\{b_n\}$  are bounded. Thus for a subsequence we get  $J(\phi_n) \rightarrow \lambda^*$ ,  $a_n \rightarrow a^*$  and  $b_n \rightarrow b^*$  for some  $\lambda^*, a^* \geq 0$  and  $b^* \leq 0$ . The sequence  $\{\phi_n\}$  is bounded in  $W_0^{1,p}(B_1)$  and hence up to a subsequence  $\phi_n \rightharpoonup \phi^*$  in  $W_0^{1,p}(B_1)$  and a.e. in  $B_1$ . Since  $a_n \tilde{u}_n + b_n \tilde{v}_n \rightarrow a^* u^* + b^* v^*$  a.e. in  $B_1$ , we must have

$$\phi^* = a^* u^* + b^* v^*.$$

Since, each  $\phi_n$  is an eigenfunction of (1.1), it is easy to verify that  $\phi^*$  is an eigenfunction corresponding to the eigenvalue  $\lambda^*$ . Thus by the regularity of  $\phi^*$ , we must have  $a^* b^* < 0$  and hence

$$a^* > 0, \quad b^* < 0.$$

Let  $B^* = B_r((1-r)e_1)$  and  $\Omega^* = B_1 \setminus B^*$ . Clearly  $u^* > 0$  on  $B^*$  and  $u^* = 0$  on  $\Omega^*$ . On the other hand,  $v^* = 0$  a.e. in  $B^*$  and  $v^* \geq 0$  a.e. on  $\Omega^*$ . Thus from the continuity of the  $\phi^*$  we get

$$\phi^*(x) > 0, \quad \forall x \in B^*, \quad \phi^*(x) \leq 0, \quad \forall x \in \Omega^*.$$

Now we apply Theorem 5 of [15] (a Hopf's lemma type result for  $p$ -Laplacian) on  $B^* \cup \{e_1\}$  to get

$$\frac{\partial \phi^*}{\partial x_1}(e_1) = c < 0.$$

Since  $\phi^* \leq 0$  on  $\Omega^*$  we also have

$$\frac{\partial \phi^*}{\partial \eta(x)}(x) \geq 0, \quad \forall x \in \partial B_1 \setminus \{e_1\},$$

where  $\eta(x)$  is the outward unit normal to  $B_1$  at  $x$ . The above two inequalities contradicts the fact that  $\phi^*$  is in  $C^1(\overline{B_1})$ . Thus we conclude that alternative (i) does not hold.  $\square$

**Proof of Theorem 1.1** Let  $\mathcal{A}_n$  be as before. Thus we have  $\sup_{v \in \mathcal{A}_n} J(v) \leq \gamma_2$ . By the above lemma, the alternative (ii) holds, i.e. there exists  $n_0 \in \mathbb{N}$  such that  $\mathcal{A}_{n_0}$  does not contain any critical points of  $J$  on  $\mathcal{S}$ . Thus by Proposition 2.2 and by (2.4), we get  $\tilde{\mathcal{A}} \in \mathcal{F}_2$  such that

$$\sup_{u \in \tilde{\mathcal{A}}} J(u) < \sup_{v \in \mathcal{A}} J(v) \leq \gamma_2.$$

Now from (2.3) we get  $\lambda_2 < \gamma_2$ .

## 4 Construction of nonradial eigenfunctions

In this section we construct an infinite sequence of nonradial eigenfunctions of (1.1). First we fix the following conventions. A vector  $x$  in  $\mathbb{R}^N$  is always taken as a  $1 \times N$  row vector, i.e  $x = (x_1, x_2 \dots x_N)$ . The transpose of  $x$ , denoted by  $x^T$ , is an  $N \times 1$  column vector. We denote the scalar product in  $\mathbb{R}^N$  by  $x \cdot y (= xy^T)$ . Let  $H$  be the hyperplane given by  $H = \{x \in \mathbb{R}^N : x \cdot a = 0\}$  for some unit vector  $a \in \mathbb{R}^N$ . Let  $\sigma_H$  be the reflection about  $H$ . Then

$$\sigma_H(x) = x - 2(x \cdot a)a = x(I - 2a^T a).$$

Next we list some of the elementary properties of  $\sigma_H$  that we use in this article.

- (i)  $\sigma_H$  is linear and  $\sigma_H = (I - 2a^T a)$ .
- (ii)  $\sigma_H^{-1} = \sigma_H$ .
- (iii)  $\sigma_H$  is symmetric and orthogonal.
- (iv)  $D\sigma_H(x) = \sigma_H$  and  $\det D\sigma_H(x) = -1, \forall x \in \mathbb{R}^N$ .

Let  $\mathcal{O}$  be a bounded domain symmetric about  $H$ , i.e,  $\sigma_H(\mathcal{O}) = \mathcal{O}$ . Let  $\mathcal{O}^+ := \{x \in \mathcal{O} : \langle x, a \rangle > 0\}$  and let  $\mathcal{O}^- = \sigma_H(\mathcal{O}^+)$ . Let  $u \in W_0^{1,p}(\mathcal{O}^+)$  be a weak solution of (2.1) on  $\Omega = \mathcal{O}^+$ . Define  $u^*$  on  $\mathcal{O}$  as below

$$u^*(x) = \begin{cases} u(x), & x \in \mathcal{O}^+, \\ 0, & x \in \partial(\mathcal{O}^+) \cup \partial(\mathcal{O}^-), \\ -u(\sigma_H(x)), & x \in \mathcal{O}^-. \end{cases}$$

Clearly  $u^* \in W_0^{1,p}(\mathcal{O})$  and we also have the following lemma:

**Lemma 4.1.** *Let  $u^*$  be defined as above. Then  $u^*$  is a weak solution of (2.1) on  $\Omega = \mathcal{O}$ .*

*Proof.* Let  $\phi \in W_0^{1,p}(\mathcal{O})$  be a test function. We show that

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx = \lambda \int_{\mathcal{O}} |u^*(x)|^{p-2} u^*(x) \phi(x) dx. \quad (4.1)$$

From the definition of  $u^*$ ,

$$\begin{aligned} \int_{\mathcal{O}} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx &= \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \phi(x) dx \\ &\quad + \int_{\mathcal{O}^-} |\nabla(-u(\sigma_H(x)))|^{p-2} \nabla(-u(\sigma_H(x))) \cdot \nabla \phi(x) dx \end{aligned}$$

Now by noting that  $D\sigma_H(x) = \sigma_H$  and  $\sigma_H$  is an isometry we get

$$\begin{aligned} &\int_{\mathcal{O}^-} |\nabla(-u(\sigma_H(x)))|^{p-2} \nabla(-u(\sigma_H(x))) \cdot \nabla \phi(x) dx \\ &= - \int_{\mathcal{O}^-} |\nabla u(\sigma_H(x)) \sigma_H|^{p-2} [\nabla u(\sigma_H(x)) \sigma_H] \cdot \nabla \phi(x) dx, \\ &= - \int_{\mathcal{O}^-} |\nabla u(\sigma_H(x))|^{p-2} \nabla u(\sigma_H(x)) \cdot [\nabla \phi(x) \sigma_H] dx, \end{aligned}$$

where the equality in the last step also uses the fact that  $\sigma_H$  is symmetric. Now the change of variable  $y = \sigma_H(x)$  along with properties (ii) and (iv) of  $\sigma_H$  will give

$$\int_{\mathcal{O}^-} |\nabla(-u(\sigma_H(x)))|^{p-2} \nabla(-u(\sigma_H(x))) \cdot \nabla \phi(x) dx = - \int_{\mathcal{O}^+} |\nabla u(y)|^{p-2} \nabla u(y) \cdot [\nabla \phi(\sigma_H(y)) \sigma_H] dy.$$

Thus

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx = \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2} \nabla u(x) \cdot [\nabla \phi(x) - [\nabla \phi(\sigma_H(x)) \sigma_H]] dx.$$

Let  $\psi(x) = \phi(x) - \phi(\sigma_H(x))$ . Then we have

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx = \int_{\mathcal{O}^+} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \psi(x) dx. \quad (4.2)$$

Further,

$$\int_{\mathcal{O}} |u^*(x)|^{p-2} u^*(x) \phi(x) dx = \int_{\mathcal{O}^+} |u(x)|^{p-2} u(x) \psi(x) dx. \quad (4.3)$$

Clearly  $\psi \in W_0^{1,p}(\mathcal{O}^+)$  and hence

$$\int_{\mathcal{O}} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx = \int_{\mathcal{O}^+} |u(x)|^{p-2} u(x) \psi(x) dx, \quad (4.4)$$

since  $u$  solves (2.1) on  $\Omega = \mathcal{O}^+$ . Now (4.1) follows from (4.2), (4.3) and (4.4).  $\square$

**Proof of Theorem 1.2:** For  $n \in \mathbb{N}$ , we consider the sectors  $S_k$  given by  $S_k = \{x \in B_1 : \frac{(k-1)\pi}{n} < \theta_{N-1} < \frac{k\pi}{n}\}$ ,  $k = 1, \dots, n$ . Let  $H_k$  be the hyperplane given by  $H_k = \{x \in \mathbb{R}^N : \theta_{N-1} = \frac{\pi k}{n}\}$ , for  $k = 1, \dots, n$ . Let  $\tau_n$  be the first eigenvalue for the  $p$ -Laplacian on  $S_1$  and  $u_1(x)$  be a corresponding eigenfunction. For  $i = 2, \dots, n$ , we define  $u_i$  recursively by  $u_i = -u_{i-1}(\sigma_{H_{i-1}}(x))$ , the odd reflection of  $u_i$  about  $H_{i-1}$ . Let  $D^+$  be the sector given by  $\{x \in B_1 : 0 < \theta_{N-1} < \pi\}$ . Now we define  $u^*$  on  $\overline{D^+}$  by

$$u^*(x) = u_i(x), \quad x \in \overline{S_i}, \quad i = 1, \dots, n.$$

From Lemma 4.1, it is clear that  $u^*$  solves (2.1) on the union of two adjacent sectors with  $\lambda = \tau_n$ . Let  $U_i = \{x \in B_1 : \frac{(i-1)\pi}{n} < \theta_{N-1} < \frac{(i+1)\pi}{n}\}$ , for  $i = 1, \dots, n-1$ . Then  $\{U_i\}_{i=1}^{n-1}$  is an open covering of  $D^+$ . Let  $\{\phi_i\}_{i=1}^{n-1}$  be a  $\mathcal{C}^\infty$  partition of the unity corresponding to this open covering. Note that for each  $i$ ,  $\phi_i$  intersects at most  $S_i$  and  $S_{i+1}$ . Since  $\sum_{i=1}^{n-1} \phi_i = 1$ , we have

$$\begin{aligned} \int_{D^+} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx &= \int_{D^+} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \left( \phi(x) \sum_{i=1}^{n-1} \phi_i(x) \right) dx \\ &= \sum_{i=1}^{n-1} \int_{D^+} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla (\phi(x) \phi_i(x)) dx. \end{aligned}$$



For a fixed  $i$ , the product  $\phi\phi_i \in W_0^{1,p}(U_i)$ . Hence by the definition of  $u^*$  and Lemma 4.1, we get

$$\begin{aligned} \int_{D^+} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla(\phi(x)\phi_i(x)) dx &= \int_{U_i} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla(\phi(x)\phi_i(x)) dx, \\ &= \tau_n \int_{U_i} |u^*(x)|^{p-2} u^*(x) (\phi(x)\phi_i(x)) dx, \\ &= \tau_n \int_{D^+} |u^*(x)|^{p-2} u^*(x) (\phi(x)\phi_i(x)) dx. \end{aligned}$$

Thus we get

$$\begin{aligned} \int_{D^+} |\nabla u^*(x)|^{p-2} \nabla u^*(x) \cdot \nabla \phi(x) dx &= \sum_{i=1}^{n-1} \tau_n \int_{D^+} |u^*(x)|^{p-2} u^*(x) (\phi(x)\phi_i(x)) dx, \\ &= \tau_n \int_{D^+} |u^*(x)|^{p-2} u^*(x) \left( \sum_{i=1}^{n-1} \phi(x)\phi_i(x) \right) dx, \\ &= \tau_n \int_{D^+} |u^*(x)|^{p-2} u^*(x) \phi(x) dx. \end{aligned}$$

Now define  $\Psi_n$  on  $B_1$  by

$$\Psi_n(x) = \begin{cases} u^*(x), & x \in D^+, \\ 0, & x \in \partial(D^+) \cup \partial(D^-), \\ -u^*(\sigma_{H_0}(x)), & x \in D^-, \end{cases}$$

where  $D^- = \{x \in B_1 : \pi < \theta_{N-1} < 2\pi\}$  is the “lower” half-ball and  $H_0$  is the hyperplane corresponding to  $\theta_{N-1} = 0$ . Applying Lemma 4.1 once again, we get that  $\Psi_n$  is a weak solution of (1.1). Thus we have constructed an eigenpair  $\{\tau_n, \Psi_n\}$  of (1.1) such that  $\Psi_n$  has  $2n$  nodal domains and each nodal domain is a sector with measure  $\frac{|B_1|}{2n}$ .  $\square$

In the next remark we list some of the interesting open problems related to the results of this paper:

**Remark 4.2.** (Open problems associated with (1.1))

1. Payne conjectured (Conjecture 5, [13]) that the nodal line of a second eigenfunction of Laplacian on a bounded domain  $\Omega \subset \mathbb{R}^2$  cannot be a closed curve. In [14], he proved his conjecture for the special case when  $\Omega$  is convex in  $x$  and symmetric about  $y$  axis. For a ball, his result was easily obtained by applying the Fourier method to the Laplacian in polar co-ordinates. We conjecture that the nodal surface of a second eigenfunction of  $p$ -Laplacian on  $B_1$  cannot be a closed surface in  $B_1$  for  $1 < p < \infty$  and for every  $N \geq 2$ .
2. For  $p = 2$ , it is easy to see that  $\lambda_2 = \tau_1$ . We anticipate the same result for  $p \neq 2$  as well. More precisely, the nodal surface of any second eigenfunction is given by the intersection of a hyperspace with  $B_1$  and the nodal domains are the half balls symmetric to this hyperspace.

3. We have just shown that all the eigenfunctions corresponding to  $\lambda_2$  are nonradial. Is it true that all the eigenfunctions corresponding to the second radial eigenvalue  $\gamma_2$  are radial?
4. Note that  $\lambda_2$  is the least eigenvalue having an eigenfunction with two nodal domains. For  $p = 2$ , it can also be seen that  $\gamma_2$  is the maximal eigenvalue having an eigenfunction with two nodal domains. In other words, the eigenfunctions corresponding to  $\lambda > \gamma_2$  must have at least three nodal domains. Is this true for  $p \neq 2$ ?

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