PROPER HOLOMORPHIC MAPPINGS BETWEEN HYPERBOLIC PRODUCT MANIFOLDS

JAIKRISHNAN JANARDHANAN

Abstract. We prove a result on the structure of finite proper holomorphic mappings between complex manifolds that are products of hyperbolic Riemann surfaces. While an important special case of our result follows from the ideas developed by Remmert and Stein, the proof of the full result relies on the interplay of the latter ideas and a finiteness theorem for Riemann surfaces.

1. Introduction

A natural problem in the study of holomorphic mappings is to give a description of the automorphism group or the set of all proper holomorphic self-maps of a given complex space. Of course, this problem is intractable in general even when posed for arbitrary domains in $\mathbb{C}^n$. One of the simplest classes of domains in $\mathbb{C}^n$ is the class of products of planar domains. The following theorem due to Remmert and Stein gives the precise structure of a proper holomorphic map between certain product domains.

Result 1.1 (Remmert–Stein [10]). *Let $D = D_1 \times \cdots \times D_n$ and $W = W_1 \times \cdots \times W_n$ be products of planar domains such that, for each $j = 1, \ldots, n$, $\mathbb{C} \setminus D_j$ has non-empty interior and $W_j \subset \mathbb{C}$ is bounded. Let $f = (f_1, \ldots, f_n)$ be a proper holomorphic map from D to W. Then, each f_j, $j = 1, \ldots, n$, is of the form $f_j(z_{p(j)})$, where p is a permutation of $\{1, \ldots, n\}$.*

Remark 1.2. The proof of this result in the case $n = 2$ was given by Remmert and Stein (Satz 12 in [10]). Their proof uses Rado's theorem. The proof of the general case requires a generalization of Rado's theorem, but all other aspects of Remmert and Stein's proof remain unchanged: see [8, pp. 71–78], for instance.

The methods of Remmert–Stein cannot, as Remark 1.2 suggests, be applied when the planar factors in Result 1.1 are replaced by compact Riemann surfaces. Yet, the phenomenon exhibited by Result 1.1 occurs in some other settings as well. Another result on mappings between product spaces is the following theorem of Peters which generalizes a well-known result by Cartan [1]:

Result 1.3 (Peters [9]). *Let X and Y be hyperbolic complex spaces. Then the natural injection $Aut(X) \times Aut(Y) \to Aut(X \times Y)$ induces an isomorphism*

$$Aut^0(X) \times Aut^0(Y) \cong Aut^0(X \times Y).$$

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Here $\mathrm{Aut}^0(X)$ denotes the connected component of the identity element of $\mathrm{Aut}(X)$.

To the best of our knowledge, there is no analogue of the above result for proper holomorphic maps in the literature, except for Result 1.1 (and a small technical improvement thereof in [8, p. 77]). It is natural to ask whether a result similar to Result 1.1, with the planar domains therein replaced by hyperbolic Riemann surfaces, holds true. However, it is far from clear whether the methods seen in the proofs of *either* of the above results are alone decisive in proving the hoped-for generalization. In our view, a key ingredient that is needed is the phenomenon illustrated by the following example.

Example 1.4. *Let $D = \mathbb{C} \setminus \{0, 1\}$, and $f = (f_1, f_2)$ be a proper holomorphic self-map of $D \times D$. Note that D is hyperbolic. Even though most of the hypotheses of the Remmert–Stein theorem are not satisfied, the conclusion still follows.*

This is not hard to see. Fix $z_0 \in \mathbb{C} \setminus \{0, 1\}$. By the big Picard theorem, it follows that $0, 1$ and ∞ are removable singularities or poles of the map $h := f_1(z_0, \cdot)$. Hence h extends as a holomorphic map to $\widehat{\mathbb{C}}$, and is therefore a rational map. If h is not proper as a map from $\mathbb{C} \setminus \{0, 1\}$ to itself, then there is a sequence $\{x_n\} \subseteq \mathbb{C} \setminus \{0, 1\}$ that converges to either $0, 1$ or ∞, such that some subsequence of the image sequence $\{h(x_n)\}$ converges to a finite point in $\mathbb{C} \setminus \{0, 1\}$. Hence h is a rational map that misses at least one of the points $0, 1$ or ∞, and must therefore be constant.

On the other hand, assume that h is a proper map from $\mathbb{C} \setminus \{0, 1\}$ to itself; then it is a non-constant rational map. Thus, if $h = \frac{P}{Q}$, where P and Q are two polynomials having no common factors, at least one of P or Q has to be non-constant. Also, note that h takes $\{0, 1, \infty\}$ to itself.

If P were non-linear, it would follow that either Q has the same degree as P, or Q is some constant C. In the latter case, both P and $P - Q$ are non-constant polynomials with disjoint zero sets. From this, it follows that P is either z^k or $(z - 1)^k$, $k > 1$. Therefore, the equation $h = 1$ has roots different from 0 and 1, which is a contradiction. If P and Q have the same degree, then it follows that $\frac{P}{Q}$ is of the form R^k, where R is a non-constant rational function, and the value 1 is attained by k distinct values, which is also a contradiction. Hence P is linear, and a similar argument shows that Q is also linear. Hence h is a fractional linear transformation that takes $\{0, 1, \infty\}$ to itself. There are only six possibilities for the map h. From this it follows that, if for some z_0, $f_1(z_0, \cdot)$ is an automorphism, then $f_1(z, \cdot)$ is the same automorphism for all $z \in \mathbb{C} \setminus \{0, 1\}$ (see Lemma 3.3). Together with the conclusion of the first paragraph, this proves that $f_1(z, \cdot)$ is either constant for all z, or is independent of z. Applying the same argument to f_2, we conclude that the conclusion of the Remmert–Stein theorem still holds.

The key fact used in the above example is that there are only finitely many proper holomorphic self-maps of $\mathbb{C} \setminus \{0, 1\}$. This is not true for the domains $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$. We are motivated by all of this to generalize the result of Remmert and Stein to complex manifolds that are products of certain hyperbolic Riemann surfaces.

Theorem 1.5. *Let R_j and S_j, $j = 1, \ldots, n$, be compact Riemann surfaces, and let X_j (resp. Y_j) be a connected, hyperbolic open subset of R_j (resp. S_j) for each $j = 1, \ldots, n$. Let $F = (F_1, \ldots, F_n) : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$ be a finite proper holomorphic map. Then, denoting $z \in X_1 \times \cdots \times X_n$ as $(z_1, \ldots, z_n)$, each F_i is of the form $F_i(z_{\pi(i)})$, where π is a permutation of $\{1, \ldots, n\}$.*

Remark 1.6. It is essential for F to be a finite map in the above theorem. Without this requirement, Theorem 1.5 is false. To see this, let X be some compact hyperbolic Riemann surface. The map $F : X^2 \to X^2$ defined by $F(z_1, z_2) := (z_1, z_1)$ is a proper map. In fact, F satisfies all the assumptions of Theorem 1.5 except finiteness.

Remark 1.7. The conclusion of the above theorem can fail if even one of the factors is non-hyperbolic. Consider $X = \mathbb{D} \times (\widehat{\mathbb{C}} \setminus \{p\})$, where $p \in \widehat{\mathbb{C}}$ and $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. We know that $\widehat{\mathbb{C}} \setminus \{p\}$ is **not** hyperbolic. We view $\widehat{\mathbb{C}} \setminus \{p\}$ as $\mathbb{C}$. It is easy to check that any $F \in Aut(\mathbb{D} \times \mathbb{C})$ is of the form

$$F(z_1, z_2) = (\psi(z_1), A(z_1)z_2 + B(z_1)),$$

where $\psi \in Aut(\mathbb{D}), A, B \in \mathcal{O}(\mathbb{D})$ and A is non-vanishing.

The novelty of our proof, from the viewpoint of function theory, lies in our use of the fact that the set of non-constant holomorphic maps between certain Riemann surfaces is at most finite. This pheneomenon is well understood in the realm of *compact* complex manifolds; see, for instance, [7, Chapters 6 & 7]. However, the factors X_j and Y_j in Theorem 1.5 are not necessarily compact. We will see that the main idea in the Remmert–Stein theorem (i.e. Result 1.1) is still useful in our more general setting. Loosely speaking, we show that, in general, the manifold $X_1 \times \cdots \times X_n$ splits into two factors, one of which is the product of those non-compact factors to which the Remmert–Stein method can be applied. The finiteness result that is essential to our proof is a result by Imayoshi [4]. This result, plus some other technical necessities are stated in Section 3. The proof of Theorem 1.5 is presented in Section 4.

2. A version of Montel's Theorem

In the proof of our main result, we need a version of Montel's theorem that is adapted to our situation. The proof of this version requires some general results about normal families. We state these results, with references, in this section. Throughout this section, M and N will denote complex manifolds, and $\mathcal{O}(M, N)$ will denote the space of holomorphic maps from M into N. We give $\mathcal{O}(M, N)$ the compact-open topology. We begin with the definition of a normal family.

Definition 2.1. A subset $\mathcal{F}$ of $\mathcal{O}(M, N)$ is said to be *normal* if every sequence of $\mathcal{F}$ contains a subsequence $\{f_n\}$ that is either convergent in $\mathcal{O}(M, N)$, or is compactly divergent. By the latter we mean that given compact sets $K \subseteq M$ and $H \subseteq N$, $f_n(K) \cap H = \varnothing$ for all sufficiently large n.

Result 2.2 (see [5], Proposition 3)**.** *Let M be a complex manifold, and let N be a complete Kobayashi hyperbolic complex manifold. Then $\mathcal{O}(M, N)$ is a normal family.*

Result 2.3 (see [6], Theorem 5.5)**.** *Let X be a hyperbolic Riemann surface. Then X is complete Kobayashi hyperbolic.*

We now state and prove the version of Montel's Theorem that we need, which is a corollary of the last two results.

Corollary 2.4. *Let X be a connected complex manifold and let R be a hyperbolic open connected subset of a compact Riemann surface S. Then, given any sequence $\{f_\nu\} \subset \mathcal{O}(X, R)$, there exists a subsequence $\{f_{\nu_k}\}$ and a holomorphic map $f_0 : X \to \overline{R}$ (the* *closure taken in S whenever R is non-compact) such that $f_{\nu_k} \to f_0$ uniformly on compact subsets of X.*

Proof. We begin by noting that if R is compact, then the result follows immediately from Results 2.3 and 2.2.

We now consider the case when R is a punctured Riemann surface. By Result 2.2, $\mathcal{O}(X, R)$ is a normal family. There is nothing to prove if there exists a subsequence $\{f_{\nu_k}\}$ that converges uniformly on compact subsets of X. Therefore, let us consider the case when we get only a compactly divergent subsequence $\{f_{\nu_k}\}$. Let $\{K_j : j \in \mathbb{Z}_+\}$ be an exhaustion of X by *connected* compact subsets, and let $\{L_j : j \in \mathbb{Z}_+\}$ be an exhaustion of R by compact subsets. Since R is obtained from S by deleting finitely many points from it, compact divergence implies that we can extract a further subsequence from $\{f_{\nu_k}\}$ — which we shall re-index again as $\{f_{\nu_k}\}$ — such that $f_{\nu_k}(K_1) \subset D^* \ \forall k$, where D^* is a deleted neighbourhood of one of the punctures, say p_0. Now, given any $j \in \mathbb{Z}_+$, there exists a $k(j) \in \mathbb{Z}_+$ such that, by the connectedness of the K_j's, we have:

$$f_{\nu_k}(K_j) \subset (D^* \setminus L_j) \ \ \forall k \geq k(j).$$

This just means that $f_{\nu_k} \to p_0$ uniformly on compacts as $k \to \infty$.

In the general case, as R is hyperbolic, we can make sufficiently many punctures in S to get a Riemann surface R' that is hyperbolic and $R \subseteq R' \subset S$. By considering each f_ν as a mapping in $\mathcal{O}(X, R')$, we can find, by the preceding argument, a subsequence $\{f_{\nu_k}\}$ and a holomorphic map $f_0 : X \to \overline{R'}$ such that $f_{\nu_k} \to f_0$ uniformly on compact subsets of X. As each $f_{\nu_k} \in \mathcal{O}(X, R)$, we must have $f_0 \in \mathcal{O}(X, \overline{R})$, and we are done. □

3. Some technical necessities

In this section we summarize several results that we need for the proof of Theorem 1.5. We state these results with appropriate references. We begin with an extension of a classical result due to de Franchis [3], which states that there are at most finitely many non-constant holomorphic mappings between two compact hyperbolic Riemann surfaces. We shall call a Riemann surface obtained by removing a finite, non-empty set of points from some compact Riemann surface a *punctured Riemann surface*. A Riemann surface obtained by removing n points from a compact Riemann surface of genus g will be called a *Riemann surface of finite type* (g, n). Imayoshi extended de Franchis' result as follows:

Result 3.1 (Imayoshi [4])**.** *Let R be a Riemann surface of finite type and let S be a Riemann surface of finite type (g, n) with $2g - 2 + n > 0$. Then the set of non-constant holomorphic maps from R into S is at most finite.*

The above result combined with the following lemma will play a key role in the proof of the main theorem. To state this lemma, we need a definition.

Definition 3.2. Let $F : X \to Y$ be a map between two sets, and suppose that $X = X_1 \times \cdots \times X_n$. We say that F is *independent of* X_j if, for each fixed $(x_1^0, \ldots, x_{j-1}^0, x_{j+1}^0, \ldots, x_n^0)$, $x_i^0 \in X_i$, the map

$$X_j \ni x_j \longmapsto F(x_1^0, \ldots, x_{j-1}^0, x_j, x_{j+1}^0, \ldots, x_n^0),$$

is a constant map. We say that F *varies along* X_j if F is **not** independent of X_j.

Lemma 3.3. *Let R and S be as in Result 3.1, and let X be a connected complex manifold. Let $F : R \times X \to S$ be a holomorphic mapping with the property that for some $x_0 \in X$, the mapping $R \ni z \mapsto F(z, x_0) \in S$ is a non-constant mapping. Then F is independent of X.*

Proof. Let d_R and d_S be metrics that induce the topology of R and S, respectively. By Result 3.1, the set of non-constant holomorphic mappings from R to S is at most finite. By our hypotheses, there is at least one such map. Let $F_1, \dots, F_k$ be the only distinct non-constant mappings in $\mathcal{O}(R, S)$. Let $x_0 \in X$ be such that the map $F(\cdot, x_0)$ is non-constant. By continuity of F, there is an X-open neighbourhood $U_0 \ni x_0$ such that $F(\cdot, x)$ is non-constant for $x \in U_0$. Choose $\varepsilon > 0$ and $r_{ij} \in R,\ 1 \le i, j \le k,\ i \neq j$, such that $d_S(F_i(r_{ij}), F_j(r_{ij})) > \varepsilon$. By the continuity of F, we can find a neighbourhood $U \subset U_0$ of x_0 such that, for each of the r_{ij}'s, we have $d_S(F(r_{ij}, x), F(r_{ij}, y)) < \varepsilon\ \forall x, y \in U$. This is possible only if $F(\cdot, x) \equiv F(\cdot, y),\ \forall x, y \in U$. It follows that

- $\exists j_0 \le k$ such that $F(\cdot, x) = F_{j_0}\ \forall x \in U$;
- For each fixed $r \in R$, the map $F(r, \cdot)$ is constant on U.

As X is connected, the Identity Theorem implies that $F(r, \cdot) \equiv F_{j_0}(r)$. This proves that F is independent of X. □

The next result is the well known Remmert's Proper Mapping Theorem. For the proof, refer to [2, p. 31].

Result 3.4 (Proper Mapping Theorem)**.** *Let X and Y be complex manifolds, and let A be an analytic subset of X. Let $f : A \to Y$ be a proper finite holomorphic map. Then, $f(A)$ is an analytic subset of Y, and at every $w \in f(A)$*

$$dim_w f(A) = \max\{dim_z A : f(z) = w\}.$$

In particular, $dim A = dim f(A)$. Furthermore, if $A = X$ and $dim(X) = dim(Y)$ then F is surjective.

The following result due to Kobayashi [7, p. 284] can be thought of as an higher dimensional analogue of the big Picard theorem. For this, we first need to make a definition.

Definition 3.5. Let Z be a complex manifold and let Y be a relatively compact complex submanifold of Z. We call a point $p \in \overline{Y}$ a *hyperbolic point* if every neighbourhood U of p contains a smaller neighbourhood V of p, $\overline{V} \subset U$, such that

$$K_Y(\overline{V} \cap Y, Y \setminus U) := \inf\{K_Y(x, y) : x \in \overline{V} \cap Y, y \in Y \setminus U\} > 0,$$

where K_Y denotes the Kobayashi pseudo-distance on Y. We say that Y is *hyperbolically imbedded* in Z if every point of $\overline{Y}$ is a hyperbolic point.

Result 3.6 (Kobayashi)**.** *Let Y and Z be complex manifolds, and let Y be hyperbolically imbedded in Z. Then every map $h \in \mathcal{O}(\mathbb{D}^*, Y)$ extends to a map $\tilde{h} \in \mathcal{O}(\mathbb{D}, Z)$.*

Lemma 3.7. *If Y is a hyperbolic open connected subset of a compact Riemann surface Z, then Y is hyperbolically imbedded in Z.*

Proof. The lemma is obvious if Y has only isolated boundary points. If not, then, as Y is hyperbolic, we can make sufficiently many punctures in Z to get a hyperbolic Riemann surface $\widetilde{Y}$ such that $Y \subset \widetilde{Y} \subset Z$. It follows that $\widetilde{Y}$ is hyperbolically imbedded in Z.

Now let $y \in \overline{Y}$. Then, $y \in \overline{\widetilde{Y}}$. Let U be a neighbourhood of y, and let V be a smaller neighbourhood of y such that

$$K_{\widetilde{Y}}(\overline{V} \cap \widetilde{Y}, \widetilde{Y} \setminus U) > 0.$$

As $K_Y \geq K_{\widetilde{Y}}$ on $Y \times Y$ and $Y \subseteq \widetilde{Y}$, it follows that y is also a hyperbolic point of Y. Consequently, Y is hyperbolically imbedded in Z. $\square$

We require one more result, a generalization of Rado's theorem that is proved in [8].

Lemma 3.8. *Let $(\phi_{\mu\nu})$, $1 \leq \mu \leq k$, $1 \leq \nu \leq l$, be a matrix of holomorphic functions on $D \subseteq U$, where D and U are connected open subsets of $\mathbb{C}$, and $U \setminus D$ is a non-empty indiscrete set. Suppose that*

$$\prod_{\nu=1}^{l} \sum_{\mu=1}^{k} |\phi_{\mu\nu}(z)|^2 \to 0 \text{ as } D \ni z \to \zeta$$

for any $\zeta \in \partial D \cap U$. Then, for some ν_0, $1 \leq \nu_0 \leq l$, we have

$$\phi_{\mu\nu_0} \equiv 0, \ \mu = 1, \ldots, k.$$

Proof. Suppose each column of $(\phi_{\mu\nu})$ has a member that is not identically 0 on D. Let f be the product of these members. We extend f to be a function on U by defining $f \equiv 0$ on $U \setminus D$. By hypothesis, f is continuous on U and holomorphic on D. Therefore by the classical Rado's theorem $f \equiv 0$, a contradiction. $\square$

4. Proof of the main theorem

We begin this section by considering a special case of Theorem 1.5 whose proof contains some technicalities. Since these technicalities would lengthen the proof of Theorem 1.5 if we were to embark on it directly, we shall isolate the technical portion of our proof in the following proposition. Its proof consists of rephrasing the Remmert–Stein argument relative to a coordinate patch; see [8, pp. 71–78]. *We shall therefore be brief* and explain in detail only those points that differ from the proof in [8].

Proposition 4.1. *Let $X = X_1 \times \cdots \times X_n$ and $Y = Y_1 \times \cdots \times Y_n$, $n \geq 2$, be complex manifolds. Assume that each X_j and each Y_j satisfy the hypothesis of Theorem 1.5 and that Y is non-compact. Further assume that, for each j, $R_j \setminus X_j$ is a non-empty indiscrete set. Let $F : X \to Y$ be a finite proper holomorphic map. Then, denoting $z \in X_1 \times \cdots \times X_n$ as $(z_1, \ldots, z_n)$, each F_i is of the form $F_i(z_{\pi(i)})$, where π is a permutation of $\{1, \ldots, n\}$.*

In particular, if there is a mapping with the above properties from X to Y, then Y cannot have any compact factors.

Proof. For $1 \leq j \leq n$, let R_j and S_j be as in Theorem 1.5. Let $p = (p_1, \ldots, p_n)$ be a point in $R_1 \times \cdots \times R_n$ such that, for each $1 \leq i \leq n$, p_i is a limit point of the set $R_i \setminus X_i$ and belongs to ∂X_i.

Let (U_k, ψ_k) be connected holomorphic co-ordinate charts of R_k, chosen in such a way that $p_i \in U_i$ and the image of $(\prod_{k=1}^{n} U_k) \cap X$ under each F_j lies in some holomorphic co-ordinate chart (V_j, ρ_j) of S_j. Let W_i be a connected component of $U_i \cap X_i$ such that $p_i \in \partial W_i$. For $(z_1, \ldots, z_n) \in \prod_{k=1}^{n} \psi_k(W_k)$, let

$$g_j(z_1, \ldots, z_n) := \rho_j \circ F_j(\psi_1^{-1}(z_1), \ldots, \psi_n^{-1}(z_n)).$$

In view of Corollary 2.4, we can rephrase the arguments in [8, p. 75] to conclude:

$$\prod_{j=1}^{n} \sum_{k=1, k\neq i}^{n} \left| \frac{\partial g_j}{\partial z_k}(z_1, \ldots, z_{i-1}, w, \ldots, z_n) \right|^2 \to 0 \ \text{ as } \ w \to \zeta \in \psi_i(\partial W_i),$$

where ζ is any arbitrary point in $\psi_i(\partial W_i)$. Let us take $D = \psi_i(W_i)$ and $U = \psi_i(U_i)$ in Lemma 3.8. Note that $\psi_i(p_i) \in U \setminus D$, whence $U \setminus D$ is indiscrete. Thus, we have that for each $(z_1, \ldots, \ z_{i-1}, \ z_{i+1}, \ldots, \ z_n) \in \prod_{k=1, k\neq i}^{n} \psi_k(W_k)$, there is a $j = j(z)$ such that

$$h_j(z_1, \ldots, z_{i-1}, w, z_{i+1}, \ldots, z_n) := \sum_{k=1, k\neq i}^{n} \left| \frac{\partial g_j}{\partial z_k}(z_1, \ldots, z_{i-1}, w, \ldots, z_n) \right|^2$$

is zero $\forall w \in D$. At this point, we can again argue exactly as in [8, p. 75] to conclude that there exists an integer $\sigma(i), 1 \leq \sigma(i) \leq n$, such that

$$\frac{\partial g_{\sigma(i)}}{\partial z_k} \equiv 0 \ \text{ on } \ \psi_1(W_1) \times \cdots \times \psi_n(W_n), \quad k = 1, \ldots, n, \quad k \neq i.$$

Therefore on $W_1 \times \cdots \times W_n$, $F_{\sigma(i)}$ is independent of $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$. By applying the Identity Theorem, we conclude that $F_{\sigma(i)}$ is independent of the same variables on X. By Remmert's Proper Mapping Theorem, F is surjective. This implies that $F_{\sigma(i)}$ varies along X_i. Since the choice of $1 \leq i \leq n$ in the preceding argument was arbitrary, for each i there exists precisely one $\sigma(i)$ such that $F_{\sigma(i)}(z) = F_{\sigma(i)}(z_i)$ $\forall z \in X$. The permutation $\pi = \sigma^{-1}$, and we are done with the proof of the first part.

To establish the final part of this result, assume that $Y_{s+1}, \ldots, Y_n$ are all compact, for some $s < n$. Fix an i as in the previous paragraph. The heart of the argument above, see [8, p. 75], consists of using Montel's theorem (Corollary 2.4 in our present set-up) to construct a map $(\phi_1, \ldots, \phi_n) : Z \to \partial Y$, where $Z := \prod_{k=1, k\neq i}^{n} X_k$. Set $E_j := \{z \in Z : \phi_j(z) \in \partial Y_j\}$. Clearly :

$$\{l : 1 \leq l \leq n, \ \mathrm{int}(E_l) \neq \varnothing\} \subseteq \{1, \ldots, s\}. \tag{4.1}$$

In view of (4.1), the argument in [8, p. 75] reveals that, for each $i, \sigma(i) \in \{1, \ldots, s\}$. Since $s < n$, by assumption, there would exist $i \neq i'$ such that $\sigma(i) = \sigma(i')$. But this would contradict the surjectivity of F, and we are done.

□

4.2. The proof of Theorem 1.5. For $1 \leq j \leq n$, let R_j and S_j be the compact Riemann surfaces in the statement of the theorem. We start off with a simple consequence of the finiteness of F.

Claim A: *For any holomorphic finite map $F : X \to Y$, given any $X_i, 1 \leq i \leq n$, there is some F_j that varies along X_i.* To see this, assume that there is a factor X_i such that all the F_j's are independent of X_i. Then for any point $x = (x_1, \ldots, x_n) \in X$, by Definition 3.2, the inverse image of $F(x)$ contains the set $\{x_1\} \times \cdots \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \{x_n\}$. But this contradicts the finiteness of F.

Let X_C and Y_C denote the product of those factors of X and Y, respectively, that are either compact, or compact with finitely many punctures, and let X_B and Y_B denote the product of the remaining factors. Since Proposition 4.1 already establishes our theorem if $X_C = \varnothing$, we may assume, without loss of generality, that $X_C := X_1 \times \ldots \times X_p$, $1 \leq p \leq n$. Note that if $X_C = \varnothing$ and there exists a proper holomorphic map $F : X \to Y$, then Y cannot be compact.

Claim B: *The maps F_i are independent of $X_1, \dots, X_p$, whenever $S_i \setminus Y_i$ is a non-empty indiscrete set.* To see this, fix $x' \in X_2 \times \cdots \times X_n$. The map $F_i(\bullet, x')$ is a holomorphic map from X_1 into a hyperbolically imbedded Riemann surface. Now, R_1 is the compact Riemann surface from which X_1 is obtained by deleting at most finitely many points. From Result 3.6, it follows that $F_i(\bullet, x')$ extends holomorphically to a map $\widetilde{f_i}$ from R_1 into S_i. If $\widetilde{f_i}$ is non-constant, then by the compactness of R_1, it follows that the image set of R_1 under $\widetilde{f_i}$ is both compact and open. But, this means that $S_i = \widetilde{f_i}(R_1)$, which is not possible as $S_i \setminus Y_i$ is a non-empty indiscrete set, and $\widetilde{f_i}(R_1)$ is obtained by adjoining at most finitely many points to Y_i. This proves that F_i is independent of X_1. Repeating the same argument for the factors $X_2, \dots, X_p$, the claim is proved.

Now note that, in view of Claim B, if $1 \leq i \leq p$ and F_j is a map that varies along X_i, then Y_j is either compact, or compact with finitely many punctures. Then, by Lemma 3.3, F_j is independent of *all the factors of X other than X_i*. Without loss of generality, we may assume that $Y_C = Y_1 \times \dots Y_k,\ 1 \leq k \leq n$. Combining our last deduction with Claim A, we infer that:

(1) $p \leq k \leq n$;
(2) Without loss of generality, there is an enumeration of the factors of Y_C such that for each $1 \leq i \leq p$, there is a unique $\sigma(i)$, $1 \leq \sigma(i) \leq p$, such that $F_{\sigma(i)}(z) = F_{\sigma(i)}(z_i)\ \forall z \in X$.

Suppose $k > p$. Then, in view of the (harmless) assumption in (2), we need to analyse the behaviour of $F_i,\ p+1 \leq i \leq k$. Note that we already know from Claim B that $F_{k+1}, \dots, F_n$ is independent of X_C. Assume that F_{p+1} varies along some $X_i, 1 \leq i \leq p$; then from Lemma 3.3, F_{p+1} is independent of all other factors of X. From Remmert's Proper Mapping Theorem (Result 3.4), F is a surjective map from X onto Y. Hence, combining the last two assertions with (2), $(F_1, \dots, F_{p+1})$ determines a surjective holomorphic map $(F_1, \dots, F_{p+1}) : X_C \to Y_1 \times \cdots \times Y_{p+1}$ from a space of dimension p to a space of dimension $p+1$, which contradicts Sard's theorem. Hence, F_{p+1} is independent of $X_1, \dots, X_p$. Repeating the same argument for each map $F_j,\ p+1 \leq j \leq k$, we conclude that each $F_j,\ p+1 \leq j \leq n$, is independent of X_C.

Whether or not $k > p$, the previous paragraph implies that $F_i,\ p+1 \leq i \leq n$, are independent of X_C, whence they determine a surjective map $F_B = (F_{p+1}, \dots, F_n) : X_B \to Y_{p+1} \times \dots \times Y_n$. This map is clearly finite. We will now show that it is proper. Consider a compact set $K \subseteq Y_{p+1} \times \dots \times Y_n$. We must show that $F_B^{-1}(K)$ is a compact subset of X_B. Let $H \subseteq Y_1 \times \cdots \times Y_p$ be some compact set. Then, by the properness of F, it follows that $F^{-1}(H \times K)$ is compact. But, given the independence of the various F_i's from certain factors of X,

$$F^{-1}(H \times K) = (F_1, \dots, F_p)^{-1}(H) \times F_B^{-1}(K).$$

Thus, $F_B^{-1}(K)$ is compact, as required.

As X_B is non-compact, and F_B is a proper map, it follows that $Y_{p+1} \times \cdots \times Y_n$ is also non-compact. We now apply Proposition 4.1 to the map F_B to get a permutation π of $\{p+1, \dots, n\}$ such that, for each $p < i \leq n$, we have $F_i(z) = F_i(z_{\pi(i)})$. Juxtaposing π with the permutation σ^{-1} of $\{1, \dots, p\}$, we are done. $\square$

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Department of Mathematics, Indian Institute of Science, Bangalore – 560012
E-mail address: jaikrishnan@math.iisc.ernet.in