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REPRESENTATIONS OF A CLASS OF REAL B^*-ALGEBRAS AS ALGEBRAS OF QUATERNION-VALUED FUNCTIONS

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ABSTRACT. For a compact Hausdorff space X, let $C(X, \mathbf{H})$ denote the set of all quaternion-valued functions on X. It is proved that if a real B^*-algebra A satisfies the following conditions: (i) the spectrum of every selfadjoint element is contained in the real line and (ii) every element in A is normal, then A is isometrically $*$-isomorphic to a closed $*$-subalgebra of $C(X, \mathbf{H})$ for some compact Hausdorff X. In particular, a real C^*-algebra in which every element is normal is isometrically $*$-isomorphic to a closed $*$-subalgebra of $C(X, \mathbf{H})$.

INTRODUCTION

Let A be a real or complex normed algebra with an involution $*$. Obviously, if A is commutative then every element in A is normal. The converse is also true for complex algebras, because the normality of $a = h + ik$ with h, k selfadjoint implies that h and k commute. Since every element in A can be expressed uniquely in the form $h+ik$, with h and k selfadjoint, the algebra A is commutative. However, this is not true for real algebras. The algebra $\mathbf{H}$ of all real quaternions with the usual involution is noncommutative, though every element in $\mathbf{H}$ is normal. The aim of this paper is to show that, under certain assumptions, the algebras of $\mathbf{H}$-valued functions are essentially the only real algebras in which every element is normal. Viswanath [9] has shown that such algebras arise naturally in the study of normal operators on real Hilbert spaces.

PRELIMINARIES

We denote the set of all real numbers by $\mathbf{R}$, the set of all complex numbers by $\mathbf{C}$, and the set of all real quaternions by $\mathbf{H}$. Let A be a real algebra with a unit element 1 and a an element of A. We adopt Kaplansky's definition of the *spectrum of a in A* denoted by $\mathrm{Sp}(a, A)$ (or simply by $\mathrm{Sp}(a)$ when no confusion is likely).

$\mathrm{Sp}(a, A) := \{s + it \in \mathbf{C}\colon (s - a)^2 + t^2 \text{ is singular in } A\}$.

If A is a Banach algebra, then the *spectral radius* of an *element a* in A is

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given by the *spectral radius formula*

$$r(a) := \lim_{n\to\infty} \|a^n\|^{1/n} = \sup\{(s^2+t^2)^{1/2} : s+it \in \mathrm{Sp}(a, A)\}.$$

(See [2, 4] for a proof.)

Our proof of the main result in this note depends on the following theorem proved in [6].

Theorem 1 [6, Theorem 11]. *Let A be a real Banach algebra with a unit 1 and suppose $\|a\| \le \alpha r(a)$ for every a in A and some constant $\alpha \ge 0$. Let $c \in A$ be such that $\mathrm{Sp}(c) \subseteq \mathbf{R}$ and b any element of A. Then $cb = bc$, that is, the set of all elements with the real spectra lies in the centre of A.*

As usual, an *involution on a real algebra A* is a map $a \to a^*$ such that for all a, b in A and s in $\mathbf{R}$: (i) $(a+b)^* = a^* + b^*$, (ii) $(sa)^* = sa^*$, (iii) $(ab)^* = b^*a^*$, and (iv) $(a^*)^* = a$. An algebra A with an involution $*$ is called a *$*$-algebra.* It is called an *auto $*$-algebra* if condition (iii) is replaced by (iii)$'$ $(ab)^* = a^*b^*$. A is called a *generalized $*$-algebra* if it is a $*$-algebra or an auto $*$-algebra (cf. [4, 7, 8]). In the sequel, we shall use many results of [4]. These results were claimed to have been proved for a generalized $*$-algebra. Magyar [7] has pointed out that the proof of Theorem 2.3 in [4] does not work for an auto $*$-algebra and has supplied a proof that works. This does not affect the results in the present paper, as we shall deal only with a $*$-algebra. Some related topics are also discussed in [7] and [8].

Let A be a real $*$-algebra. A subalgebra that is closed under the involution $*$ is called a $*$-subalgebra. An element a is called *selfadjoint* if $a^* = a$, *skew* if $a^* = -a$, and *normal* if $a^*a = aa^*$.

Let $\mathrm{Sym}(A) := \{a \in A : a^* = a\}$ and $\mathrm{Skew}(A) := \{a \in A : a^* = -a\}$. A B^*-algebra is a Banach algebra A with an involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every a in A.

For $q = q_0 + q_1 i + q_2 j + q_3 k$ in $\mathbf{H}$, q^* is defined as $q^* = q_0 - q_1 i - q_2 j - q_3 k$ and $|q|$ is defined as

$$|q| := (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}.$$

Note that $|q|^2 = q^*q = qq^*$. $q_0 = (q+q^*)/2$ is called the real part of q denoted by $\mathrm{Re}(q)$. $\mathbf{H}$ is a real B^*-algebra. For a compact Hausdorff space X and a normed linear space E, $C(X, E)$ denotes the set of all continuous E-valued functions on X. For f in $C(X, E)$, let

$$\|f\| := \sup\{\|f(x)\| : x \in X\}.$$

For f in $C(X, \mathbf{H})$, let $f^*(x) := (f(x))^*$ for all x in X. Then $*$ is an involution on $C(X, \mathbf{H})$, and $\|\ \|$ is a norm on $C(X, \mathbf{H})$ making it a real B^* algebra.

In the remaining part of this note A is a real B^*-algebra with a unit 1, satisfying the following conditions:

(I) $\mathrm{Sp}(a) \subseteq \mathbf{R}$ for every a in $\mathrm{Sym}(A)$,
(II) Every element in A is normal.

B^* condition implies that $\mathrm{Sp}(a)$ contains no nonzero real number if $a^* = -a$ [4, Theorem 2.4]. This along with (I) yields that $\mathrm{Sp}(a^*a) \subseteq [0, \infty)$ for every a in A by Theorem 2.3 of [4]. (See also [5].)

Note that $\mathbf{H}$ and $C(X, \mathbf{H})$ satisfy (I) and (II). So does any $*$-subalgebra of $C(X, \mathbf{H})$. It will be proved that A is isometrically $*$ isomorphic to a closed $*$-subalgebra of $C(X, \mathbf{H})$ for some compact Hausdorff space X.

Lemma 2. (i) $\|a\| = r(a)$ *for all* a *in* A *and* $\mathrm{Sym}(A)$ *is a commutative real Banach algebra.*

(ii) *For every nonzero homomorphism* φ *of* $\mathrm{Sym}(A)$ *to* $\mathbf{R}$, *there exists a nonzero homomorphism* π *of* A *into* $\mathbf{H}$ *such that* $\pi(a) = \varphi(a)$ *for every* a *in* $\mathrm{Sym}(A)$.

Proof. (i) First B^* condition implies that $\|h^2\| = \|h\|^2$ for every h in $\mathrm{Sym}(A)$. Since $a^*a \in \mathrm{Sym}(A)$ for each a in A, we can use (II) to obtain

$$\begin{aligned}(\|a\|^2)^2 = \|a^*a\|^2 = \|(a^*a)^2\| = \|a^*(aa^*)a\| = \|a^*a^*aa\| \\ = \|(a^2)^*a^2\| = \|a^2\|^2,\end{aligned}$$

so that $\|a\|^2 = \|a^2\|$ for each a in A. Next, by induction,

$$\|a\|^{2^n} = \|a^{2^n}\| \quad \text{for } n = 1, 2, 3, \ldots.$$

By taking the 2^nth root and applying the Spectral radius formula, we get, $\|a\| = r(a)$ for every a in A.

Now (I) and Theorem 1 imply that every selfadjoint element lies in the centre of A. In particular, $\mathrm{Sym}(A)$ is a real commutative Banach algebra with 1.

(ii) In view of (i), there exists a nonzero homomorphism φ of $\mathrm{Sym}(A)$ into $\mathbf{R}$. (In fact, $\mathrm{Sym}(A)$ is isometrically isomorphic to $C(Y, \mathbf{R})$ for some compact Hausdorff space y by a theorem of Arens [2–4].) Let φ be such a homomorphism. We have $\varphi(1) = 1$ and

$$\|\varphi\| := \sup\{|\varphi(a)| : a \in \mathrm{Sym}(A), \|a\| \le 1\} = 1.$$

Also since $\mathrm{Sp}(a^*a) \subseteq [0, \infty)$, we have, for every a in A, $\varepsilon + a^*a$ is invertible in A for every $\varepsilon > 0$. Clearly $(\varepsilon + a^*a)^{-1} \in \mathrm{Sym}(A)$. Thus $\varphi(a^*a+\varepsilon) \ne 0$; that is, $\varphi(a^*a) \ne -\varepsilon$ for every $\varepsilon > 0$. Hence $\varphi(a^*a) \ge 0$. We define $\psi: A \to \mathbf{R}$ as $\psi(a) := \varphi((a + a^*)/2)$ for all a in A. It is easy to see that ψ is a continuous liear functional on A, $\psi(1) = 1 = \|\psi\|$, $\psi(a^*) = \psi(a)$ for every a in A and $\psi(a^*a) = \varphi(a^*a) \ge 0$ for every a in A. In other words ψ is a normalised real state on A. Hence by Proposition 14.3 of [2], $\psi(b^*a)^2 \le \psi(a^*a)\psi(b^*b)$ for every a, b in A. Hence

$$\begin{aligned}\psi(ab)^2 &= \psi((a^*)^*b)^2 \le \psi(aa^*)\psi(b^*b) \\ &= \psi(a^*a)\psi(b^*b) \quad \text{by (II)}.\end{aligned} \tag{1}$$

Now let $N_\psi = \{a \in A : \psi(a^*a) = \varphi(a^*a) = 0\}$. The inequality (1) above implies that N_ψ is a *two-sided* ideal in A. It is closed as ψ is continuous. Hence $M_\psi := A/N_\psi$ is a Banach algebra (with the quotient norm) with the unit $1 + N_\psi$.

Claim. M_ψ is a division algebra. Suppose for some b in A, $b + N_\psi \ne N_\psi$. This means $\psi(b^*b) \ne 0$. Let $c := b^*/\psi(b^*b)$ and $h := cb - 1$. Then $h \in \mathrm{Sym}(A)$ and $\psi(h) = 0 = \varphi(h)$. Hence $\psi(h^*h) = \psi(h^2) = \varphi(h^2) = 0$ as φ is a homomorphism on $\mathrm{Sym}(A)$. Hence $h \in N_\psi$. Then $(c + N_\psi)(b + N_\psi) = 1 + N_\psi$. Similarly, using normality of b, we can prove $(bc - 1) \in N_\psi$; that is, $(b + N_\psi)(c + N_\psi) = 1 + N_\psi$. This proves the claim.

Now by a theorem of Mazur and Arens [2, Theorem 9.7; 1, Theorem 14.7], there is an isomorphism θ of M_ψ onto $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Then $\pi: A \to \mathbf{H}$ defined as $\pi(a) := \theta(a + N_\psi)$ is a homomorphism. It is nonzero because $\pi(1) = \theta(1 + N_\psi) = 1$.

Now suppose $a \in \mathrm{Sym}(A)$ and $\varphi(a) = 0$. Then

$$\psi(a^*a) = \varphi(a^*a) = \varphi(a^2) = (\varphi(a))^2 = 0.$$

Hence $a \in N_\psi$ and $\pi(a) = 0$. Further, if $b \in \mathrm{Sym}(A)$ with $\varphi(b) \neq 0$, then we consider $a = 1 - b/\varphi(b)$. Clearly, $\varphi(a) = 0$. Hence, by what we have proved just now, $\pi(a) = 0$; that is, $\varphi(b) = \pi(b)$. Thus for all c in $\mathrm{Sym}(A)$, $\varphi(c) = \pi(c)$. □

Throughout this paper, by a "homomorphism," we mean a morphism of real algebras. It is worth mentioning here that if φ is a nonzero morphism of rings from $\mathbf{R}$ into itself then φ is the identity map on $\mathbf{R}$. Hence if π is a nonzero morphism of rings from A into $\mathbf{H}$ such that π takes the (real) scalar multiples of 1 to reals, then π is a morphism of real algebras, that is, a homomorphism in our sense. Now let X be the set of all such nonzero homomorphisms of A into $\mathbf{H}$. X is nonempty by Lemma 2. For a in A, we define a map $\hat{a}: X \to \mathbf{H}$ by $\hat{a}(\pi) = \pi(a)$ for all π in X. Let $\widehat{A} := \{\hat{a}: a \in A\}$. We give X the weak $\widehat{A}$ topology (that is, the weakest topology on X making $\hat{a}$ continuous for each a in A). Now we are in a position to prove the main theorem.

Theorem 3. (i) *For each π in X,*

$$\|\pi\| := \sup\{|\pi(a)|: a \in A, \|a\| \leq 1\} = 1.$$

(ii) *X is a compact Hausdorff space (with respect to the weak $\widehat{A}$ topology).*
(iii) *For each π in X and a in* $\mathrm{Sym}(A)$, *$\pi(a)$ is real.*
(iv) *For each π in X and a in* $\mathrm{Skew}(A)$, $\mathrm{Re}(\pi(a)) = 0$.
(v) *For each π in X and a in A, $\pi(a^*) = (\pi(a))^*$.*
(vi) *$\widehat{A}$ is a closed $*$-subalgebra of $C(X, \mathbf{H})$ and the map $a \to \hat{a}$ is an isometric $*$-isomorphism of A onto $\widehat{A}$.*

Proof. (i) We have already noted in Lemma 2 that $r(a) = \|a\|$ for all a in A. Since H also satisfies the conditions assumed of A, $r(q) = |q|$ for all q in $\mathbf{H}$. Further, if $(a - s)^2 + t^2$ is invertible for a in A and $s + it$ in $\mathbf{C}$, then for a homomorphism π in X, $\pi((a-s)^2 + t^2) = (\pi(a) - s)^2 + t^2$ is invertible. Hence $\mathrm{Sp}(\pi(a)) \subseteq \mathrm{Sp}(a)$. This shows that $|\pi(a)| = r(\pi(a)) \leq r(a) = \|a\|$. Thus $\|\pi\| \leq 1$. Since $\pi(1) = 1$, we have $\|\pi\| = 1$.

(ii) Let π_1, π_2 be distinct homomorphisms in X. Then $\pi_1(a) \neq \pi_2(a)$ for some a in A. Hence we can find disjoint open sets G_1 and G_2 containing $\pi_1(a)$ and $\pi_2(a)$, respectively. Then the inverse images of G_1 and G_2 under $\hat{a}$ are disjoint open sets (in the weak $\widehat{A}$ topology) in X containing π_1 and π_2, respectively. This shows that X is Hausdorff. Next, we define $K_a := \{q \in \mathbf{H}: |q| \leq \|a\|\}$. Then K_a is compact in the topology of $\mathbf{H}$. Let K be the topological product of K_a for all a in A. Then K is compact by the Tychonoff theorem. Now let $\pi \in X$. Then, from (i), $\pi(a) \in K_a$ for each a in A. Thus $\pi \in K$. Hence X is a subset of K. Now it is straightforward to show that the relative topology on X is the same as the weak $\widehat{A}$ topology and that X is a closed subset of K. Hence X is compact.

(iii) Let $a \in \mathrm{Sym}(A)$, $\pi \in X$, and $\pi(a) = s + t$, where s is real and $t = t_1 i + t_2 j + t_3 k$. If $t \neq 0$, let $b := (a - s)/(2\|a - s\|) \neq 0$. Then $b \in \mathrm{Sym}(A)$ and $\|b\| < 1$. By Ford's square-root lemma [1, Proposition 12.11], there exists c in $\mathrm{Sym}(A)$ such that $1 - b^2 = c^2$. Thus,

$$|1 - \pi(b^2)| = |\pi(c^2)| \leq \|c\|^2 \quad \text{by (i).}$$

Since $\mathrm{Sym}(A)$ is isometrically isomorphic to $C(Y, \mathbf{R})$ for some compact Hausdorff space Y and $b, c \in \mathrm{Sym}(A)$, we have $\|c\|^2 \leq \|c^2 + b^2\| = \|1\| = 1$. Thus,

$$|1 - t^2/(4\|a - s\|^2)| = |1 - \pi(b^2)| \leq \|c\|^2 \leq 1.$$

This shows that $t^2 \geq 0$. But $t^2 = -(t_1^2 + t_2^2 + t_3^2) \leq 0$. Hence $t = 0$ and $\pi(a) = s$.

(iv) Let $a \in \mathrm{Skew}(A)$, $\pi \in X$, and $\pi(a) = s + t$, where s is real and $t = t_1 i + t_2 j + t_3 k$. We shall show that $s = 0$. Consider $b = a + \alpha$ for α in $\mathbf{R}$. Then $b^* = -a + \alpha$ and

$$\begin{aligned}(s + \alpha)^2 + t_1^2 + t_2^2 + t_3^2 = |\pi(b)|^2 \leq \|b\|^2 \quad \text{by (i)} \\ = \|b^* b\| = \|\alpha^2 - a^2\| \leq \alpha^2 + \|a\|^2.\end{aligned}$$

Since this is true for every real α, we must have $s = 0$.

(v) Let $a \in A$. Then $a = b + c$, where $b = (a + a^*)/2 \in \mathrm{Sym}(A)$ and $c = (a - a^*)/2 \in \mathrm{Skew}(A)$. Hence for every π in X, $\pi(b)$ is real by (iii) and $\mathrm{Re}\,\pi(c) = 0$ from (iv). Hence $(\pi(b))^* = \pi(b)$ and $(\pi(c))^* = -\pi(c)$. Thus

$$\begin{aligned}\pi(a^*) = \pi(b - c) = \pi(b) - \pi(c) = (\pi(b))^* + (\pi(c))^* \\ = (\pi(b + c))^* = (\pi(a))^*.\end{aligned}$$

(vi) It is obvious that $\widehat{A}$ is a subalgebra of $C(X, \mathbf{H})$ and the map $a \to \hat{a}$ is a homomorphism. It follows from (v) that $(a^*)^\wedge = (\hat{a})^*$ for each a in A. Thus $\widehat{A}$ is a $*$-subalgebra and the map $a \to \hat{a}$ is a $*$-homomorphism. Further,

$$\begin{aligned}\|\hat{a}\| := \sup\{|\hat{a}(\pi)| : \pi \in X\} \\ = \sup\{|\pi(a)| : \pi \in X\} \leq \|a\| \quad \text{by (i).}\end{aligned}$$

For every a in A, $a^* a \in \mathrm{Sym}(A)$. Since $\mathrm{Sym}(A)$ is isometrically isomorphic to $C(Y, \mathbf{R})$ for some compact Hausdorff space Y, there exists a nonzero homomorphism φ of $\mathrm{Sym}(A)$ into $\mathbf{R}$ such that $|\varphi(a^* a)| = \|a^* a\|$. Now by Lemma 2, there exists π in X such that $\pi = \varphi$ on $\mathrm{Sym}(A)$. Thus,

$$\begin{aligned}\|a\|^2 = \|a^* a\| = |\varphi(a^* a)| = |\pi(a^* a)| = |\pi(a^*)\pi(a)| \\ = |(\pi(a))^* \pi(a)| = |\pi(a)|^2,\end{aligned}$$

that is, $\|a\| = |\hat{a}(\pi)|$. Hence $\|\hat{a}\| = \|a\|$ for every a in A. This shows that the map $a \to \hat{a}$ is an isometry from A to $\widehat{A}$. In particular, it is 1-1, and hence an isometric $*$-isomorphism. This also implies that $\widehat{A}$ is complete and hence closed in $C(X, \mathbf{H})$. □

Remark 4. The presence of a unit in A is not essential in Theorem 3. If A does not have a unit then the spectrum of an element a in A is defined as

$$\begin{aligned}\mathrm{Sp}(a, A) = \{0\} \cup \{s + it \in \mathbf{C}\backslash\{0\} : (2sa - a^2)/(s^2 + t^2) \\ \text{is quasi-singular in } A\}.\end{aligned}$$

Suppose A is a real B^*-algebra without unit and let $a \to L_a$ be the left regular representation of A on A.

Let $B = \{L_a + sI : a \in A, s \in \mathbf{R}\}$, where I denotes the identity operator. We define an involution on B by

$$(L_a + sI)^* = L_{a^*} + sI \quad \text{for all } a \in A, s \in \mathbf{R}.$$

Yood has shown that B is a B^*-algebra with a unit I and the map $a \to L_a$ is an isometric $*$-monomorphism of A into B (see [1, p. 67]). Further, it is straightforward to check that if A satisfies the conditions (I) and (II), then so does B.

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