

SOME REGULARITY THEOREMS FOR CR MAPPINGS

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ABSTRACT. The purpose of this article is to study Lipschitz CR mappings from an h -extendible (or semi-regular) hypersurface in \mathbb{C}^n . Under various assumptions on the target hypersurface, it is shown that such mappings must be smooth. A rigidity result for proper holomorphic mappings from strongly pseudoconvex domains is also proved.

1. INTRODUCTION

The purpose of this note is to prove regularity results for Lipschitz CR mappings from h -extendible hypersurfaces in \mathbb{C}^n .

Theorem 1.1. *Let $f : M \rightarrow M'$ be a non-constant Lipschitz CR mapping between C^∞ smooth pseudoconvex finite type (in the sense of D'Angelo) hypersurfaces in \mathbb{C}^n . Let $p \in M$ and $p' = f(p) \in M'$. Assume that M is h -extendible at p , and that there is an open neighbourhood $U' \subset \mathbb{C}^n$ of p' and a C^∞ smooth defining function $r'(z')$ for $U' \cap M'$ in U' which is plurisubharmonic on the pseudoconvex side of M' near p' . If the Levi rank of M' at p' is at least $n - 2$ then f is C^∞ smooth in a neighbourhood of p .*

This theorem, which is purely local in nature, is motivated by the regularity and rigidity results for CR mappings obtained in [20], [21], [22] and [32]. Let $B(p, \epsilon), B(p', \epsilon')$ be small open balls around p, p' respectively. The pseudoconvex side of M near p is that component of $B(p, \epsilon) \setminus M$ which is a pseudoconvex domain for small $\epsilon > 0$. Denote this by $B^-(p, \epsilon)$ while the other component (the pseudoconcave part) will be denoted by $B^+(p, \epsilon)$. The same practice will be followed while referring to the respective components of the complement of M' in $B(p', \epsilon')$. Since M and M' are both pseudoconvex and of finite type near p and p' respectively, it follows that f admits a holomorphic extension to the pseudoconvex side of M near p and this extension (which will still be denoted by f) maps the pseudoconvex side of M near p into the pseudoconvex side of M' near p' . In fact, since $f : M \rightarrow M'$ is assumed to be Lipschitz, it follows that the extension is also Lipschitz on $B^-(p, \epsilon) \cup M$ near p . Fix $\epsilon, \epsilon' > 0$ so that

$$f : B^-(p, \epsilon) \rightarrow B^-(p', \epsilon')$$

is holomorphic, f extends Lipschitz continuously up to $M \cap B(p, \epsilon)$ and satisfies $f(M \cap B(p, \epsilon)) \subset M' \cap B(p', \epsilon')$. For brevity, let D and D' denote the pseudoconvex domains $B^-(p, \epsilon)$ and $B^-(p', \epsilon')$ respectively so that $M \cap B(p, \epsilon)$ and $M' \cap B(p', \epsilon')$ are smooth open finite type pieces on their boundaries. We may also assume that $\epsilon' > 0$ is small enough so that the defining function $r'(z') \in C^\infty(B(p', \epsilon'))$ and $D' = \{r'(z') < 0\}$ where $r'(z')$ is plurisubharmonic in D' . Note that while this condition holds for strongly pseudoconvex

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and convex finite type domains in \mathbb{C}^n , not all (see [24]) pseudoconvex domains satisfy this condition. Now observe that the map $f : D \rightarrow D'$ is not known to be proper though it does extend continuously up to $M \cap B(p, \epsilon)$. A sufficient condition for f to be proper from D onto D' was considered by Bell-Catlin in [9] – namely, if $f^{-1}(p') = f^{-1}(f(p))$ is compact in $M \cap B(p, \epsilon)$ then it is possible to choose D, D' in such a way that f extends as a proper holomorphic mapping between these domains. Once this is established, the C^∞ smoothness of f is a consequence of the techniques in [9]. The proof of theorem 1.1 therefore consists of showing that the fibre over p' is compact in $M \cap B(p, \epsilon)$. To do this, we adapt the method of uniform scaling from [20],[21] and [22]. The domains D, D' and the map f are scaled by composing f with a suitable family of automorphisms of \mathbb{C}^n that enlarge these domains near p, p' respectively. This produces a scaled family $f^\nu : D_\nu \rightarrow D'_\nu$ of maps between scaled domains. The scaled domains D_ν converge in the Hausdorff sense to a model domain of the form

$$(1.1) \quad D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P('z, '\bar{z}) < 0\}$$

where $'z = (z_1, z_2, \dots, z_{n-1})$ and $P('z, '\bar{z})$ is a weighted homogeneous (the weights being determined by the Catlin multitype of ∂D at p) real valued plurisubharmonic polynomial without pluriharmonic terms of total weight 1, while the domains D'_ν converge to

$$(1.2) \quad D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\}.$$

Here $2m'$ is the type of $\partial D'$ at p' and $Q_{2m'}(z_1, \bar{z}_1)$ is a subharmonic polynomial of degree at most $2m'$ without harmonic terms. D'_∞ is manifestly of finite type while D_∞ also has the same property since p is assumed to be h -extendible. The local geometry of a smooth pseudoconvex finite type hypersurface whose Levi rank is at least $n - 2$ guarantees the existence of special polydiscs around points sufficiently close to the hypersurface on which the defining function does not change by more than a prescribed amount – these are the analogues of Catlin's bidiscs and have been considered earlier in [18]. The holomorphic mappings $f_\nu : D_\nu \rightarrow D'_\nu$ are shown to form a normal family by using these polydiscs as in [43]; the limit map $F : D_\infty \rightarrow D'_\infty$ is therefore holomorphic. The assumptions that f is Lipschitz and that the defining function of $\partial D'$ is plurisubharmonic on D' force F to be non-degenerate. Furthermore, since f extends Lipschitz continuously up to M near p , it is natural to expect that F imbibes some regularity near $0 \in \partial D_\infty$. This indeed happens and it is possible to show that F is Hölder continuous up to ∂D_∞ near the origin. The main ingredient needed to do this is a stable rate of blow up of the Kobayashi metric on D'_ν for all $\nu \gg 1$ and this follows by analyzing the behaviour of analytic discs in D'_ν whose centers lie close to the origin. In particular, theorem 2.3 provides a stable lower bound for the Kobayashi metric in D'_ν near the origin using ideas from [43]. Once F is known to be Hölder continuous up to ∂D_∞ near the origin, Webster's theorem ([46]) implies that F must be algebraic. Moreover, if f has a noncompact fibre over p' , it can be shown that the same must hold for that of F over $F(0)$. This violates the invariance property of Segre varieties associated to ∂D_∞ and $\partial D'_\infty$ and proves the compactness of the fibre of f over $p' \in \partial D'$.

It is known that smooth convex finite type hypersurfaces can also be scaled and moreover the stability of the Kobayashi metric on the scaled domains is also understood (see [21]). Hence the same line of reasoning yields the following:

Corollary 1.2. *With M as in theorem 1.1, let $M' \subset \mathbb{C}^n$ be a smooth convex finite type hypersurface and $f : M \rightarrow M'$ a Lipschitz CR mapping. As before, let $p \in M$ and $p' = f(p) \in M'$. Then f is C^∞ smooth in a neighbourhood of p .*

More can be said when at least one of the hypersurfaces is strongly pseudoconvex. We first consider the following local situation – let $M \subset \mathbb{C}^n$ be a C^∞ smooth pseudoconvex finite type hypersurface and let $p \in M$ be an h -extendible point. Let the Catlin multitype of M at p be $(1, m_2, \dots, m_n)$ where the m_i 's form an increasing sequence of even integers. Then there exists a holomorphic coordinate system around $p = 0$ in which the defining function for M takes the form:

$$r(z) = 2\Re z_n + P('z, ' \bar{z}) + R(z)$$

where $P('z, ' \bar{z})$ is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ -homogeneous plurisubharmonic polynomial of total weight one without pluriharmonic terms and the error $R(z)$ has weight strictly bigger than one. As usual, if $J = (j_1, j_2, \dots, j_{n-1})$ is a multiindex of length $n - 1$, then $'z^J$ denotes the monomial $z_1^{j_1} z_2^{j_2} \dots z_{n-1}^{j_{n-1}}$ and $'\bar{z}^J = \bar{z}_1^{j_1} \bar{z}_2^{j_2} \dots \bar{z}_{n-1}^{j_{n-1}}$.

Theorem 1.3. *Let $f : M \rightarrow M'$ be a nonconstant Lipschitz CR mapping between real hypersurfaces in \mathbb{C}^n . Fix $p \in M$ and $p' = f(p) \in M'$. Suppose that M is C^∞ smooth pseudoconvex and of finite type near p and that M' is C^2 strongly pseudoconvex near p' . If M is h -extendible at p , then the weighted homogeneous polynomial in the defining function for M near $p = 0$ can be expressed as*

$$P('z, ' \bar{z}) = |P_1('z)|^2 + |P_2('z)|^2 + \dots + |P_{n-1}('z)|^2$$

where each $P_i('z)$ for $1 \leq i \leq n - 1$ is a weighted holomorphic polynomial of total weight $1/2$. Moreover, the algebraic variety

$$V = \{ 'z \in \mathbb{C}^{n-1} : P_1('z) = P_2('z) = \dots = P_{n-1}('z) = 0 \}$$

contains $'0 \in \mathbb{C}^{n-1}$ as an isolated point. In particular, there exist constants $c_j > 0$ for $1 \leq j \leq n - 1$ such that

$$P('z, ' \bar{z}) = c_1 |z_1|^{m_n} + c_2 |z_2|^{m_{n-1}} + \dots + c_{n-1} |z_{n-1}|^{m_2} + \text{mixed terms}$$

where the phrase ‘mixed terms’ denotes a sum of weight one monomials annihilated by at least one of the natural quotient maps $\mathbb{C}['z, ' \bar{z}] \rightarrow \mathbb{C}['z, ' \bar{z}] / (z_j \bar{z}_k)$ for $1 \leq j, k \leq n - 1$, $j \neq k$.

As in theorem 1.1, the limit map $F : D_\infty \rightarrow D'_\infty$ is holomorphic, nonconstant and hence algebraic. Since $D'_\infty \simeq \mathbb{B}^n$, it is possible to show that F extends holomorphically across $0 \in \partial D_\infty$ and $F(0) = 0' \in \partial D'_\infty$. The explicit description of the weighted homogeneous polynomial $P('z, ' \bar{z})$ follows from working with the extended mapping near the origin. On the other hand, it is also natural to consider the case of CR mappings from strongly pseudoconvex hypersurfaces and we have both a global and a local version – in the spirit of theorem 1.1, for this case. First recall that a domain $D \subset \mathbb{C}^n$ is said to be regular at $p \in \partial D$ if there is a pair of open neighbourhoods $V \subset U$ of p , constants $M > 0$, $0 < \alpha \leq 1$ and $\beta > 1$ such that for any $\zeta \in V \cap \partial D$, there is a function ϕ_ζ which is continuous on $U \cap \bar{D}$, plurisubharmonic on $U \cap D$ and satisfies

$$-M |z - \zeta|^\alpha \leq \phi_\zeta(z) \leq -|z - \zeta|^\beta$$

for all $z \in U \cap \overline{D}$. It is known that the class of regular points includes open pieces of strongly pseudoconvex boundaries, smooth weakly pseudoconvex finite type pieces in \mathbb{C}^2 , those that are smooth convex finite type in \mathbb{C}^n (see [30]) and finally smooth pseudoconvex finite type boundaries in \mathbb{C}^n (see [17]). D is said to be regular if each of its boundary points is regular.

Theorem 1.4. *Let $D \subset \mathbb{C}^n$ be a bounded regular domain, $D' \subset \mathbb{C}^n$ a possibly unbounded domain and $f : D \rightarrow D'$ a proper holomorphic mapping. Let $p \in \partial D$ be a C^2 strongly pseudoconvex point and $p' \in \partial D'$ be such that the boundary $\partial D'$ is C^∞ smooth pseudoconvex and of finite type near p' . Suppose that the Levi rank of $\partial D'$ at p' is $n - 2$ and assume that $p' \in cl_f(p)$, the cluster set of p . Then p' is also a strongly pseudoconvex point.*

This fits in the paradigm, observed earlier by many other authors (for example [25]), that a proper mapping does not increase the type of a boundary point. Note furthermore that there are no other assumptions on the defining function for $\partial D'$ near p' as in theorem 1.1 except the Levi rank condition. The difficulty created by the lack of this assumption as explained above is circumvented by the global properness of f – indeed, it is possible to scale f to get a holomorphic limit $F : \mathbb{B}^n \rightarrow D'_\infty$ where D'_∞ is as in (1.2). Now using the fact that $p \in \partial D$ is strongly pseudoconvex and hence regular, it is possible to peel off a local correspondence from the global one $f^{-1} : D' \rightarrow D$ that extends continuously up to $\partial D'$ near p' and contains p in its cluster set by [13]. This local correspondence can be scaled, using the Schwarz lemma for correspondences from [45] and the behaviour of the scaled balls in the Kobayashi metric from [37]. This gives a well defined correspondence from D'_∞ with values in \mathbb{B}^n and this turns out to be the inverse for F . Thus $F : \mathbb{B}^n \rightarrow D'_\infty$ is proper and this is sufficient to conclude that $p' \in \partial D'$ must be strongly pseudoconvex.

Corollary 1.5. *Let $f : M \rightarrow M'$ be a nonconstant Lipschitz CR mapping between real hypersurfaces in \mathbb{C}^n . Fix $p \in M$ and $p' = f(p) \in M'$. Suppose that M is C^2 strongly pseudoconvex near p and that there is an open neighbourhood $U' \subset \mathbb{C}^n$ of p' and a C^∞ smooth defining function $r'(z')$ for $U' \cap M'$ in U' which is plurisubharmonic on the pseudoconvex side of M' near p' . If the Levi rank of M' at p' is at least $n - 2$ then p' is a strongly pseudoconvex point.*

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2. PROOF OF THEOREM 1.1

Consider a smooth pseudoconvex finite type hypersurface $M \subset \mathbb{C}^n$. Associated to each $p \in M$ are two well known invariants: one is the D'Angelo type

$$\Delta(p) = (\Delta_n(p), \Delta_{n-1}(p), \dots, \Delta_1(p))$$

where the integer $\Delta_q(p)$ is the q -type of M at p and is a measure of the maximal order of contact of q dimensional varieties with M at p . To recall the definition (see [23]), let r be a local defining function for M near p and let $\tilde{r}(z) = r(z + p)$. Then for $1 \leq q \leq n$,

$$\Delta_q(p) = \inf_L \sup_\tau \{ \nu(\tilde{r} \circ L \circ \tau) / \nu(\tau) \}$$

where the infimum is taken over all linear embeddings $L : \mathbb{C}^{n-q+1} \rightarrow \mathbb{C}^n$ and the supremum is taken over all germs of holomorphic curves $\tau : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{n-q+1}, 0)$ mapping the

origin in \mathbb{C} to the origin in \mathbb{C}^{n-q+1} and $\nu(f)$ denotes the order of vanishing of f at the origin. The smoothness of M at p implies that $\Delta_n(p) = 1$ and it can be seen that $2 \leq \Delta_{n-1}(p) \leq \Delta_{n-2}(p) \leq \dots \leq \Delta_1(p) < \infty$.

To quickly recall the Catlin multitype of M at p (see [16]), let Γ_n be the collection of n -tuples of reals $m = (m_1, m_2, \dots, m_n)$ such that $0 < m_1 \leq m_2 \leq \dots \leq m_n \leq \infty$. Order Γ_n lexicographically. An element m of Γ_n is called distinguished provided there is a holomorphic coordinate system $w = \phi(z)$ around p with mapped to the origin such that if

$$(\alpha_1 + \beta_1)/m_1 + (\alpha_2 + \beta_2)/m_2 + \dots + (\alpha_n + \beta_n)/m_n < 1$$

then $D^\alpha \bar{D}^\beta r \circ \phi^{-1}(0) = 0$; here $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are n -tuples and D^α and \bar{D}^β are the partial derivatives

$$\partial^{|\alpha|} / \partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_n^{\alpha_n} \text{ and } \partial^{|\beta|} / \partial \bar{z}_1^{\beta_1} \partial \bar{z}_2^{\beta_2} \dots \partial \bar{z}_n^{\beta_n}$$

respectively. The Catlin multitype $\mathcal{M}(p) = (m_1, m_2, \dots, m_n)$ of M at p is defined to be the largest amongst all distinguished elements. Since $r(z)$ is a smooth defining function for M near p , it can be seen that the first entry in $\mathcal{M}(p)$ is always one. If $\mathcal{M}(p)$ is finite, i.e., $m_n < \infty$, then there is a coordinate system around $p = 0$ such that the defining function is of the form

$$(2.1) \quad r(z) = 2\Re z_n + P('z, ' \bar{z}) + R(z)$$

where $P('z, ' \bar{z})$ is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ homogeneous polynomial of total weight one which is plurisubharmonic and does not contain pluriharmonic terms and

$$(2.2) \quad |R(z)| \lesssim (|z_1|^{m_n} + |z_2|^{m_{n-1}} + \dots + |z_n|^{m_1})^\gamma$$

for some $\gamma > 1$. The homogeneity of $P('z, ' \bar{z})$ with total weight one mentioned above means that for $t \geq 0$, $P \circ \pi_t('z) = tP('z, ' \bar{z})$ where

$$\pi_t(z_1, z_2, \dots, z_{n-1}) = (t^{1/m_n} z_1, t^{1/m_{n-1}} z_2, \dots, t^{1/m_2} z_{n-1}).$$

Other total weights $\mu > 0$ occur when t^μ (instead of t) can be factored from $P \circ \pi_t('z)$. The plurisubharmonicity of $P('z, ' \bar{z})$ implies that each m_k for $2 \leq k \leq n$ must be even. Thus the variable z_i is assigned a weight of $1/m_{n-i+1}$ for $1 \leq i \leq n-1$ and by definition the weight of the monomial $'z^J ' \bar{z}^K$ is

$$(j_1 + k_1)/m_n + (j_2 + k_2)/m_{n-1} + \dots + (j_{n-1} + k_{n-1})/m_2$$

for $(n-1)$ -multiindices $J = (j_1, j_2, \dots, j_{n-1})$ and $K = (k_1, k_2, \dots, k_{n-1})$. A basic relation between $\Delta(p)$ and $\mathcal{M}(p)$ proved in [16] is that $m_{n+1-q} \leq \Delta_q$ for all $1 \leq q \leq n$.

Call $p \in M$ an h -extendible (or semi-regular) point if $\mathcal{M}(p)$ is finite and $\Delta(p) = \mathcal{M}(p)$. This happens if and only if (see [47]) there is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ homogeneous C^1 smooth real function $a('z)$ on $\mathbb{C}^{n-1} \setminus \{0\}$ such that $a('z) > 0$ whenever $'z \neq 0$ and $P('z, ' \bar{z}) - a('z)$ is plurisubharmonic on \mathbb{C}^{n-1} and among other things, this is equivalent to the model domain D_∞ (as in (1.1)) being of finite type. We shall henceforth assume that $p = 0$ and $p' = 0'$ and that the respective defining functions $r(z)$ and $r'(z')$ satisfy $\partial r / \partial z_n(0) \neq 0$ and $\partial r' / \partial z'_n(0') \neq 0$. The holomorphic map $f : D \rightarrow D'$ is not necessarily proper, but Hölder continuous with exponent $\delta \in (0, 1)$ on \bar{D} near $0 \in \partial D$ by [42] and $f(0) = 0'$. Thus the assumption that f is Lipschitz is stronger than what is a priori known.

2.1. The Scaling Method applied to (D, D', f) . For $z \in D$ close to the origin, note that

$$\text{dist}(z, \partial D) \lesssim \text{dist}(f(z), \partial D') \lesssim \text{dist}(z, \partial D)$$

where the inequality on the right follows since f admits a Lipschitz extension to D near the origin, while the left inequality follows by applying the Hopf lemma to $r' \circ f(z)$ which is a negative plurisubharmonic function on D . To scale D , choose a sequence of points $p^\nu = (0, -\delta_\nu)$ in D along the inner normal at the origin, where $\delta_\nu > 0$ and $\delta_\nu \searrow 0$. Let T^ν be the dilation defined by

$$T^\nu : (z_1, z_2, \dots, z_{n-1}, z_n) \mapsto (\delta_\nu^{-1/m_n} z_1, \delta_\nu^{-1/m_{n-1}} z_2, \dots, \delta_\nu^{-1/m_2} z_{n-1}, \delta_\nu^{-1} z_n)$$

and note that $T^\nu(p^\nu) = (0, -1)$ while the domains $D_\nu = T^\nu(D)$ are defined by

$$r_\nu = \delta_\nu^{-1} r \circ (T^\nu)^{-1}(z) = 2\Re z_n + P(z', \bar{z}') + \delta_\nu^{-1} R \circ (T^\nu)^{-1}(z)$$

where

$$|\delta_\nu^{-1} R \circ (T^\nu)^{-1}(z)| \lesssim \delta_\nu^{\gamma-1} (|z_1|^{m_n} + |z_2|^{m_{n-1}} + \dots + |z_{n-1}|^{m_2} + |z_n|)^\gamma$$

by (2.1). On each compact set in \mathbb{C}^n this error term converges to zero since $\gamma > 1$ and hence the sequence of domains D_ν converges in the Hausdorff metric to

$$D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P(z', \bar{z}') < 0\}.$$

Let $r_\infty(z) = 2\Re z_n + P(z', \bar{z}')$. To scale D' recall that by [18], for each ζ near $0' \in \partial D'$ there is a unique polynomial automorphism $\Phi_\zeta(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\Phi_\zeta(\zeta) = 0$ such that

$$\begin{aligned} r(\Phi_\zeta^{-1}(z)) &= r(\zeta) + 2\Re z_n + \sum_{\substack{j+k \leq 2m \\ j, k > 0}} a_{jk}(\zeta) z_1^j \bar{z}_1^k + \sum_{\alpha=1}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq m \\ j, k > 0}} \Re \left((b_{jk}^\alpha(\zeta) z_1^j \bar{z}_1^k) z_\alpha \right) \\ &\quad + O(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1}). \end{aligned}$$

where for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we denote $z_* = (z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-2}$. These automorphisms converge to the identity uniformly on compact subsets of \mathbb{C}^n as $\zeta \rightarrow 0$. Furthermore, if for $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in D'$ as above we consider the point $\tilde{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_n + \epsilon)$ where $\epsilon > 0$ is chosen to ensure that $\tilde{\zeta} \in \partial D'$, then the actual form of $\Phi_\zeta(z)$ shows that $\Phi_{\tilde{\zeta}}(\zeta) = (0, \dots, 0, -\epsilon)$ – since the explicit description of these automorphisms will come up later, we shall be content at this stage with merely collecting the relevant properties needed to describe the scaling of D' . To define the distinguished polydiscs around ζ (more precisely, biholomorphic images of polydiscs), let

$$A_l(\zeta) = \max\{|a_{jk}(\zeta)| : j + k = l\}, \quad 2 \leq l \leq 2m'$$

and

$$B_{l'}(\zeta) = \max\{|b_{jk}^\alpha(\zeta)| : j + k = l', 2 \leq \alpha \leq n-1\}, \quad 2 \leq l' \leq m'.$$

For each $\delta > 0$ define

$$\tau(\zeta, \delta) = \min \left\{ (\delta/A_l(\zeta))^{1/l}, (\delta^{1/2}/B_{l'}(\zeta))^{1/l'} : 2 \leq l \leq 2m', 2 \leq l' \leq m' \right\}$$

Since the type of $\partial D'$ at the origin is $2m'$ it follows that $A_{2m'}(0) > 0$ and hence $A_{2m'}(\zeta)$ is positive for all ζ sufficiently close to the origin. Thus

$$\delta^{1/2} \lesssim \tau(\zeta, \delta) \lesssim \delta^{1/2m'}$$

for ζ close to the origin – the upper bound being a consequence of the non-vanishing of $A_{2m'}(\zeta)$ near the origin while the lower bound follows since the greatest possible exponent of δ in the definition of $\tau(\zeta, \delta)$ is $1/2$. Set

$$\tau_1(\zeta, \delta) = \tau(\zeta, \delta) = \tau, \tau_2(\zeta, \delta) = \dots = \tau_{n-1}(\zeta, \delta) = \delta^{1/2}, \tau_n(\zeta, \delta) = \delta$$

and define

$$R(\zeta, \delta) = \{z \in \mathbb{C}^n : |z_k| < \tau_k(\zeta, \delta), 1 \leq k \leq n\}$$

which is a polydisc around the origin in \mathbb{C}^n with polyradii $\tau_k(\zeta, \delta)$ along the z_k direction for $1 \leq k \leq n$ and let

$$Q(\zeta, \delta) = \Phi_\zeta^{-1}(R(\zeta, \delta))$$

which is a distorted polydisc around ζ . It was shown in [43] that these domains satisfy the engulfing property, i.e., for all ζ in a small fixed neighbourhood of the origin, there is a uniform constant $C > 0$ such that if $\eta \in Q(\zeta, \delta)$, then $Q(\eta, \delta) \subset Q(\zeta, C\delta)$ and $Q(\zeta, \delta) \subset Q(\eta, C\delta)$.

Consider the sequence $p^\nu = f(p^\nu) \in D'$ that converges to the origin and denote by w^ν the point on $\partial D'$ chosen such that if $p^\nu = (p_1^\nu, p_2^\nu, \dots, p_n^\nu)$ then $w^\nu = (p_1^\nu, p_2^\nu, \dots, p_n^\nu + \gamma_\nu)$ for some $\gamma_\nu > 0$. Note that

$$\gamma_\nu \approx \text{dist}(p^\nu, \partial D')$$

for all large ν . Hence

$$\delta_\nu = \text{dist}(p^\nu, \partial D) \approx \text{dist}(p^\nu, \partial D') \approx \gamma_\nu$$

for all large ν . Let $g^\nu = \Phi_{w^\nu}(\cdot)$ be the polynomial automorphism of \mathbb{C}^n corresponding to $w^\nu \in D'$ as described above. Let us consider the holomorphic mappings

$$f^\nu = g^\nu \circ f : D \rightarrow g^\nu(D')$$

and define a dilation of coordinates in the target space by

$$B^\nu : (z'_1, z'_2, \dots, z'_n) \mapsto ((\tau_1^\nu)^{-1} z'_1, (\tau_2^\nu)^{-1} z'_2, \dots, (\tau_n^\nu)^{-1} z'_n)$$

where $\tau_1^\nu = \tau(w^\nu, \gamma_\nu)$, $\tau_j^\nu = \gamma_{1/2}^\nu$ for $2 \leq j \leq n-1$ and $\tau_n^\nu = \gamma_\nu$. Let $D^\nu = T^\nu(D)$ and $D'^\nu = (B^\nu \circ g^\nu)(D')$ be the scaled domains and the scaled maps between them are

$$F^\nu = B^\nu \circ f^\nu \circ (T^\nu)^{-1} : D_\nu \rightarrow D'_\nu.$$

To understand the Hausdorff limit of the domains D'_ν , note first that $B^\nu \circ g^\nu(p^\nu) = ('0, -1)$, which implies that $F^\nu('0, -1) = ('0, -1)$ for all ν , and that r'_ν , the defining function for D'_ν is given by

$$\gamma_\nu^{-1} r' \circ (B^\nu \circ g^\nu)^{-1}(z) = 2\Re z_n + Q_\nu(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \Re(S_\nu^\alpha(z_1, \bar{z}_1) z_\alpha) + O(\tau_1^\nu)$$

where

$$Q_\nu(z_1, \bar{z}_1) = \sum_{\substack{j+k \leq 2m' \\ j, k > 0}} a_{jk}(w^\nu) \gamma_\nu^{-1} (\tau_1^\nu)^{j+k} z_1^j \bar{z}_1^k$$

and

$$S_\nu^\alpha(z_1, \bar{z}_1) = \sum_{\substack{j+k \leq m' \\ j, k > 0}} b_{jk}^\alpha(w^\nu) \gamma_\nu^{-1/2} (\tau_1^\nu)^{j+k} z_1^j \bar{z}_1^k.$$

By the definition of $A_l, B_{l'}$ and τ_1^ν it follows that the largest coefficient in both Q_ν and S_ν^α is at most one in modulus. It was shown in [18] that there exists a uniform $\epsilon > 0$ such that

$$|b_{jk}^\alpha(w^\nu)\gamma_\nu^{-1/2}(\tau_1^\nu)^{j+k}| \lesssim (\tau_1^\nu)^\epsilon$$

for all possible indices j, k, α and all large ν . Therefore some subsequence of this family of defining functions converges together with all derivatives on compact sets to

$$r'_\infty(z) = 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2$$

where $Q_{2m'}(z_1, \bar{z}_1)$ is a polynomial of degree at most $2m'$ without harmonic terms. Hence the domains D'_ν converge to

$$D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\}$$

which, being the smooth limit of pseudoconvex domains, is itself pseudoconvex. In particular, it follows that $Q_{2m'}(z_1, \bar{z}_1)$ is subharmonic.

It is known that analytic discs in a bounded regular domain in \mathbb{C}^n satisfy the so-called attraction property (see [11]), i.e., if the centre of a given disc is close to a boundary point, then a given subdisc around the origin cannot wander too far away from the same boundary point. A quantitative version of this was proved by Berteloot-Coeuré in [14] and forms the basis for controlling families of scaled mappings in \mathbb{C}^2 . Using ideas from [12] and [18], the following analogue was proved in [43] and will be useful in this situation as well – we include the statement for the sake of completeness.

Proposition 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $\zeta_0 \in \partial\Omega$ and suppose there is an open neighbourhood V such that $V \cap \partial\Omega$ is C^∞ smooth pseudoconvex of finite type and the Levi rank is at least $n - 2$ on $V \cap \partial\Omega$. Fix a domain $\omega \subset \mathbb{C}^m$.*

Then for any fixed point $z_0 \in \omega$ and a compact $K \subset \omega$ containing z_0 , there exist constants $\epsilon(K), C(K) > 0$ such that for any $\xi \in V \cap \partial\Omega$ and $0 < \epsilon < \epsilon(K)$, every holomorphic mapping $F : \omega \rightarrow \Omega$ with

$$|F(z_0) - \zeta_0| < \epsilon(K) \quad \text{and} \quad F(z_0) \in Q(\xi, \epsilon)$$

also satisfies $F(K) \subset Q(\xi, C(K), \epsilon)$.

Now let $\{K_j\}$ be an increasing sequence of relatively compact domains that exhaust D_∞ such that each contains the base point $(0, -1)$. Fix $K = K_\mu$ and let $\tilde{f}^\nu = f \circ (T^\nu)^{-1}$. Then

$$\tilde{f}^\nu(0, -1) = f(0, -\delta_\nu) = f(p^\nu) = p^\nu$$

and hence $\tilde{f}^\nu(0, -1) \rightarrow 0 \in \partial D'$. In particular, $\tilde{f}^\nu(0, -1) = p^\nu \in Q(w^\nu, 2\gamma_\nu)$ for all large ν , by the construction of these distorted polydiscs. By the previous proposition

$$\tilde{f}^\nu(K) \subset Q(w^\nu, C(K)\gamma_\nu)$$

and therefore

$$F^\nu(K) = B^\nu \circ g^\nu \circ \tilde{f}^\nu(K) \subset B^\nu \circ g^\nu(Q(w^\nu, C(K)\gamma_\nu)).$$

However,

$$B^\nu \circ g^\nu(Q(w^\nu, C(K)\gamma_\nu)) = B^\nu(R(w^\nu, C(K)\gamma_\nu))$$

which by definition is contained in a polydisk around the origin with polyradii

$$r_k = \tau_k(w^\nu, C(K)\gamma_\nu) / \tau_k(w^\nu, \gamma_\nu)$$

for $1 \leq k \leq n$. Note that $r_n = C(K)$ and that $r_k = (C(K))^{1/2}$ for $2 \leq k \leq n-1$. Since $C(K) > 1$, and this may be assumed without loss of generality, the definition of $\tau(\zeta, \delta)$ shows that for $\delta' < \delta''$,

$$(\delta'/\delta'')^{1/2} \tau(\zeta, \delta'') \leq \tau(\zeta, \delta') \leq (\delta'/\delta'')^{1/2m'} \tau(\zeta, \delta').$$

Therefore

$$\tau_1(w^\nu, C(K)\gamma_\nu)/\tau_1(w^\nu, \gamma_\nu) \leq (C(K))^{1/2}$$

and hence $r_1 \leq (C(K))^{1/2}$. Thus $\{F^\nu\}$ is uniformly bounded on each compact set in D_∞ and is therefore normal. Let $F : D_\infty \rightarrow D'_\infty$ be a holomorphic limit of some subsequence in $\{F^\nu\}$ and since $F^\nu('0, -1) = ('0, -1)$ for all ν by construction, it follows that $F('0, -1) = ('0, -1)$. The maximum principle shows that $F(D_\infty) \subset D'_\infty$.

Let $R > 0$ be arbitrary and fix $z \in D_\infty \cap B(0, R)$. Note that since f preserves the distance to the boundary, it follows that

$$|(r'_\nu \circ F^\nu)(z)| = \gamma_\nu^{-1} |(r' \circ f \circ T_\nu^{-1})(z)| \lesssim (\delta_\nu/\gamma_\nu) |r_\nu(z)|$$

and hence

$$|r'_\infty \circ F(z)| \lesssim |r_\infty(z)|$$

since the constants appearing in the first estimate are independent of ν . This shows that the cluster set of a finite boundary point of D_∞ does not intersect D'_∞ and therefore F is non-degenerate.

2.2. Boundedness of $F^\nu(0)$. A crucial point in the proof of theorem 1.1 is to show that near the origin, the family $\{F^\nu\}$ of the scaled maps is uniformly Hölder continuous upto the boundary i.e., the Hölder constant and exponent are stable under the scaling. A first step in establishing this is to show that the images of the origin under these scaled maps is stable.

Proposition 2.2. *The sequence $\{F^\nu(0)\}$ is bounded.*

Proof. Recall the scaled maps

$$F^\nu = B^\nu \circ g^\nu \circ f \circ (T^\nu)^{-1}$$

where B^ν, T^ν were linear maps and $f(0) = 0$. So $F^\nu(0) = (B^\nu \circ g^\nu \circ f)(0) = (B^\nu \circ g^\nu)(0)$. Now let us recall the explicit form of the map $g^\nu = \Phi_{w^\nu}$ from [43], which is a polynomial automorphism of \mathbb{C}^n that reduces the defining function to a certain normal form as stated in section 2.1. Let us recall this reduction procedure for any given domain Ω in \mathbb{C}^n that is smooth pseudoconvex of finite type $2m$ and of Levi rank (at least) $n-2$ on a (small) open boundary piece Σ in ∂D_∞ . Let r be a smooth defining function for Σ with $\partial r/\partial z_n(z) \neq 0$ for all z in a small neighbourhood U in \mathbb{C}^n of Σ such that the vector fields

$$L_n = \partial/\partial z_n, \quad L_j = \partial/\partial z_j + b_j(z, \bar{z})\partial/\partial z_n$$

where $b_j = (\partial r/\partial z_n)^{-1} \partial r/\partial z_j$, form a basis of $\mathbb{C}T^{(1,0)}(U)$ and satisfy $L_j r \equiv 0$ for $1 \leq j \leq n-1$ and $\partial \bar{\partial} r(z)(L_i, \bar{L}_j)_{2 \leq i, j \leq n-1}$ has all its eigenvalues positive for each $z \in U$.

The reduction now consists of five steps – for $\zeta \in U$, the map $\Phi_\zeta = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ where each ϕ_j is described below.

ϕ_1 is the affine map given by

$$\phi_1(z_1, \dots, z_n) = (z_1 - \zeta_1, \dots, z_{n-1} - \zeta_{n-1}, z_n + \sum_{j=1}^{n-1} b_j^\zeta z_j - (\zeta_n + \sum_{j=1}^{n-1} b_j^\zeta \zeta_j)).$$

where the coefficients $b_j^\zeta = b_j(\zeta, \bar{\zeta})$ are clearly smooth functions of ζ on U . Therefore, ϕ_1 translates ζ to the origin and

$$r(\phi^{-1}(z)) = r(\zeta) + \Re z_n + \text{terms of higher order.}$$

where the constant term disappears when $\zeta \in \Sigma$.

The remaining reductions remove the occurrence of harmonic (not just pluriharmonic) monomials in the variables z_1, \dots, z_{n-1} of weights upto one in the weighted homogeneous expansion of the defining function with respect to the weights given by the multitype for Σ , i.e., the variable z_1 is assigned a weight of $1/2m$, z_n a weight of 1 while the others are assigned $1/2$ each. Now, since the Levi form restricted to the subspace

$$L_* = \text{span}_{\mathbb{C}^n} \langle L_2, \dots, L_{n-1} \rangle$$

of $T_\zeta^{(1,0)}(\partial\Omega)$ is positive definite, we may diagonalize it via a unitary transform ϕ_2 and a dilation ϕ_3 will then ensure that the quadratic part involving only z_2, z_3, \dots, z_{n-2} in the Taylor expansion of r is $|z_2|^2 + |z_3|^2 + \dots + |z_{n-2}|^2$. The entries of the matrix that represents the composite of the last two linear transformations are smooth functions of ζ and in the new coordinates still denoted by z_1, \dots, z_n , the defining function is in the form

$$(2.3) \quad r(z) = r(\zeta) + \Re z_n + \sum_{\alpha=2}^{n-1} \sum_{j=1}^m \Re((a_j^\alpha z_1^j + b_j^\alpha \bar{z}_1^j) z_\alpha) + \Re \sum_{\alpha=2}^{n-1} c_\alpha z_\alpha^2 \\ + \sum_{2 \leq j+k \leq 2m} a_{j,k} z_1^j \bar{z}_1^k + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{2 \leq j+k \leq m} \Re(b_{j,k}^\alpha z_1^j \bar{z}_1^k z_\alpha) \\ + O(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1})$$

A change in the normal variable z_n to absorb the pluriharmonic terms here i.e., $z_1^k, \bar{z}_1^k, z_\alpha^2$ as well as $z_1^k z_\alpha, \bar{z}_1^k \bar{z}_\alpha$, can be done according to the following standard change of coordinates

$$z_j = t_j \quad (1 \leq j \leq n-1), \\ z_n = t_n - P_1(t_1, \dots, t_{n-1})$$

where

$$P_1(t_1, \dots, t_{n-1}) = \sum_{k=2}^{2m} a_{k0} t_1^k - \sum_{\alpha=2}^{n-1} \sum_{k=1}^m a_k^\alpha t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha t_\alpha^2$$

with coefficients that are smooth functions of ζ .

Finally, just as we absorbed into the simplest pure term $\Re z_n$ that occurs in the Taylor expansion, other pure terms not divisible by this variable in the last step, we may absorb into the simplest of non-harmonic monomials that occur there namely, $|z_\alpha|^2$ where $2 \leq \alpha \leq n-1$ other non-harmonic terms of degree > 2 not divisible by them. Let us do this

to those of weight at most one, remaining in (2.3) rewritten in the t -coordinates, which are of the form $\bar{t}_1^j t_\alpha$ by applying the transform

$$\begin{aligned} t_1 &= w_1, \quad t_n = w_n, \\ t_\alpha &= w_\alpha - P_2(w_1) \quad (2 \leq \alpha \leq n-1) \end{aligned}$$

where

$$P_2(w_1) = \sum_{k=1}^m b_k^\alpha w_1^k$$

with coefficients smooth in ζ , as before (since all these coefficients are simply the derivatives of some order of the smooth defining function r evaluated at ζ). We then have the sought for simplification of the Taylor expansion.

Now, suppose that we already have the reduced form holding at the origin, then (since no further normalization would be required) Φ_0 may be taken to be the identity. Note that the normalizing map Φ_ζ is not uniquely determined – even among the class of all maps of the same form – owing to the (only) ambiguity in the choice of the diagonalizing map ϕ_2 ; however, this choice can certainly be done in a manner such that the coefficients of that unitary matrix are smooth in ζ and satisfying the ‘initial condition’ that ϕ_2 for the origin is the identity. Thus in all, the map Φ_ζ is smooth in the parameter ζ with Φ_0 being the identity map. This implies that the family $\Phi_\zeta(\cdot)$ is uniformly Lipschitz (where $\zeta \in U$) and converges uniformly on compact subsets of \mathbb{C}^n to the identity as ζ approaches the origin as mentioned in section 2.1. Next, the consequence of the simpler fact that the map $\Phi(\zeta, 0) = \Phi_\zeta(0)$ is Lipschitz in a neighbourhood of the origin, to our setting is that

$$|g^\nu(0)| \leq C_1 |w^\nu|$$

with C_1 independent of ν . Now

$$F^\nu(0) = B^\nu \circ g^\nu(0) = \left((\tau_1^\nu)^{-1}(g^\nu(0))_1, \dots, (\tau_{n-1}^\nu)^{-1}(g^\nu(0))_{n-1}, (\tau_n^\nu)^{-1}(g^\nu(0))_n \right).$$

From the fact that $\tau_1^\nu \gtrsim \gamma_\nu^{1/2}$ and $\tau_j^\nu = \gamma_\nu^{1/2}$ for $2 \leq j \leq n-1$ while $\tau_n^\nu = \gamma_\nu$, we get that

$$\left| (B^\nu \circ g^\nu(0)) \right| \leq C_2 |w^\nu| / \gamma_\nu$$

with C_2 again independent of ν . Now, since f is Lipschitz upto M and $\delta_\nu \lesssim \gamma_\nu$ we have

$$\begin{aligned} |w^\nu| &\leq \text{dist}(w^\nu, p^\nu) + \text{dist}(p^\nu, 0') \\ &= \text{dist}(p^\nu, \partial D') + |f(p^\nu) - f(0)| \\ &\leq C_3(\gamma_\nu + |p^\nu|) \\ &= C_3(\gamma_\nu + \delta_\nu) \\ &\leq C_4 \gamma_\nu \end{aligned}$$

with constants independent of ν as before, implying that $\{F^\nu(0)\}$ is bounded.

2.3. Stability of the Kobayashi metric. For Ω, Σ, U with $0 \in \Sigma$, as in the previous section, recall from [43] (see also [12]), the M -metric defined for $\zeta \in U \cap \Omega$ by

$$M_\Omega(\zeta, X) = \sum_{k=1}^n |(D\Phi_\zeta(\zeta)X)_k|/\tau_k(\zeta, \epsilon(\zeta)) = |D(B_\zeta \circ \Phi_\zeta)(\zeta)(X)|_{\mu}$$

with $B_\zeta = B_\zeta^{\epsilon(\zeta)}$ where $\epsilon(\zeta) > 0$ is such that $\zeta + (0, \dots, 0, \epsilon(\zeta))$ lies on Σ and

$$B_\zeta^\delta(z_1, \dots, z_n) = ((\tau_1)^{-1}z_1, \dots, (\tau_n)^{-1}z_n)$$

where $\tau_1 = \tau(\zeta, \delta)$, $\tau_j = \delta^{1/2}$ for $2 \leq j \leq n-1$ and $\tau_n = \delta$. Now let us get to our scaled domains $D'_\nu = B^\nu \circ g^\nu$, where we recall $B^\nu = B_{p^\nu}$ and $g^\nu = \Phi_{w^\nu}$ and denote by ${}^\nu\Phi_\zeta$ and ${}^\nu B_\zeta$ the normalizing map and the dilation respectively, associated to (D'_ν, ζ) ; we shall drop the left superscript ν for the domain D' . Since all the coefficients of the polynomial automorphisms ${}^\nu\Phi_\zeta(\cdot)$ depend smoothly on the derivatives of the defining functions r_ν , which converge in the C^∞ -topology on U , there exists $L > 1$ such that

$$(2.4) \quad 1/L < |\det(D({}^\nu\Phi_\zeta(z)))| < L$$

for all ν and $\zeta \in U$. The main result of this section is the following stability theorem for the Kobayashi metric.

Theorem 2.3. *There exists a neighbourhood U of the origin such that*

$$K_{D'_\nu}(z, X) \geq CM_{D'_\nu}(z, X)$$

for all $z \in U \cap D'_\nu$, $\nu \gg 1$ where C is a positive constant independent of ν .

The proof of this requires two steps and follows the lines of the argument presented in [43]. The first step consists in uniformly localizing the Kobayashi metric of the scaled domains near the origin which translates to verifying the following uniform version of the attraction of analytic discs near a local plurisubharmonic peak point.

Lemma 2.4. *There exists a neighbourhood $V \subseteq U$ of the origin and $\delta > 0$ independent of j such that for any j large enough and any analytic disc $f : \Delta \rightarrow D'_j$ with $f(0) = \zeta \in V \cap D'_j$, we have*

$$f(\Delta_\delta) \subseteq U \cap D'_j$$

Proof. If the lemma were false, then for any neighbourhood $V \subset U$ and any $\delta > 0$, there exists a sequence of analytic discs $f_j : \Delta \rightarrow D'_j$ with $f_j(0) = \zeta^j \in V \cap D'_j$, converging to the origin but $f_j(\Delta_\delta) \not\subseteq U$. Consider

$$\tilde{f}_j = S_j^{-1} \circ f_j : \Delta \rightarrow D'_j$$

where $S_j = B^j \circ \Phi_{w^j}$ are the scaling maps. Then $\tilde{\zeta}^j = \tilde{f}_j(0) = S_j^{-1}(\zeta^j)$ converges to the origin. Indeed, the family $S_j^{-1} = \Phi_{w^j}^{-1} \circ (B^j)^{-1}$ is equicontinuous at the origin since their derivatives are

$$D(\Phi_{w^j}^{-1}) \circ (B^j)^{-1}$$

with $D(\Phi_{w^j}^{-1})$ being bounded in a neighbourhood of the origin and $(B^j)^{-1}$ converging to the zero map. Now, $0 \in \partial D'$ being a plurisubharmonic peak point by [17], the simplest version of the attraction property of analytic discs (see for instance lemma 2.1.1 in [33]), gives for all $j \gg 1$ that

$$\tilde{f}_j(\Delta_\gamma) \subseteq U \cap D'$$

for some $\gamma \in (0, 1)$. Rescale the \tilde{f}_j 's, so that we may take $\gamma = 1$. Now recall the sharper version of the attraction property given by lemma 3.6 of [43] namely,

Lemma 2.5. *Let $W \Subset U$ be a neighbourhood of the origin. There exist constants $\alpha, A \in [0, 1]$ and $K \geq 1$ such that for any analytic disc $f : \Delta_N \rightarrow U$ that satisfies $M_{D'}(f(t), f'(t)) \leq A$, $f(0) \in W$ and $K^{N-1}\epsilon(f(0)) < \alpha$ also satisfies,*

$$\overline{f(\Delta_N)} \subset Q\left(f(0), K^N \epsilon(f(0))\right)$$

We intend to apply this lemma to the analytic discs $\tilde{g}_j(t) = \tilde{f}_j(r_0 t)$ for which we only need to verify

$$M_{D'}(\tilde{g}_j(t), \tilde{g}'_j(t)) \leq A$$

since $\tilde{g}_j(0) = \tilde{f}_j(0)$ converges to the origin. Then

$$\begin{aligned} M_{D'}(\tilde{g}_j(t), \tilde{g}'_j(t)) &= M_{D'}(\tilde{f}_j(r_0 t), r_0 \tilde{f}'_j(r_0 t)) \\ &= r_0 M_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \\ &\leq r_0 C_5 K_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \end{aligned}$$

by theorem 3.10 of [43], where C_5 is a positive constant independent of j . At the last step, we use the fact that \tilde{f}_j 's map Δ into so small a neighbourhood of the origin where $K_{D'} \approx M_{D'}$. As the Kobayashi metric decreases under holomorphic mappings, we have

$$r_0 C_5 K_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \leq r_0 C_5 K_\Delta(t, \partial/\partial t)$$

So, if we choose r_0 such that

$$r_0 C_5 \left(\sup_{|t| \leq r_0} K_\Delta(t, \partial/\partial t) \right) \leq A,$$

we will have completed the required verification of the hypothesis for the discs $g_j : \Delta \rightarrow D' \cap U$ and the aforementioned lemma gives

$$(2.5) \quad \tilde{f}_j(\Delta_{r_0}) = \tilde{g}_j(\Delta) \subset Q(\tilde{g}_j(0), \epsilon(\tilde{g}_j(0))) = Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0))).$$

At this point observe also that r_0 does not depend on δ . Next, fix any compact subdisc $K \subset \Delta_{r_0}$ and note by (2.5) that,

$$\tilde{f}_j(K) \subset Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0)))$$

and subsequently,

$$\begin{aligned} (S_j \circ \tilde{f}_j)(K) &\subset S_j(Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0)))) \\ &= (B^j \circ \Phi_{w^j}) \circ (B_{\tilde{\zeta}^j} \circ \Phi_{\tilde{\zeta}^j})^{-1}(\Delta^n) \\ (2.6) \quad &= \left(B_{p^j} \circ (\Phi_{w^j} \circ \Phi_{\tilde{\zeta}^j}^{-1}) \circ B_{\tilde{\zeta}^j}^{-1} \right)(\Delta^n) \end{aligned}$$

where $\tilde{\zeta}^j = \tilde{\zeta}^j + (0, \dots, 0, \epsilon(\tilde{\zeta}^j))$. Now the fact that $\epsilon(\tilde{\zeta}^j) \approx \epsilon(p^j)$, as both p^j and $\tilde{\zeta}^j$ converge to the origin and corollary 2.8 of [18] – the fact that the distinguished radius τ at any two points in U are comparable – together imply the boundedness of the sequence $\tau(\tilde{\zeta}^j, \epsilon(\tilde{\zeta}^j))/\tau(w^j, \epsilon(p^j))$, by C_6 say. This when combined with (2.4) and (2.6), yields the

stability of our discs f_j on compact subdiscs in Δ i.e., we have for a positive constant $C_K > C_6 L^2$ (which may depend on K but not on j), that

$$f_j(K) = (S_j \circ \tilde{f}_j)(K) \subseteq \Delta_{\sqrt{C_K}} \times \dots \times \Delta_{\sqrt{C_K}} \times \Delta_{C_K}$$

Hence by Montel's theorem, f_j 's converge to a map $f_\infty : \Delta_{r_0} \rightarrow \overline{D'}_\infty$, where $f_\infty(0) = 0 \in \partial D'_\infty$. If ϕ is a local plurisubharmonic peak function at the origin, then the maximum principle applied to the subharmonic function $\psi = \phi \circ f_\infty$ implies that $f_\infty \equiv \text{constant}$, since ϕ peaks precisely at one point. This contradicts our assumption that $f_j(\Delta_\delta) \not\subseteq U$ (as soon as δ is taken smaller than r_0) and completes the proof of the lemma. \square

Remark 2.6. The same line of argument gives proposition 2.1 by replacing the discs by balls in \mathbb{C}^m , covering any given compact $K \subset \omega$ by balls and using the engulfing property of the special polydiscs Q .

The second and more technical step, is a quantitative form of the Schwarz lemma at the boundary which generalizes lemma 3.6 of [43] mentioned earlier. To do this we require a stable version of the engulfing property for the distinguished polydiscs Q^ν associated to the scaled domain D'_ν . Fix a pair of neighbourhoods $0 \in W \Subset V \Subset U$.

Lemma 2.7. *There exist constants $\alpha, A \in [0, 1]$ and $K > 1$ such that for every analytic disc*

$$f : \Delta_N \rightarrow V \cap D'_\nu$$

that satisfies $M_{D^\nu}(f(t), f'(t)) < A$, $f(0) \in W$ and $K^{N-1}\epsilon(f(0)) < \alpha$ also satisfies,

$$\overline{f(\Delta_N)} \subset Q^\nu(f(0), K^N \epsilon(f(0))).$$

Recall the definition of $A_l(z)$ and $\tau(z, \delta)$ from section 2.1. Let $A_l^\nu(z)$ and $\tau^\nu(z, \delta)$ denote the corresponding quantities for the domain D'_ν and $A_l^\infty(z)$ and $\tau^\infty(z, \delta)$ that for D'_∞ . Then it is clear that $A_l^\nu(z) \rightarrow A_l^\infty(z)$ and consequently $\tau^\nu(z, \delta) \rightarrow \tau^\infty(z, \delta)$, both convergences being uniform for z in V . Let $\delta > 0$ be given. Let

$$M = \sup \{ (\tau^\nu(z, \delta))^l : z \in V, 2 \leq l \leq 2m', \nu \geq 1 \}$$

and $0 < \epsilon < \delta/M$. Then there exists by the uniform convergence of the A 's, an N_δ independent of $z \in V$ such that for all $2 \leq l \leq 2m$ and all $\nu > N_\delta$,

$$(2.7) \quad A_l^\infty(z) - \epsilon < A_l^\nu(z) < A_l^\infty(z) + \epsilon$$

The right inequality gives for all $2 \leq l \leq 2m$ that

$$(\tau^\infty(z, \delta))^l A_l^\nu(z) < (\tau^\infty(z, \delta))^l A_l^\infty(z) + (\tau^\infty(z, \delta))^l \epsilon < \delta + \delta$$

by the definition of τ^∞ and ϵ . The definition of τ^ν now makes this read as

$$(2.8) \quad \tau^\infty(z, \delta) < 2^{1/2} \tau^\nu(z, \delta)$$

for all $\nu > N_\delta$ and $z \in V$. Meanwhile the left inequality at (2.7) similarly gives,

$$(\tau^\nu(z, \delta))^l A_l^\infty(z) < (\tau^\nu(z, \delta))^l A_l^\nu(z) + (\tau^\nu(z, \delta))^l \epsilon < 2\delta$$

which implies $\tau^\nu(z, \delta) < 2^{1/2} \tau^\infty(z, \delta)$. Combining this with corollary 2.8 of [18] applied to τ^∞ , we get for $z \in Q^\infty(z', \delta)$ and all ν large that

$$(2.9) \quad 1/C \tau^\nu(z', \delta) < \tau^\nu(z, \delta) < C \tau^\nu(z', \delta)$$

for some $C > 0$, independent of z' and ν . This will lead to the following uniform engulfing property of these polydiscs.

Lemma 2.8. *There exists a positive constant C such that for all ν large, $z' \in U$ and $z'' \in Q^\nu(z', \delta)$ we have*

$$\begin{aligned} Q^\nu(z'', \delta) &\subset Q^\nu(z', C\delta) \\ Q^\nu(z', \delta) &\subset Q^\nu(z'', C\delta) \end{aligned}$$

Proof. Let us recall what Q^ν is and rewrite for instance the first statement as

$$\left({}^\nu B_{z''}^\delta \circ {}^\nu \Phi_{z''} \right)^{-1}(\Delta^n) \subset \left({}^\nu B_{z'}^{C\delta} \circ {}^\nu \Phi_{z'} \right)^{-1}(\Delta^n)$$

which is equivalent to saying that the map

$$(2.10) \quad {}^\nu B_{z'}^{C\delta} \circ {}^\nu \Phi_{z'} \circ \left({}^\nu \Phi_{z''} \right)^{-1} \circ \left({}^\nu B_{z''}^\delta \right)^{-1}$$

maps the unit polydisc into itself. Now recall from (2.4), that the sequence of Jacobians of ${}^\nu \Phi_{z'} \circ {}^\nu \Phi_{z''}^{-1}$ is uniformly bounded above by L^2 . Unravelling the definition of the scalings ${}^\nu B_{z'}^{C\delta}$ and ${}^\nu B_{z''}^\delta$, then shows that the map defined in (2.10) carries the unit polydisc into the polydisc of polyradius given by

$$(2.11) \quad \left(L^2 \tau^\nu(z'', \delta) / \tau^\nu(z', C\delta), L^2 \delta^{1/2} / (C\delta)^{1/2}, \dots, L^2 \delta^{1/2} / (C\delta)^{1/2}, L^2 \delta / C\delta \right)$$

Now the uniform comparability of the τ 's obtained in (2.9) gives $C_7 > 0$ independent of ν and z' , such that

$$\tau^\nu(z'', \delta) < C_7 \tau^\nu(z', \delta)$$

whereas looking at the definition of the τ directly gives

$$\tau^\nu(z', C\delta) > C^{1/2m} \tau^\nu(z', \delta)$$

Combining these two we see that the first component at (2.11) is bounded above by $L^2 C_7 / C^{1/2m}$. Therefore, choosing C to be such that $C^{1/2} > L^2$ and $C^{1/2m} > L^2 C_7$ ensures that the map at (2.10) leaves the unit polydisc invariant, completing the verification of the lemma. \square

Before passing, let us record one more stable estimate concerning the τ 's to be of use later, namely

$$(2.12) \quad \tau^\nu(z, \delta) \leq C_8 \delta^{1/2m'}$$

for $C_8 > 0$ (independent of ν), which follows as in section 2.1 from the fact that the $A_{2m'}^\nu(z)$'s are uniformly bounded below by a positive constant on a neighbourhood of the origin (which as usual we may assume to be U).

Next, observing that all the constants in the calculations in the proof of lemma 3.6 of [43] are stable owing to (2.9), lemma 2.8 and the stability of the estimates on the various coefficients of ${}^\nu \Phi$'s given in lemma 3.4 of [43] – because all these coefficients are nothing but the derivatives of some order (less than $2m$) of the defining functions of the scaled domains, which converge in the C^∞ -topology on U – lemma 2.7 follows as well.

Proof of theorem 2.3. If the theorem were false, there would exist a subsequence ν_j of \mathbb{N} and $\zeta^j \in D^{\nu_j}$ converging to the origin such that

$$K_{D^{\nu_j}} \left(\zeta^j, X / M_{D^{\nu_j}}(\zeta^j, X) \right) < 1/j^2$$

which after re-indexing the D^{ν_j} 's as D^j and using lemma 2.4 reads as :

There exists a sequence $\zeta^j \in U \cap D^j$ with ζ^j converging to the origin and a sequence of analytic discs

$$f_j : \Delta \rightarrow U \cap D'_j$$

such that

$$f_j(0) = \zeta^j \text{ and } f'_j(0) = R_j \left(X/M_{D'_j}(\zeta^j, X) \right)$$

with $R_j \geq j^2$.

The idea as in [12] and [43] now consists of scaling these analytic discs appropriately to enlarge their domains to discs of growing radius while ensuring their normality also, so that taking their limit produces an entire curve lying inside D'_∞ to contradict its Brody hyperbolicity. To work this out, define $M_j(t) : \overline{\Delta}_{1/2} \rightarrow \mathbb{R}^+$ by

$$M_j(t) = M_{D'_j}(f_j(t), f'_j(t))$$

Then

$$\begin{aligned} M_j(0) &= M_{D'_j}(f_j(0), f'_j(0)) = M_{D'_j}(\zeta^j, R_j X/M_{D'_j}(\zeta^j, X)) \\ &= R_j \geq j^2 \end{aligned}$$

Recall the following lemma from [12].

Lemma 2.9. *Let (X, d) be a complete metric space and let $M : X \rightarrow \mathbb{R}^+$ be a locally bounded function. Then for all $\sigma > 0$ and for all $u \in X$ satisfying $M(u) > 0$, there exists $v \in X$ such that*

- (i) $d(u, v) \leq 2/\sigma M(u)$
- (ii) $M(v) \geq M(u)$
- (iii) $M(x) \leq 2M(v)$ if $d(x, v) \leq 1/\sigma M(v)$

Apply this lemma to $M_j(t)$ on $\overline{\Delta}_{1/2}$ with $u = 0$ and $\sigma = 1/j$, to get $a_j \in \overline{\Delta}_{1/2}$ such that $|a_j| \leq 2j/M_j(0)$ and

$$(2.13) \quad M_j(a_j) \geq M_j(0) \geq j^2$$

and furthermore,

$$(2.14) \quad M_j(t) \leq 2M_j(a_j) \text{ on } \Delta(a_j, j/M_j(a_j))$$

(for all j big enough so that the discs above lie inside $\Delta_{1/2}$). Now, scaling the unit disc with respect to the points a_j which approach the origin (since the contention is that the derivatives of the f_j 's near the origin, in the M -metric, are blowing up rapidly), we get the discs $\Delta_j = s_j(\Delta)$ where $s_j = b^j \circ \phi_{a_j}$ with ϕ_{a_j} the translation that transfers a_j to the origin and b^j being the map that dilates by a factor of $2A^{-1}M_j(a_j)$ with A the constant of lemma 2.7. The scaled analytic discs $g_j : \Delta_j \rightarrow U \cap D'_j$ are then given by

$$g_j(t) = (f_j \circ s_j^{-1})(t) = f_j(a_j + c_j t),$$

where $c_j = A/2M_j(a_j)$. Note that

$$g'_j(t) = c_j f'_j(a_j + c_j t)$$

which implies

$$\begin{aligned} M_{D^j}(g_j(t), g_j'(t)) &= c_j M_j\left(f_j(a_j + c_j t), f_j'(a_j + c_j t)\right) \\ &= c_j M_j(a_j + c_j t) \end{aligned}$$

Now note that $a_j + c_j t$ lies in $\Delta(a_j, j/M_j(a_j))$ for all $|t| < j$ as $0 < A < 1$ and therefore by (2.13), (2.14) and the definition of c_j , we have for $t \in \Delta_j$ that

$$\begin{aligned} M_{D^j}(g_j(t), g_j'(t)) &= c_j M_j(a_j + c_j t) \\ &< c_j (2M_j(a_j)) \\ (2.15) \qquad \qquad \qquad &= A \end{aligned}$$

We wish to apply lemma 2.7 to these g_j 's. First we must that verify their centres lie close enough to the origin; to that end write

$$|g_j(0)| \leq |f_j(a_j) - f_j(0)| + |f_j(0)|$$

and note that the first term on the right can be made arbitrarily small by the equicontinuity of the f_j 's which map into the bounded neighbourhood U . After passing to a subsequence if necessary to ensure that $\epsilon(g_j(0)) \leq \alpha/K^{j-1}$, we have with (2.15), verified all the criteria of lemma 2.7 and therefore

$$(2.16) \qquad g_j(\Delta_N) \subset Q^j\left(g_j(0), K^N \epsilon(g_j(0))\right)$$

for all $j \geq N$. Let $\eta_j = g_j(0)$ and $\eta_j' = \eta_j + (0, \dots, 0, \epsilon_j)$ with $\epsilon_j > 0$ such that $r_j(\eta_j') = 0$ and note that

$$\eta_j \in Q^j(\eta_j', C\epsilon_j)$$

for a uniform constant $C \geq 1$ since $\epsilon_j \approx r_j(\eta_j)$, uniformly in j . Consequently using lemma 2.7, (2.16) becomes

$$(2.17) \qquad g_j(\Delta_N) \subset Q^j\left(\eta_j', CK^N \epsilon_j\right)$$

for all $j \geq N$. Now if we let

$$h_j = {}^j B_{\eta_j} \circ {}^j \Phi_{\eta_j'} \circ g_j : \Delta_j \rightarrow \tilde{S}_j(U \cap D')$$

where \tilde{S}_j is the map

$${}^j B_{\eta_j} \circ {}^j \Phi_{\eta_j'} \circ B_{p^j} \circ \Phi_{w^j}$$

then what (2.17) translates to, for the map h_j is that

$$h_j(\Delta_N) \subset \Delta_{(CK^N)^{1/2}} \times \dots \times \Delta_{(CK^N)^{1/2}} \times \Delta_{CK^N}$$

for all $N \leq j$, exactly as in the proof of lemma 2.4. Also, note that the domains $\tilde{S}_j(U \cap D')$ after passing to a subsequence if necessary, converge to a domain \tilde{D}'_∞ of the same form as D'_∞ , by the same argument as in section 2.1 together with (2.12). Thus Montel's theorem, a diagonal sequence argument and an application of the maximum principle to the limit, gives rise to an entire curve

$$h : \mathbb{C} \rightarrow \tilde{D}'_\infty.$$

Indeed, note that

$$h_j(0) = ({}^j B_{\eta_j} \circ {}^j \Phi_{\eta_j'})(\eta_j) = (0, \dots, 0, -1)$$

for all j and so $h(0) = (0, \dots, 0, -1)$, which lies in \tilde{D}'_∞ all of whose boundary points – including the point at infinity – are peak points. Now, to check that h is nonconstant, we examine the sequence of their derivatives at the origin. We write X_j for $g'_j(0) = c_j f'_j(a_j)$ (which tend to 0 but are bounded below in the M -metric).

$$\begin{aligned}
(2.18) \quad |h'_j(0)|_{l^1} &= |({}^j B_{\eta'_j} \circ D({}^j \Phi_{\eta'_j}))(\eta_j)(X_j)|_{l^1} \\
&\geq |({}^j B_{\eta_j} \circ D({}^j \Phi_{\eta_j}))(\eta_j)(X_j)|_{l^1} \\
&= M_{D^j}(g_j(0), X_j) \\
&= c_j M_j(a_j) = A/2
\end{aligned}$$

where the lower bound here is a consequence of (2.9) applied to τ^j for the points η_j and η'_j and the fact that the family

$$D({}^\nu \Phi_{z'}) \circ D({}^\mu \Phi_{z''}^{-1})$$

is uniformly bounded below in norm. Passing to the limit in (2.18), now gives

$$|h'(0)|_{l^1} \geq C(A/2) > 0$$

and we reach the contradiction mentioned earlier namely to the Brody hyperbolicity of $\partial \tilde{D}'_\infty$ (see for instance, lemma 3.8 of [43]). \square

Finally, let us note the consequence of theorem (2.3) in the form that we shall make use of

Corollary 2.10. *There is neighbourhood U of the origin and a positive constant C such that*

$$K_{D^\nu}(z, X) \geq C |X| / (\text{dist}(z, \partial D^\nu))^{1/2m'}$$

for all $z \in U \cap D'_\nu$ and all $\nu \gg 1$.

This comes from the facts that $\epsilon_\nu(z) \approx \text{dist}(z, \partial D^\nu)$ and for some $C_9 > 0$ we have for all ν that

$$M_{D^\nu}(z, X) \geq C_9 |X|_{l^1} / (\epsilon^\nu(z))^{1/2m'}$$

which in turn follows from (2.12).

2.4. Uniform Hölder continuity of the scaled maps near the origin. In this section, we recall from [20], the arguments which show that the family of scaled maps is uniformly Hölder continuous upto the boundary – we already know that each scaled map is Hölder continuous upto the origin.

Theorem 2.11. *There exist positive constants r, C such that*

$$|F^\nu(z') - F^\nu(z'')| \leq C |z' - z''|^{1/2m'}$$

for any $z', z'' \in \bar{D}_\nu \cap B(0, r)$ and all large ν .

By the previous section we may assume that $\{F^\nu(0)\}$ converges to a (finite) boundary point $q \in \partial D'_\infty$. Recall that the origin (respectively the $\Re z_n$ axis) is a common boundary point (respectively, the common normal at the origin) for all the scaled domains. At this special boundary point, we first show that the family of scaled mappings is equicontinuous – we already know that this holds on compact subsets of D_∞ – or equivalently that, given any neighbourhood of q , there exists a small ball about the origin, such that every scaled map carries the piece of its domain intercepted by this ball into that given neighbourhood.

Then, the uniform lower bound on the Kobayashi metric near q will sharpen the uniform boundary distance decreasing property to uniform Hölder continuity, of the scaled maps. In particular then, the family $\{F^\nu\} \cup \{F\}$ is equicontinuous near the origin, upto the respective boundary.

Lemma 2.12. *For any $\epsilon > 0$ there exists $\delta > 0$, such that $|F^\nu(z) - q| < \epsilon$ for any $z \in D_\nu \cap B(0, \delta)$ and all large ν .*

Proof. Suppose to obtain a contradiction, that the assertion were false. Then there exists $\epsilon_0 > 0$ and a sequence $a_\nu \in D_\nu$ such that $a_\nu \rightarrow 0$ and $|F^\nu(a_\nu) - q| \geq \epsilon_0$. Since every F^ν is continuous upto the boundary ∂D_ν , we can also choose a sequence of points $b_\nu \in D_\nu$ lying on the common inner normal to D_ν at the origin i.e., $b_\nu = (0, -\beta_\nu)$ with $\beta_\nu > 0$, such that $b_\nu \rightarrow 0$ and $|F^\nu(b_\nu) - q| \rightarrow 0$. Let $s_\nu = |a_\nu - b_\nu|$. It is not difficult to see that there exists a constant $C > 0$ such that for all ν , there is a smooth path $\gamma_\nu : [0, 3s_\nu] \rightarrow D_\nu$ with the following properties:

- (i) $\gamma_\nu(0) = a_\nu, \gamma_\nu(3s_\nu) = b_\nu$
- (ii) $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq Ct$, for $t \in [0, s_\nu]$,
 $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq Cs_\nu$, for $t \in [s_\nu, 2s_\nu]$,
 $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq C(3s_\nu - t)$, for $t \in [2s_\nu, 3s_\nu]$,
- (iii) $|d\gamma_\nu(t)/dt| \leq C$, for $t \in [0, s_\nu]$.

By corollary (2.10) there is a positive constant C such that for any $w \in D'_\nu \cap B(q, 2\alpha)$ for some $\alpha > 0$, and any vector $X \in \mathbb{C}^n$ the lower bound

$$(2.19) \quad K_{D'_\nu}(w, X) \geq C \text{dist}(w, \partial D'_\nu)^{-1/2m'} |X|$$

holds for all large ν . Let $\eta = \min\{\alpha/4, \epsilon_0/4\}$. Since $F^\nu(b_\nu)$ lies in $B(q, \alpha)$ for ν large enough, we can choose $t_\nu \in [0, 3s_\nu]$ such that $F^\nu \circ \gamma_\nu(t_\nu) \in \partial B(q, 2\eta)$ and $F^\nu \circ \gamma_\nu((t_\nu, 3s_\nu))$ is contained in $B(q, 2\eta)$. From section 2.1 we know that

$$(2.20) \quad \text{dist}(F^\nu(z), \partial D'_\nu) \leq C(R) \text{dist}(z, \partial D_\nu)$$

for any $R > 0$ and $z \in D_\nu \cap B(0, R)$ with $F^\nu(z) \in D'_\nu \cap (B(q, 2\alpha))$. Fix $r > 0$ and let $z \in D_\nu \cap B(0, r)$ be such that $F^\nu(z) \in D'_\nu \cap B(q, 2\tau)$. Since the Kobayashi metric is decreasing under holomorphic mappings, we get from (2.19) and (2.20) that

$$\begin{aligned} K_{D_\nu}(z, X) &\geq K_{D'_\nu}(F^\nu(z), dF^\nu(z)X) \\ &\geq C \text{dist}(F^\nu(z), \partial D'_\nu)^{-1/2m'} |dF^\nu(z)X| \\ &\geq C \text{dist}(z, \partial D_\nu)^{-1/2m'} |dF^\nu(z)X|. \end{aligned}$$

On the other hand it can be seen that for all ν ,

$$K_{D_\nu}(z, X) \leq |X|/\text{dist}(z, \partial D_\nu)$$

and this implies the uniform estimate

$$(2.21) \quad |dF^\nu(z)| \leq C \text{dist}(z, \partial D_\nu)^{-1+1/2m'}$$

for all large ν and $z \in D_\nu \cap B(0, r)$ such that $F^\nu(z) \in D_\nu \cap B(p, 2\alpha)$. Therefore,

$$\begin{aligned} |F^\nu(\gamma^\nu(t_\nu)) - F^\nu(b_\nu)| &\leq \int_{t_\nu}^{3s_\nu} |dF^\nu(\gamma^\nu(t))| |d\gamma_\nu/dt| dt \\ &\leq C \int_{t_\nu}^{3s_\nu} \text{dist}(\gamma_\nu(t), \partial D_\nu)^{-1+1/2m'} dt \\ &\leq C s_\nu^{1/2m'} \rightarrow 0 \end{aligned}$$

as $\nu \rightarrow \infty$, which is a contradiction.

Proof of theorem 2.11. It was just shown that $F^\nu(z)$ lies in $B(q, 2\alpha)$ for any $z \in \bar{D}_\nu \cap B(0, \delta)$ if $\delta > 0$ is chosen small enough. Hence (2.21) holds for any $w = F^\nu(z)$ and a similar integration argument as above, then gives the estimate asserted in the theorem with a uniform constant. \square

2.5. Compactness of $f^{-1}(0)$. Recall that to establish theorem 1.1, we were to prove that $f^{-1}(0)$ is compact in M . Suppose to obtain a contradiction that this were false. Then the intersection

$$f^{-1}(0) \cap M \cap \partial B(0, \epsilon) \neq \emptyset.$$

for all $\epsilon > 0$ small. Since the scaled mappings differ from f by a biholomorphic change of coordinates on the domain and the target, the same holds for them as well, i.e.,

$$(F^\nu)^{-1}(F^\nu(0)) \cap \partial D_\nu \cap \partial B(0, \epsilon) \neq \emptyset.$$

Let us show that this property passes to the limit as well, i.e.,

$$F^{-1}(q) \cap \partial D_\infty \cap \partial B(0, \epsilon) \neq \emptyset$$

for all $\epsilon > 0$ small – so small that Theorem 2.11 holds for $r = \epsilon$. Let a^ν be in $(F^\nu)^{-1}(F^\nu(0)) \cap \partial D_\nu \cap \partial B(0, \epsilon)$ and $a^\nu \rightarrow a \in \partial D_\infty \cap \partial B(0, \epsilon)$. Let b be a point in D_∞ near a . Since $D_\nu \rightarrow D_\infty$, there exists an integer N such that for all $\nu > N$, b lies in D^ν and

$$|F(a) - q| \leq |F(a) - F(b)| + |F(b) - F^\nu(b)| + |F^\nu(b) - F^\nu(a^\nu)| + |F^\nu(a^\nu) - q|$$

By taking ν large enough, the second term can be made arbitrarily small by the convergence of F^ν at b and the last term as well by the convergence of $F^\nu(a^\nu) = F^\nu(0)$ to q . The equicontinuity of the family $\{F^\nu\} \cup \{F\}$ given by theorem 2.11, then assures that the same can be done with the third and the first terms, by taking b sufficiently close to a and ν larger if necessary. Thus, $F^{-1}(q)$ is not compact in any neighbourhood of the origin in ∂D_∞ .

On the other hand, by starting from the standpoint of F being a holomorphic mapping between algebraic domains that extends continuously upto a boundary piece Σ of ∂D_∞ , we may apply the theorem of Webster to establish the algebraicity of F . Thus F extends as an analytic set and by the invariance property of Segre varieties (see [19]) as a locally finite to one holomorphic map near $0 \in \partial D_\infty$. Contradiction.

To make this work, we only have to show that F does not map an open piece of ∂D_∞ into the weakly pseudoconvex points on $\partial D'_\infty$. Let us denote by $w(\Gamma)$ the set of all weakly pseudoconvex points of a given smooth hypersurface Γ . Since

$$\partial D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P(z, \bar{z}) = 0\}$$

it follows that

$$w(\partial D_\infty) = (Z \times \mathbb{C}) \cap \partial D_\infty$$

where Z is the real algebraic variety in \mathbb{C}^{n-1} defined by the vanishing of the determinant of the complex Hessian of P in \mathbb{C}^{n-1} . The finite type assumption on ∂D_∞ implies that $w(\partial D_\infty)$ is a real algebraic set of dimension at most $2n - 2$. Recall that

$$F(\Sigma) \subset \partial D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 = 0\},$$

and denote $Q_{2m'}$ by Q for brevity.

Proposition 2.13. *The map F extends to an algebraic map in \mathbb{C}^n .*

Proof. Suppose now that there exists a strictly pseudoconvex point $a \in \partial D_\infty$ such that $F(a)$ is also a strictly pseudoconvex point in $\partial D'_\infty$. Then by [41], F is a smooth CR-diffeomorphism near a and extends locally biholomorphically across a by the reflection principle in [39]. Then Webster's theorem [46] assures us that F is algebraic. So, we may as well let Σ be the set of strictly pseudoconvex points, assume that F maps it into $w(\partial D'_\infty)$ and argue only to obtain a contradiction. As before note that

$$w(\partial D'_\infty) = \{z \in \mathbb{C}^n : \Delta Q(z_1, \bar{z}_1) = 0\} \cap \partial D'_\infty.$$

Then $w(\partial D'_\infty)$ admits a semi-analytic stratification by real analytic manifolds of dimension $2n - 2$ and $2n - 3$. Using the specific form of $\partial D'_\infty$, this can be explicitly described as follows. Let us denote by $V = V(\Delta Q)$ the real algebraic variety in \mathbb{C} defined by the polynomial ΔQ . The set of singular points $\text{Sng}(V)$, near which V may fail to be a smooth curve is finite. Let $a \in w(\partial D'_\infty)$ be such that $\pi_1(a) \in \text{Reg}(V)$ where π_1 is the natural surjection onto the z_1 -axis. Then V is a smooth curve near $\pi_1(a)$ and after a biholomorphic change of coordinates, we may assume that in its vicinity V coincides with $\{\Im z_1 = 0\}$. Note that the fibre of π_1 over a in $w(\partial D'_\infty)$ namely $\pi_1^{-1}(\pi_1(a)) \cap w(\partial D'_\infty)$, is given by

$$\left\{ z \in \mathbb{C}^n : 2\Re z_n + |z_2|^2 + \dots + |z_{n-1}|^2 = -Q(\pi_1(a), \overline{\pi_1(a)}) \right\} \cap \left\{ z \in \mathbb{C}^n : z_1 = \pi_1(a) \right\}$$

which is evidently equivalent to $\partial \mathbb{B}^{n-1} \subset \mathbb{C}^{n-1}$. This description of the fibres of π_1 restricted to $w(\partial D'_\infty)$ persists in a neighbourhood of a and then the real analytic strata $S_{2n-2} \subset w(\partial D'_\infty)$ of dimension $2n - 2$ is locally biholomorphic to $\partial \mathbb{B}^{n-1} \times \{\Im z_1 = 0\}$. The complement of S_{2n-2} in $w(\partial D'_\infty)$, which we will denote by S_{2n-3} is locally biholomorphic to $\partial \mathbb{B}^{n-1} \times \text{Sng}(V)$ and this evidently has dimension $2n - 3$. Observe also that S_{2n-2} is generic since its complex tangent space has dimension $n - 2$ as a complex vector space. Now, if

$$F(\Sigma) \cap S_{2n-2} \neq \emptyset$$

then by the continuity of F , there is an open piece of Σ that is mapped by F into S_{2n-2} . Denote this piece by $\tilde{\Sigma}$. We may assume that $0 \in \tilde{\Sigma}$ and that $F(0) = 0 \in S_{2n-2}$. Using an idea from Lemma 3.2 in [28], let L be a 2 dimensional complex plane that intersects $w(\partial D'_\infty)$ in a totally real submanifold of real dimension 2 – this is possible by the genericity of S_{2n-2} – and this also holds for all translates $L_{a'}$ of L passing through a' in a sufficiently small neighbourhood U' of the origin. We may therefore find a non-negative, strictly plurisubharmonic function $\phi_{a'}$ on U' that vanishes on

$$S_{a'} = L'_{a'} \cap w(\partial D'_\infty) \cap U'.$$

Indeed by a change of coordinates, we may assume that

$$L = \text{span}_{\mathbb{C}} \langle \partial/\partial z_1, \partial/\partial z_n \rangle = \{z_2 = \dots = z_{n-1} = 0\}$$

and then

$$\phi_{a'} = |z_2 - a'_2|^2 + \dots + |z_{n-1} - a'_{n-1}|^2 + |\Im z_1|^2 + (r'_\infty(z, \bar{z}))^2$$

furnishes an example. By the continuity of F and the openness of $\tilde{\Sigma}$ in ∂D_∞ , we can pick $b \in D_\infty$ so near the origin that $F(b) \in U'$ and $\partial A_b \subset \tilde{\Sigma}$ where

$$A_b = \{z \in \mathbb{C}^n : z_n = b_n\} \cap F^{-1}(L_{F(b)})$$

Since the pull-back of an analytic set under a holomorphic map is again analytic of no lesser dimension, A_b is a positive dimensional analytic set. Also, $b \in A_b$ and $F(\partial A_b) \subset S_{F(b)}$. Therefore, $\psi_b = \phi_{F(b)} \circ F$ is a non-negative, plurisubharmonic function on A_b that vanishes on ∂A_b . By the maximum principle $\psi_b \equiv 0$ on A_b which implies that F maps all of A_b into $S_{F(b)}$. Since F maps D_∞ into D'_∞ , this is a contradiction.

To finish, note that the the remaining possibility is $F(\Sigma) \subset S_{2n-3}$. The above argument can be repeated in this case as well – we will only need to replace $|\Im z_1|^2$ as a subharmonic function vanishing on a curve-segment of $\text{Reg}(V)$ by $|p(z_1)|^2$ where p is a holomorphic polynomial that vanishes on the finite set $\text{Sng}(V)$. \square

3. PROOF OF THEOREM 1.3

We begin with the scaling template for (D, D', f) in section 2.1, maintaining as far as possible the notations therein. For clarity and completeness, let us briefly describe the scaling of D' which is simpler this time as M' is strongly pseudoconvex. As before, let $p^\nu = (0, -\delta_\nu) \in D$ and note that $p^\nu = f(p^\nu)$ converges to the origin which is a strongly pseudoconvex point on $\partial D'$. Let $w^\nu \in \partial D'$ be such that

$$|w^\nu - p^\nu| = \text{dist}(p^\nu, \partial D') = \gamma_\nu$$

Furthermore, since $\partial D'$ is strongly pseudoconvex near the origin, we may choose a strongly plurisubharmonic function in a neighbourhood of the origin that serves as a defining function for D' . Arguing as in Section 2.1, it follows that f preserves the distance to the boundary, i.e.,

$$\delta_\nu \approx \gamma_\nu$$

for $\nu \gg 1$. For each w^ν , lemma 2.2 in [40] provides a degree two polynomial automorphism of \mathbb{C}^n that firstly, transfers w^ν and the normal to $\partial D'$ there, to the origin and the $\Re z_n$ -axis respectively and secondly, ensures that the second order terms in the Taylor expansion of the defining function $r' \circ (g^\nu)^{-1}$ of the domain $g^\nu(D')$ constitute a hermitian form that coincides with the standard one i.e., $|z_1|^2 + \dots + |z_{n-1}|^2$, upon restriction to the complex tangent space. Define the dilations

$$B^\nu(z, z_n) = (\gamma_\nu^{-1/2} z, \gamma_\nu^{-1} z_n)$$

and note that the scaled domains $D'_\nu = (B^\nu \circ g^\nu)^{-1}(D')$ are defined by

$$\gamma_\nu^{-1} r'_\nu(z) = 2\Re(g^\nu(z))_n + |(g^\nu(z))_1|^2 + \dots + |(g^\nu(z))_{n-1}|^2 + O(\gamma_\nu^{1/2}).$$

These converge in the Hausdorff metric to

$$D'_\infty = \mathbb{H} = \{z \in \mathbb{C}^n : 2\Re z_n + |z_1|^2 + \dots + |z_{n-1}|^2 < 0\},$$

which is the unbounded manifestation of the ball, in view of the fact that the g^ν 's converge uniformly on compact subsets of \mathbb{C}^n to the identity map.

Standard arguments as in [40] show that the scaled maps

$$F^\nu = B^\nu \circ g^\nu \circ f \circ (T^\nu)^{-1}$$

converge to a map $F : D_\infty \rightarrow \overline{\mathbb{H}}$. If some point z_0 of D_∞ is sent by F to $w_0 \in \partial\mathbb{H} \cup \infty$, then composing F with a local peak function at w_0 , we get a function holomorphic on a neighbourhood of z_0 and peaking precisely at z_0 . By the maximum principle, $F(z) \equiv w_0$. However, $F^\nu('0, -1) = B^\nu \circ g^\nu \circ f \circ T^\nu('0, -1) = B^\nu \circ g^\nu(f(p^\nu)) = B^\nu('0, -\gamma_\nu) = ('0, -1)$. Hence, $F('0, -1) = ('0, -1) \neq w_0$. Thus, F maps D_∞ into D'_∞ and is again as in section 2, a non-degenerate, locally proper map extending continuously upto the boundary in a neighbourhood of the origin. By theorem 2.1 in [19], F extends holomorphically across the origin. By composing with a suitable automorphism of the ball, we may also assume $F(0) = 0$. Then, $\Re(F_n(z))$ is a pluriharmonic function that is negative on D_∞ and attains a maximum at the origin. So, by the Hopf lemma, we must have $\alpha = \partial(\Re F_n)/\partial x_n(0) > 0$ which combined with the fact that DF preserves the complex tangent space and thereby the complex normal at the origin (to the hypersurfaces ∂D_∞ and $\partial D'_\infty$, which themselves correspond under F near the origin), implies that

$$F_n(z) = \alpha z_n + g(z)$$

for some holomorphic function g with $g(z) = o(|z|)$. Now, let us compare the two defining functions for D_∞ , near the origin:

$$2\Re(F_n(z)) + |F_1(z)|^2 + \dots + |F_{n-1}(z)|^2 = h(z, \bar{z})(2\Re z_n + P('z, ' \bar{z}))$$

for a non-vanishing real analytic function $h(z, \bar{z})$. Contemplate a weighted homogeneous expansion of the above equation with respect to the weight $(1/m_n, \dots, 1/m_2)$ given by the multitype of ∂D_∞ . Note firstly that on the left, pluriharmonic terms arise precisely from $\Re(F_n(z))$. Next, note that the lowest possible weight for any term on the right is one and the non-pluriharmonic component of this weight is $h(0)P('z, ' \bar{z})$. What this means for the left, is that each F_j must expand as

$$F_j(z) = P_j('z) + (\text{terms of weight } > 1/2)$$

where each P_j is either weighted homogeneous of weight $1/2$ or identically zero and

$$(3.1) \quad h(0)P('z, ' \bar{z}) = |P_1('z)|^2 + \dots + |P_{n-1}('z)|^2$$

Clearly, all the P_j 's cannot be zero as P is non-zero. In fact, the finite type character of ∂D_∞ forces all of them to be non-zero, as follows. After a rearrangement if necessary, assume that $P_j \in \mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_{n-1}]$ is non-zero precisely when $1 \leq j \leq m \leq n-1$. Then the common zero set $V \subset \mathbb{C}^{n-1}$, of these P_j 's gives rise to the complex analytic variety $i(V)$ in ∂D_∞ where $i : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^n$ is the natural inclusion. The finite type constraint compels this variety and needless to say V , to be discrete. Furthermore, the weighted homogeneity reduces it to $\{0\}$: If (z_1, \dots, z_{n-1}) is a non-trivial zero of the P_j 's and $t \in \mathbb{C}$, then

$$P_j(e^{t/m_n} z_1, \dots, e^{t/m_2} z_{n-1}) = e^{t/2} P_j(z_1, \dots, z_{n-1}) = 0$$

for all $1 \leq j \leq m$ and so the entire curve defined by

$$\gamma(t) = (e^{t/m_n} z_1, \dots, e^{t/m_2} z_{n-1})$$

lies inside V . Now, consider the ideal I generated by these polynomials P_1, \dots, P_m , which is a \mathbb{C} -algebra whose transcendence degree cannot exceed $m \leq n - 1$. On the other hand, by the Nullstellensatz, for a large integer N the algebraically independent monomials, z_1^N, \dots, z_{n-1}^N must all lie in I , forcing its transcendence degree over \mathbb{C} and hence m to equal $n - 1$. The upshot therefore, is that P is the squared norm of a weighted homogeneous polynomial endomorphism \hat{P} of \mathbb{C}^{n-1} with $\hat{P}^{-1}(0) = 0$. Now, put $z_2 = \dots = z_{n-1} = 0$ in (3.1). This cannot reduce the left hand side there to zero, otherwise the z_1 -axis will lie inside the zero set of P and hence in ∂D_∞ . What this means for the right hand side of (3.1), is that pure z_1 -term(s) must occur. Since all polynomials therein are homogeneous of same weight, we conclude that every pure z_1 -term there must only be of the form $c|z_1|^{m_n}$ for some $c > 0$. Needless to say, the same holds for all the other variables z_j for $2 \leq z_j \leq n - 1$, as well and we have the last statement of theorem 1.3, namely

$$P(z, \bar{z}) = c_1|z_1|^{m_n} + c_2|z_2|^{m_{n-1}} + \dots + c_{n-1}|z_{n-1}|^{m_2} + \text{mixed terms}$$

with all c_j 's being positive and the mixed terms comprising of weight one monomials in $'z, \bar{z}$ each of which is annihilated by at least one of the natural quotient maps $\mathbb{C}['z, \bar{z}] \rightarrow \mathbb{C}['z, \bar{z}]/(z_j \bar{z}_k)$, $1 \leq j, k \leq n - 1$ where $j \neq k$.

4. PROOF OF THEOREM 1.4

Proof. As before by the lower semi-continuity of rank, $\partial D'$ is of rank at least $n - 2$ in a neighbourhood $\Gamma' \subset \partial D'$ of p' which we may assume also to be of finite type and pseudoconvex, consequently regular. Let us apply Theorem A of [13] to the proper holomorphic correspondence $f^{-1} : D' \rightarrow D$. The hypothesis on $cl_{f^{-1}}(\Gamma')$ there holds, since ∂D is globally regular. Therefore by that theorem, f^{-1} extends continuously upto Γ' as a proper correspondence. However, we are not certain of the splitting of f^{-1} near p into branches – for that p we will have to lie away from the branch locus of f^{-1} . Nevertheless, there exist neighbourhoods U' of p' and U of p and a (local) correspondence

$$f_{loc}^{-1} : U' \cap D' \rightarrow U \cap D,$$

extending continuously upto the boundary such that the graph of f_{loc}^{-1} is contained in that of f^{-1} and $cl_{f_{loc}^{-1}}(p') = \{p\}$ where the last condition comes from the fact that f is finite to one upto the boundary, by that theorem again. Assume that both $p = 0$ and $p' = 0$ and choose a sequence $p'' = (0, -\delta_\nu) \in D' \cap U'$ on the inner normal approaching the origin. Since $f : D \rightarrow D'$ is proper and $0 \in cl_f(0)$ there exists a sequence $p^\nu \in D$ with $p^\nu \rightarrow 0$ such that $f(p^\nu) = p''$. Moreover, by the continuity of f_{loc}^{-1} upto the boundary and the condition $cl_{f_{loc}^{-1}}(0) = \{0\}$, we may assume after shrinking U, U' if necessary, that $p^\nu \in D \cap U$ with $f_{loc}^{-1}(p^\nu) = \{p''\}$. Now scale D with respect to $\{p^\nu\}$ and D' with respect to $\{p''\}$ – to scale D' , we only consider the dilations

$$T^\nu(z_1, \dots, z_n) = ((\tau_1^\nu)^{-1}z_1, (\tau_2^\nu)^{-1}, \dots, (\tau_n^\nu)^{-1}z_n)$$

where $\tau_1^\nu = \tau(0, \delta_\nu)$, $\tau_j^\nu = \delta_\nu^{1/2}$ for $2 \leq j \leq n - 1$ and $\tau_n^\nu = \delta_\nu$, while for D we use the composition $B^\nu \circ g^\nu$ where g^ν and B^ν are as in section 3. As before, the limiting domains for $D_\nu = (B^\nu \circ g^\nu)(U \cap D)$ and $D'_\nu = T^\nu(U' \cap D')$ are the ball \mathbb{B}^n and

$$D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\},$$

respectively, where this time $Q_{2m'}$ is homogeneous of degree $2m'$ and coincides with the polynomial of this degree in the (homogeneous) Taylor expansion of the defining function r' for $\partial D'$ near the origin. Normality of the scaled mappings

$$F^\nu = T^\nu \circ f \circ (B^\nu \circ g^\nu)^{-1}$$

follows as before by proposition 2.1. But section 2 can no longer guarantee nondegeneracy of a limit map, owing to the lack of a clear cut boundary distance conservation property of f . However, the existence of the inverse as a proper correspondence paves the way for a different approach. To begin with, recall from [35], the notion of normality for correspondences and theorem 3 therein, the version of Montel's theorem for proper holomorphic correspondences with varying domains and ranges. Let K_μ be an exhaustion of D'_∞ by compact subsets containing $(0, -1)$. To establish the normality of scaled correspondences

$$(f_{loc}^{-1})^\nu = (B^\nu \circ g^\nu) \circ f_{loc}^{-1} \circ (T^\nu)^{-1},$$

it suffices by theorem 3 of [35] to show that $(f_{loc}^{-1})^\nu(K_\mu) \Subset D_\infty$ for each $\mu \in \mathbb{N}$. To this end, fix a K_μ , let $(\tilde{f}_{loc}^{-1})^\nu$ denote the correspondence $f_{loc}^{-1} \circ (T^\nu)^{-1}$ and note that $(\tilde{f}_{loc}^{-1})^\nu(K_\mu)$ is connected (since any of its components must contain p^ν). Now (viewing the origin in ∂D as the sequence p^ν that approaches it from the interior of D), we recall the Schwarz lemma for correspondences given by theorem 1.2 of [45], which assures us that the images of K_μ under these correspondences $(\tilde{f}_{loc}^{-1})^\nu$, will be contained in Kobayashi balls about p^ν of a fixed size, i.e., there exists $R > 0$ independent of ν such that

$$(\tilde{f}_{loc}^{-1})^\nu(K_\mu) \subset B_D^K(p^\nu, R),$$

where $B_\Omega^K(p, R)$ denotes the Kobayashi ball centered at $p \in \Omega$ of radius R , for any given domain Ω . Then, since biholomorphisms preserve Kobayashi balls,

$$(f_{loc}^{-1})^\nu(K_\mu) \subset (B^\nu \circ g^\nu)((\tilde{f}_{loc}^{-1})^\nu(K_\mu)) \subset (B^\nu \circ g^\nu)(B_D^K(p^\nu, R)) = B_{D_\nu}^K((0, -1), R)$$

while $B_{D_\nu}^K((0, -1), R) \subset B_{\mathbb{B}^n}^K((0, -1), R+1)$ for all ν large by lemma 4.4 of [37], giving the stability of the images of K_μ under scaling. Noting that $((0, -1), (0, -1)) \in \text{Graph}(f_{loc}^{-1})^\nu$ for all ν , we conclude that $\{(f_{loc}^{-1})^\nu\}$ must admit a subsequence that converges to a correspondence that is inverse to F , where $F : \mathbb{B}^n \rightarrow D'_\infty$ is a limit of $\{F^\nu\}$ (which means that the composition of these correspondences **contains** the graph of the identity). By [13], both F and the inverse correspondence F^{-1} extend continuously upto the respective boundaries and in particular F is finite to one on the boundary. Then, F is smooth upto the boundary by [9], preserves boundaries and then is algebraic again as in section 2.6. and thereafter by [5] extends holomorphically past the boundary. Now recall that the determinant of the Levi forms are related as

$$\lambda_{\mathbb{B}^n}(z) = \lambda_{D'_\infty}(F(z)) |J_F(z)|^2$$

for all $z \in \partial \mathbb{B}^n$, which readily gives the strict pseudoconvexity of $\partial D'_\infty$. In particular therefore, $Q_{2m'}(z_1, \bar{z}_1)$ must be $|z_1|^2$ which gives the strict pseudoconvexity of p' in $\partial D'$. \square

Proof of corollary 1.5. By the proof of Theorem 1.1 in section 2, we have that $f^{-1}(f(p))$ is compact in M and as described in [9], it is possible to choose neighbourhoods U of p and U' of p' in \mathbb{C}^n such that if D and D' are the pseudoconvex sides of $U \cap M$ and $U' \cap M'$ respectively then f extends to be a proper map from D into D' , putting us in the situation of theorem 1.4. \square

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SOME REGULARITY THEOREMS FOR CR MAPPINGS

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ABSTRACT. The purpose of this article is to study Lipschitz CR mappings from an h -extendible (or semi-regular) hypersurface in \mathbb{C}^n . Under various assumptions on the target hypersurface, it is shown that such mappings must be smooth. A rigidity result for proper holomorphic mappings from strongly pseudoconvex domains is also proved.

1. INTRODUCTION

The purpose of this note is to prove regularity results for Lipschitz CR mappings from h -extendible hypersurfaces in \mathbb{C}^n .

Theorem 1.1. *Let $f : M \rightarrow M'$ be a non-constant Lipschitz CR mapping between C^∞ smooth pseudoconvex finite type (in the sense of D'Angelo) hypersurfaces in \mathbb{C}^n . Let $p \in M$ and $p' = f(p) \in M'$. Assume that M is h -extendible at p , and that there is an open neighbourhood $U' \subset \mathbb{C}^n$ of p' and a C^∞ smooth defining function $r'(z')$ for $U' \cap M'$ in U' which is plurisubharmonic on the pseudoconvex side of M' near p' . If the Levi rank of M' at p' is at least $n - 2$ then f is C^∞ smooth in a neighbourhood of p .*

This theorem, which is purely local in nature, is motivated by the regularity and rigidity results for CR mappings obtained in [20], [21], [22] and [32]. Let $B(p, \epsilon), B(p', \epsilon')$ be small open balls around p, p' respectively. The pseudoconvex side of M near p is that component of $B(p, \epsilon) \setminus M$ which is a pseudoconvex domain for small $\epsilon > 0$. Denote this by $B^-(p, \epsilon)$ while the other component (the pseudoconcave part) will be denoted by $B^+(p, \epsilon)$. The same practice will be followed while referring to the respective components of the complement of M' in $B(p', \epsilon')$. Since M and M' are both pseudoconvex and of finite type near p and p' respectively, it follows that f admits a holomorphic extension to the pseudoconvex side of M near p and this extension (which will still be denoted by f) maps the pseudoconvex side of M near p into the pseudoconvex side of M' near p' . In fact, since $f : M \rightarrow M'$ is assumed to be Lipschitz, it follows that the extension is also Lipschitz on $B^-(p, \epsilon) \cup M$ near p . Fix $\epsilon, \epsilon' > 0$ so that

$$f : B^-(p, \epsilon) \rightarrow B^-(p', \epsilon')$$

is holomorphic, f extends Lipschitz continuously up to $M \cap B(p, \epsilon)$ and satisfies $f(M \cap B(p, \epsilon)) \subset M' \cap B(p', \epsilon')$. For brevity, let D and D' denote the pseudoconvex domains $B^-(p, \epsilon)$ and $B^-(p', \epsilon')$ respectively so that $M \cap B(p, \epsilon)$ and $M' \cap B(p', \epsilon')$ are smooth open finite type pieces on their boundaries. We may also assume that $\epsilon' > 0$ is small enough so that the defining function $r'(z') \in C^\infty(B(p', \epsilon'))$ and $D' = \{r'(z') < 0\}$ where $r'(z')$ is plurisubharmonic in D' . Note that while this condition holds for strongly pseudoconvex

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and convex finite type domains in \mathbb{C}^n , not all (see [24]) pseudoconvex domains satisfy this condition. Now observe that the map $f : D \rightarrow D'$ is not known to be proper though it does extend continuously up to $M \cap B(p, \epsilon)$. A sufficient condition for f to be proper from D onto D' was considered by Bell-Catlin in [9] – namely, if $f^{-1}(p') = f^{-1}(f(p))$ is compact in $M \cap B(p, \epsilon)$ then it is possible to choose D, D' in such a way that f extends as a proper holomorphic mapping between these domains. Once this is established, the C^∞ smoothness of f is a consequence of the techniques in [9]. The proof of theorem 1.1 therefore consists of showing that the fibre over p' is compact in $M \cap B(p, \epsilon)$. To do this, we adapt the method of uniform scaling from [20],[21] and [22]. The domains D, D' and the map f are scaled by composing f with a suitable family of automorphisms of \mathbb{C}^n that enlarge these domains near p, p' respectively. This produces a scaled family $f^\nu : D_\nu \rightarrow D'_\nu$ of maps between scaled domains. The scaled domains D_ν converge in the Hausdorff sense to a model domain of the form

$$(1.1) \quad D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P('z, '\bar{z}) < 0\}$$

where $'z = (z_1, z_2, \dots, z_{n-1})$ and $P('z, '\bar{z})$ is a weighted homogeneous (the weights being determined by the Catlin multitype of ∂D at p) real valued plurisubharmonic polynomial without pluriharmonic terms of total weight 1, while the domains D'_ν converge to

$$(1.2) \quad D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\}.$$

Here $2m'$ is the type of $\partial D'$ at p' and $Q_{2m'}(z_1, \bar{z}_1)$ is a subharmonic polynomial of degree at most $2m'$ without harmonic terms. D'_∞ is manifestly of finite type while D_∞ also has the same property since p is assumed to be h -extendible. The local geometry of a smooth pseudoconvex finite type hypersurface whose Levi rank is at least $n - 2$ guarantees the existence of special polydiscs around points sufficiently close to the hypersurface on which the defining function does not change by more than a prescribed amount – these are the analogues of Catlin's bidiscs and have been considered earlier in [18]. The holomorphic mappings $f_\nu : D_\nu \rightarrow D'_\nu$ are shown to form a normal family by using these polydiscs as in [43]; the limit map $F : D_\infty \rightarrow D'_\infty$ is therefore holomorphic. The assumptions that f is Lipschitz and that the defining function of $\partial D'$ is plurisubharmonic on D' force F to be non-degenerate. Furthermore, since f extends Lipschitz continuously up to M near p , it is natural to expect that F imbibes some regularity near $0 \in \partial D_\infty$. This indeed happens and it is possible to show that F is Hölder continuous up to ∂D_∞ near the origin. The main ingredient needed to do this is a stable rate of blow up of the Kobayashi metric on D'_ν for all $\nu \gg 1$ and this follows by analyzing the behaviour of analytic discs in D'_ν whose centers lie close to the origin. In particular, theorem 2.3 provides a stable lower bound for the Kobayashi metric in D'_ν near the origin using ideas from [43]. Once F is known to be Hölder continuous up to ∂D_∞ near the origin, Webster's theorem ([46]) implies that F must be algebraic. Moreover, if f has a noncompact fibre over p' , it can be shown that the same must hold for that of F over $F(0)$. This violates the invariance property of Segre varieties associated to ∂D_∞ and $\partial D'_\infty$ and proves the compactness of the fibre of f over $p' \in \partial D'$.

It is known that smooth convex finite type hypersurfaces can also be scaled and moreover the stability of the Kobayashi metric on the scaled domains is also understood (see [21]). Hence the same line of reasoning yields the following:

Corollary 1.2. *With M as in theorem 1.1, let $M' \subset \mathbb{C}^n$ be a smooth convex finite type hypersurface and $f : M \rightarrow M'$ a Lipschitz CR mapping. As before, let $p \in M$ and $p' = f(p) \in M'$. Then f is C^∞ smooth in a neighbourhood of p .*

More can be said when at least one of the hypersurfaces is strongly pseudoconvex. We first consider the following local situation – let $M \subset \mathbb{C}^n$ be a C^∞ smooth pseudoconvex finite type hypersurface and let $p \in M$ be an h -extendible point. Let the Catlin multitype of M at p be $(1, m_2, \dots, m_n)$ where the m_i 's form an increasing sequence of even integers. Then there exists a holomorphic coordinate system around $p = 0$ in which the defining function for M takes the form:

$$r(z) = 2\Re z_n + P('z, 'z) + R(z)$$

where $P('z, 'z)$ is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ -homogeneous plurisubharmonic polynomial of total weight one without pluriharmonic terms and the error $R(z)$ has weight strictly bigger than one. As usual, if $J = (j_1, j_2, \dots, j_{n-1})$ is a multiindex of length $n - 1$, then $'z^J$ denotes the monomial $z_1^{j_1} z_2^{j_2} \dots z_{n-1}^{j_{n-1}}$ and $'z^J = \bar{z}_1^{j_1} \bar{z}_2^{j_2} \dots \bar{z}_{n-1}^{j_{n-1}}$.

Theorem 1.3. *Let $f : M \rightarrow M'$ be a nonconstant Lipschitz CR mapping between real hypersurfaces in \mathbb{C}^n . Fix $p \in M$ and $p' = f(p) \in M'$. Suppose that M is C^∞ smooth pseudoconvex and of finite type near p and that M' is C^2 strongly pseudoconvex near p' . If M is h -extendible at p , then the weighted homogeneous polynomial in the defining function for M near $p = 0$ can be expressed as*

$$P('z, 'z) = |P_1('z)|^2 + |P_2('z)|^2 + \dots + |P_{n-1}('z)|^2$$

where each $P_i('z)$ for $1 \leq i \leq n - 1$ is a weighted holomorphic polynomial of total weight $1/2$. Moreover, the algebraic variety

$$V = \{ 'z \in \mathbb{C}^{n-1} : P_1('z) = P_2('z) = \dots = P_{n-1}('z) = 0 \}$$

contains $'0 \in \mathbb{C}^{n-1}$ as an isolated point. In particular, there exist constants $c_j > 0$ for $1 \leq j \leq n - 1$ such that

$$P('z, 'z) = c_1 |z_1|^{m_n} + c_2 |z_2|^{m_{n-1}} + \dots + c_{n-1} |z_{n-1}|^{m_2} + \text{mixed terms}$$

where the phrase ‘mixed terms’ denotes a sum of weight one monomials annihilated by at least one of the natural quotient maps $\mathbb{C}['z, 'z] \rightarrow \mathbb{C}['z, 'z] / (z_j \bar{z}_k)$ for $1 \leq j, k \leq n - 1$, $j \neq k$.

As in theorem 1.1, the limit map $F : D_\infty \rightarrow D'_\infty$ is holomorphic, nonconstant and hence algebraic. Since $D'_\infty \simeq \mathbb{B}^n$, it is possible to show that F extends holomorphically across $0 \in \partial D_\infty$ and $F(0) = 0' \in \partial D'_\infty$. The explicit description of the weighted homogeneous polynomial $P('z, 'z)$ follows from working with the extended mapping near the origin. On the other hand, it is also natural to consider the case of CR mappings from strongly pseudoconvex hypersurfaces and we have both a global and a local version – in the spirit of theorem 1.1, for this case. First recall that a domain $D \subset \mathbb{C}^n$ is said to be regular at $p \in \partial D$ if there is a pair of open neighbourhoods $V \subset U$ of p , constants $M > 0$, $0 < \alpha \leq 1$ and $\beta > 1$ such that for any $\zeta \in V \cap \partial D$, there is a function ϕ_ζ which is continuous on $U \cap \bar{D}$, plurisubharmonic on $U \cap D$ and satisfies

$$-M |z - \zeta|^\alpha \leq \phi_\zeta(z) \leq -|z - \zeta|^\beta$$

for all $z \in U \cap \overline{D}$. It is known that the class of regular points includes open pieces of strongly pseudoconvex boundaries, smooth weakly pseudoconvex finite type pieces in \mathbb{C}^2 , those that are smooth convex finite type in \mathbb{C}^n (see [30]) and finally smooth pseudoconvex finite type boundaries in \mathbb{C}^n (see [17]). D is said to be regular if each of its boundary points is regular.

Theorem 1.4. *Let $D \subset \mathbb{C}^n$ be a bounded regular domain, $D' \subset \mathbb{C}^n$ a possibly unbounded domain and $f : D \rightarrow D'$ a proper holomorphic mapping. Let $p \in \partial D$ be a C^2 strongly pseudoconvex point and $p' \in \partial D'$ be such that the boundary $\partial D'$ is C^∞ smooth pseudoconvex and of finite type near p' . Suppose that the Levi rank of $\partial D'$ at p' is $n - 2$ and assume that $p' \in cl_f(p)$, the cluster set of p . Then p' is also a strongly pseudoconvex point.*

This fits in the paradigm, observed earlier by many other authors (for example [25]), that a proper mapping does not increase the type of a boundary point. Note furthermore that there are no other assumptions on the defining function for $\partial D'$ near p' as in theorem 1.1 except the Levi rank condition. The difficulty created by the lack of this assumption as explained above is circumvented by the global properness of f – indeed, it is possible to scale f to get a holomorphic limit $F : \mathbb{B}^n \rightarrow D'_\infty$ where D'_∞ is as in (1.2). Now using the fact that $p \in \partial D$ is strongly pseudoconvex and hence regular, it is possible to peel off a local correspondence from the global one $f^{-1} : D' \rightarrow D$ that extends continuously up to $\partial D'$ near p' and contains p in its cluster set by [13]. This local correspondence can be scaled, using the Schwarz lemma for correspondences from [45] and the behaviour of the scaled balls in the Kobayashi metric from [37]. This gives a well defined correspondence from D'_∞ with values in \mathbb{B}^n and this turns out to be the inverse for F . Thus $F : \mathbb{B}^n \rightarrow D'_\infty$ is proper and this is sufficient to conclude that $p' \in \partial D'$ must be strongly pseudoconvex.

Corollary 1.5. *Let $f : M \rightarrow M'$ be a nonconstant Lipschitz CR mapping between real hypersurfaces in \mathbb{C}^n . Fix $p \in M$ and $p' = f(p) \in M'$. Suppose that M is C^2 strongly pseudoconvex near p and that there is an open neighbourhood $U' \subset \mathbb{C}^n$ of p' and a C^∞ smooth defining function $r'(z')$ for $U' \cap M'$ in U' which is plurisubharmonic on the pseudoconvex side of M' near p' . If the Levi rank of M' at p' is at least $n - 2$ then p' is a strongly pseudoconvex point.*

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2. PROOF OF THEOREM 1.1

Consider a smooth pseudoconvex finite type hypersurface $M \subset \mathbb{C}^n$. Associated to each $p \in M$ are two well known invariants: one is the D'Angelo type

$$\Delta(p) = (\Delta_n(p), \Delta_{n-1}(p), \dots, \Delta_1(p))$$

where the integer $\Delta_q(p)$ is the q -type of M at p and is a measure of the maximal order of contact of q dimensional varieties with M at p . To recall the definition (see [23]), let r be a local defining function for M near p and let $\tilde{r}(z) = r(z + p)$. Then for $1 \leq q \leq n$,

$$\Delta_q(p) = \inf_L \sup_\tau \{ \nu(\tilde{r} \circ L \circ \tau) / \nu(\tau) \}$$

where the infimum is taken over all linear embeddings $L : \mathbb{C}^{n-q+1} \rightarrow \mathbb{C}^n$ and the supremum is taken over all germs of holomorphic curves $\tau : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{n-q+1}, 0)$ mapping the

origin in \mathbb{C} to the origin in \mathbb{C}^{n-q+1} and $\nu(f)$ denotes the order of vanishing of f at the origin. The smoothness of M at p implies that $\Delta_n(p) = 1$ and it can be seen that $2 \leq \Delta_{n-1}(p) \leq \Delta_{n-2}(p) \leq \dots \leq \Delta_1(p) < \infty$.

To quickly recall the Catlin multitype of M at p (see [16]), let Γ_n be the collection of n -tuples of reals $m = (m_1, m_2, \dots, m_n)$ such that $0 < m_1 \leq m_2 \leq \dots \leq m_n \leq \infty$. Order Γ_n lexicographically. An element m of Γ_n is called distinguished provided there is a holomorphic coordinate system $w = \phi(z)$ around p with mapped to the origin such that if

$$(\alpha_1 + \beta_1)/m_1 + (\alpha_2 + \beta_2)/m_2 + \dots + (\alpha_n + \beta_n)/m_n < 1$$

then $D^\alpha \bar{D}^\beta r \circ \phi^{-1}(0) = 0$; here $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are n -tuples and D^α and \bar{D}^β are the partial derivatives

$$\partial^{|\alpha|} / \partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_n^{\alpha_n} \text{ and } \partial^{|\beta|} / \partial \bar{z}_1^{\beta_1} \partial \bar{z}_2^{\beta_2} \dots \partial \bar{z}_n^{\beta_n}$$

respectively. The Catlin multitype $\mathcal{M}(p) = (m_1, m_2, \dots, m_n)$ of M at p is defined to be the largest amongst all distinguished elements. Since $r(z)$ is a smooth defining function for M near p , it can be seen that the first entry in $\mathcal{M}(p)$ is always one. If $\mathcal{M}(p)$ is finite, i.e., $m_n < \infty$, then there is a coordinate system around $p = 0$ such that the defining function is of the form

$$(2.1) \quad r(z) = 2\Re z_n + P('z, ' \bar{z}) + R(z)$$

where $P('z, ' \bar{z})$ is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ homogeneous polynomial of total weight one which is plurisubharmonic and does not contain pluriharmonic terms and

$$(2.2) \quad |R(z)| \lesssim (|z_1|^{m_n} + |z_2|^{m_{n-1}} + \dots + |z_n|^{m_1})^\gamma$$

for some $\gamma > 1$. The homogeneity of $P('z, ' \bar{z})$ with total weight one mentioned above means that for $t \geq 0$, $P \circ \pi_t('z) = tP('z, ' \bar{z})$ where

$$\pi_t(z_1, z_2, \dots, z_{n-1}) = (t^{1/m_n} z_1, t^{1/m_{n-1}} z_2, \dots, t^{1/m_2} z_{n-1}).$$

Other total weights $\mu > 0$ occur when t^μ (instead of t) can be factored from $P \circ \pi_t('z)$. The plurisubharmonicity of $P('z, ' \bar{z})$ implies that each m_k for $2 \leq k \leq n$ must be even. Thus the variable z_i is assigned a weight of $1/m_{n-i+1}$ for $1 \leq i \leq n-1$ and by definition the weight of the monomial $'z^J ' \bar{z}^K$ is

$$(j_1 + k_1)/m_n + (j_2 + k_2)/m_{n-1} + \dots + (j_{n-1} + k_{n-1})/m_2$$

for $(n-1)$ -multiindices $J = (j_1, j_2, \dots, j_{n-1})$ and $K = (k_1, k_2, \dots, k_{n-1})$. A basic relation between $\Delta(p)$ and $\mathcal{M}(p)$ proved in [16] is that $m_{n+1-q} \leq \Delta_q$ for all $1 \leq q \leq n$.

Call $p \in M$ an h -extendible (or semi-regular) point if $\mathcal{M}(p)$ is finite and $\Delta(p) = \mathcal{M}(p)$. This happens if and only if (see [47]) there is a $(1/m_n, 1/m_{n-1}, \dots, 1/m_2)$ homogeneous C^1 smooth real function $a('z)$ on $\mathbb{C}^{n-1} \setminus \{0\}$ such that $a('z) > 0$ whenever $'z \neq 0$ and $P('z, ' \bar{z}) - a('z)$ is plurisubharmonic on \mathbb{C}^{n-1} and among other things, this is equivalent to the model domain D_∞ (as in (1.1)) being of finite type. We shall henceforth assume that $p = 0$ and $p' = 0'$ and that the respective defining functions $r(z)$ and $r'(z')$ satisfy $\partial r / \partial z_n(0) \neq 0$ and $\partial r' / \partial z'_n(0') \neq 0$. The holomorphic map $f : D \rightarrow D'$ is not necessarily proper, but Hölder continuous with exponent $\delta \in (0, 1)$ on \bar{D} near $0 \in \partial D$ by [42] and $f(0) = 0'$. Thus the assumption that f is Lipschitz is stronger than what is a priori known.

2.1. The Scaling Method applied to (D, D', f) . For $z \in D$ close to the origin, note that

$$\text{dist}(z, \partial D) \lesssim \text{dist}(f(z), \partial D') \lesssim \text{dist}(z, \partial D)$$

where the inequality on the right follows since f admits a Lipschitz extension to D near the origin, while the left inequality follows by applying the Hopf lemma to $r' \circ f(z)$ which is a negative plurisubharmonic function on D . To scale D , choose a sequence of points $p^\nu = (0, -\delta_\nu)$ in D along the inner normal at the origin, where $\delta_\nu > 0$ and $\delta_\nu \searrow 0$. Let T^ν be the dilation defined by

$$T^\nu : (z_1, z_2, \dots, z_{n-1}, z_n) \mapsto (\delta_\nu^{-1/m_n} z_1, \delta_\nu^{-1/m_{n-1}} z_2, \dots, \delta_\nu^{-1/m_2} z_{n-1}, \delta_\nu^{-1} z_n)$$

and note that $T^\nu(p^\nu) = (0, -1)$ while the domains $D_\nu = T^\nu(D)$ are defined by

$$r_\nu = \delta_\nu^{-1} r \circ (T^\nu)^{-1}(z) = 2\Re z_n + P(z', \bar{z}') + \delta_\nu^{-1} R \circ (T^\nu)^{-1}(z)$$

where

$$|\delta_\nu^{-1} R \circ (T^\nu)^{-1}(z)| \lesssim \delta_\nu^{\gamma-1} (|z_1|^{m_n} + |z_2|^{m_{n-1}} + \dots + |z_{n-1}|^{m_2} + |z_n|)^\gamma$$

by (2.1). On each compact set in \mathbb{C}^n this error term converges to zero since $\gamma > 1$ and hence the sequence of domains D_ν converges in the Hausdorff metric to

$$D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P(z', \bar{z}') < 0\}.$$

Let $r_\infty(z) = 2\Re z_n + P(z', \bar{z}')$. To scale D' recall that by [18], for each ζ near $0' \in \partial D'$ there is a unique polynomial automorphism $\Phi_\zeta(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\Phi_\zeta(\zeta) = 0$ such that

$$\begin{aligned} r(\Phi_\zeta^{-1}(z)) &= r(\zeta) + 2\Re z_n + \sum_{\substack{j+k \leq 2m \\ j, k > 0}} a_{jk}(\zeta) z_1^j \bar{z}_1^k + \sum_{\alpha=1}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq m \\ j, k > 0}} \Re \left((b_{jk}^\alpha(\zeta) z_1^j \bar{z}_1^k) z_\alpha \right) \\ &\quad + O(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1}). \end{aligned}$$

where for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we denote $z_* = (z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-2}$. These automorphisms converge to the identity uniformly on compact subsets of \mathbb{C}^n as $\zeta \rightarrow 0$. Furthermore, if for $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in D'$ as above we consider the point $\tilde{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_n + \epsilon)$ where $\epsilon > 0$ is chosen to ensure that $\tilde{\zeta} \in \partial D'$, then the actual form of $\Phi_\zeta(z)$ shows that $\Phi_{\tilde{\zeta}}(\zeta) = (0, \dots, 0, -\epsilon)$ – since the explicit description of these automorphisms will come up later, we shall be content at this stage with merely collecting the relevant properties needed to describe the scaling of D' . To define the distinguished polydiscs around ζ (more precisely, biholomorphic images of polydiscs), let

$$A_l(\zeta) = \max\{|a_{jk}(\zeta)| : j + k = l\}, \quad 2 \leq l \leq 2m'$$

and

$$B_{l'}(\zeta) = \max\{|b_{jk}^\alpha(\zeta)| : j + k = l', 2 \leq \alpha \leq n-1\}, \quad 2 \leq l' \leq m'.$$

For each $\delta > 0$ define

$$\tau(\zeta, \delta) = \min \left\{ (\delta/A_l(\zeta))^{1/l}, (\delta^{1/2}/B_{l'}(\zeta))^{1/l'} : 2 \leq l \leq 2m', 2 \leq l' \leq m' \right\}$$

Since the type of $\partial D'$ at the origin is $2m'$ it follows that $A_{2m'}(0) > 0$ and hence $A_{2m'}(\zeta)$ is positive for all ζ sufficiently close to the origin. Thus

$$\delta^{1/2} \lesssim \tau(\zeta, \delta) \lesssim \delta^{1/2m'}$$

for ζ close to the origin – the upper bound being a consequence of the non-vanishing of $A_{2m'}(\zeta)$ near the origin while the lower bound follows since the greatest possible exponent of δ in the definition of $\tau(\zeta, \delta)$ is $1/2$. Set

$$\tau_1(\zeta, \delta) = \tau(\zeta, \delta) = \tau, \tau_2(\zeta, \delta) = \dots = \tau_{n-1}(\zeta, \delta) = \delta^{1/2}, \tau_n(\zeta, \delta) = \delta$$

and define

$$R(\zeta, \delta) = \{z \in \mathbb{C}^n : |z_k| < \tau_k(\zeta, \delta), 1 \leq k \leq n\}$$

which is a polydisc around the origin in \mathbb{C}^n with polyradii $\tau_k(\zeta, \delta)$ along the z_k direction for $1 \leq k \leq n$ and let

$$Q(\zeta, \delta) = \Phi_\zeta^{-1}(R(\zeta, \delta))$$

which is a distorted polydisc around ζ . It was shown in [43] that these domains satisfy the engulfing property, i.e., for all ζ in a small fixed neighbourhood of the origin, there is a uniform constant $C > 0$ such that if $\eta \in Q(\zeta, \delta)$, then $Q(\eta, \delta) \subset Q(\zeta, C\delta)$ and $Q(\zeta, \delta) \subset Q(\eta, C\delta)$.

Consider the sequence $p^\nu = f(p^\nu) \in D'$ that converges to the origin and denote by w^ν the point on $\partial D'$ chosen such that if $p^\nu = (p_1^\nu, p_2^\nu, \dots, p_n^\nu)$ then $w^\nu = (p_1^\nu, p_2^\nu, \dots, p_n^\nu + \gamma_\nu)$ for some $\gamma_\nu > 0$. Note that

$$\gamma_\nu \approx \text{dist}(p^\nu, \partial D')$$

for all large ν . Hence

$$\delta_\nu = \text{dist}(p^\nu, \partial D) \approx \text{dist}(p^\nu, \partial D') \approx \gamma_\nu$$

for all large ν . Let $g^\nu = \Phi_{w^\nu}(\cdot)$ be the polynomial automorphism of \mathbb{C}^n corresponding to $w^\nu \in D'$ as described above. Let us consider the holomorphic mappings

$$f^\nu = g^\nu \circ f : D \rightarrow g^\nu(D')$$

and define a dilation of coordinates in the target space by

$$B^\nu : (z'_1, z'_2, \dots, z'_n) \mapsto ((\tau_1^\nu)^{-1} z'_1, (\tau_2^\nu)^{-1} z'_2, \dots, (\tau_n^\nu)^{-1} z'_n)$$

where $\tau_1^\nu = \tau(w^\nu, \gamma_\nu)$, $\tau_j^\nu = \gamma_{1/2}^\nu$ for $2 \leq j \leq n-1$ and $\tau_n^\nu = \gamma_\nu$. Let $D^\nu = T^\nu(D)$ and $D'^\nu = (B^\nu \circ g^\nu)(D')$ be the scaled domains and the scaled maps between them are

$$F^\nu = B^\nu \circ f^\nu \circ (T^\nu)^{-1} : D_\nu \rightarrow D'_\nu.$$

To understand the Hausdorff limit of the domains D'_ν , note first that $B^\nu \circ g^\nu(p^\nu) = ('0, -1)$, which implies that $F^\nu('0, -1) = ('0, -1)$ for all ν , and that r'_ν , the defining function for D'_ν is given by

$$\gamma_\nu^{-1} r' \circ (B^\nu \circ g^\nu)^{-1}(z) = 2\Re z_n + Q_\nu(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \Re(S_\nu^\alpha(z_1, \bar{z}_1) z_\alpha) + O(\tau_1^\nu)$$

where

$$Q_\nu(z_1, \bar{z}_1) = \sum_{\substack{j+k \leq 2m' \\ j, k > 0}} a_{jk}(w^\nu) \gamma_\nu^{-1} (\tau_1^\nu)^{j+k} z_1^j \bar{z}_1^k$$

and

$$S_\nu^\alpha(z_1, \bar{z}_1) = \sum_{\substack{j+k \leq m' \\ j, k > 0}} b_{jk}^\alpha(w^\nu) \gamma_\nu^{-1/2} (\tau_1^\nu)^{j+k} z_1^j \bar{z}_1^k.$$

By the definition of $A_l, B_{l'}$ and τ_1^ν it follows that the largest coefficient in both Q_ν and S_ν^α is at most one in modulus. It was shown in [18] that there exists a uniform $\epsilon > 0$ such that

$$|b_{jk}^\alpha(w^\nu)\gamma_\nu^{-1/2}(\tau_1^\nu)^{j+k}| \lesssim (\tau_1^\nu)^\epsilon$$

for all possible indices j, k, α and all large ν . Therefore some subsequence of this family of defining functions converges together with all derivatives on compact sets to

$$r'_\infty(z) = 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2$$

where $Q_{2m'}(z_1, \bar{z}_1)$ is a polynomial of degree at most $2m'$ without harmonic terms. Hence the domains D'_ν converge to

$$D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\}$$

which, being the smooth limit of pseudoconvex domains, is itself pseudoconvex. In particular, it follows that $Q_{2m'}(z_1, \bar{z}_1)$ is subharmonic.

It is known that analytic discs in a bounded regular domain in \mathbb{C}^n satisfy the so-called attraction property (see [11]), i.e., if the centre of a given disc is close to a boundary point, then a given subdisc around the origin cannot wander too far away from the same boundary point. A quantitative version of this was proved by Berteloot-Coeuré in [14] and forms the basis for controlling families of scaled mappings in \mathbb{C}^2 . Using ideas from [12] and [18], the following analogue was proved in [43] and will be useful in this situation as well – we include the statement for the sake of completeness.

Proposition 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $\zeta_0 \in \partial\Omega$ and suppose there is an open neighbourhood V such that $V \cap \partial\Omega$ is C^∞ smooth pseudoconvex of finite type and the Levi rank is at least $n - 2$ on $V \cap \partial\Omega$. Fix a domain $\omega \subset \mathbb{C}^m$.*

Then for any fixed point $z_0 \in \omega$ and a compact $K \subset \omega$ containing z_0 , there exist constants $\epsilon(K), C(K) > 0$ such that for any $\xi \in V \cap \partial\Omega$ and $0 < \epsilon < \epsilon(K)$, every holomorphic mapping $F : \omega \rightarrow \Omega$ with

$$|F(z_0) - \zeta_0| < \epsilon(K) \quad \text{and} \quad F(z_0) \in Q(\xi, \epsilon)$$

also satisfies $F(K) \subset Q(\xi, C(K), \epsilon)$.

Now let $\{K_j\}$ be an increasing sequence of relatively compact domains that exhaust D_∞ such that each contains the base point $(0, -1)$. Fix $K = K_\mu$ and let $\tilde{f}^\nu = f \circ (T^\nu)^{-1}$. Then

$$\tilde{f}^\nu(0, -1) = f(0, -\delta_\nu) = f(p^\nu) = p^\nu$$

and hence $\tilde{f}^\nu(0, -1) \rightarrow 0 \in \partial D'$. In particular, $\tilde{f}^\nu(0, -1) = p^\nu \in Q(w^\nu, 2\gamma_\nu)$ for all large ν , by the construction of these distorted polydiscs. By the previous proposition

$$\tilde{f}^\nu(K) \subset Q(w^\nu, C(K)\gamma_\nu)$$

and therefore

$$F^\nu(K) = B^\nu \circ g^\nu \circ \tilde{f}^\nu(K) \subset B^\nu \circ g^\nu(Q(w^\nu, C(K)\gamma_\nu)).$$

However,

$$B^\nu \circ g^\nu(Q(w^\nu, C(K)\gamma_\nu)) = B^\nu(R(w^\nu, C(K)\gamma_\nu))$$

which by definition is contained in a polydisk around the origin with polyradii

$$r_k = \tau_k(w^\nu, C(K)\gamma_\nu) / \tau_k(w^\nu, \gamma_\nu)$$

for $1 \leq k \leq n$. Note that $r_n = C(K)$ and that $r_k = (C(K))^{1/2}$ for $2 \leq k \leq n-1$. Since $C(K) > 1$, and this may be assumed without loss of generality, the definition of $\tau(\zeta, \delta)$ shows that for $\delta' < \delta''$,

$$(\delta'/\delta'')^{1/2} \tau(\zeta, \delta'') \leq \tau(\zeta, \delta') \leq (\delta'/\delta'')^{1/2m'} \tau(\zeta, \delta').$$

Therefore

$$\tau_1(w''', C(K)\gamma_\nu)/\tau_1(w''', \gamma_\nu) \leq (C(K))^{1/2}$$

and hence $r_1 \leq (C(K))^{1/2}$. Thus $\{F^\nu\}$ is uniformly bounded on each compact set in D_∞ and is therefore normal. Let $F : D_\infty \rightarrow D'_\infty$ be a holomorphic limit of some subsequence in $\{F^\nu\}$ and since $F^\nu('0, -1) = ('0, -1)$ for all ν by construction, it follows that $F('0, -1) = ('0, -1)$. The maximum principle shows that $F(D_\infty) \subset D'_\infty$.

Let $R > 0$ be arbitrary and fix $z \in D_\infty \cap B(0, R)$. Note that since f preserves the distance to the boundary, it follows that

$$|(r'_\nu \circ F^\nu)(z)| = \gamma_\nu^{-1} |(r' \circ f \circ T_\nu^{-1})(z)| \lesssim (\delta_\nu/\gamma_\nu) |r_\nu(z)|$$

and hence

$$|r'_\infty \circ F(z)| \lesssim |r_\infty(z)|$$

since the constants appearing in the first estimate are independent of ν . This shows that the cluster set of a finite boundary point of D_∞ does not intersect D'_∞ and therefore F is non-degenerate.

2.2. Boundedness of $F^\nu(0)$. A crucial point in the proof of theorem 1.1 is to show that near the origin, the family $\{F^\nu\}$ of the scaled maps is uniformly Hölder continuous upto the boundary i.e., the Hölder constant and exponent are stable under the scaling. A first step in establishing this is to show that the images of the origin under these scaled maps is stable.

Proposition 2.2. *The sequence $\{F^\nu(0)\}$ is bounded.*

Proof. Recall the scaled maps

$$F^\nu = B^\nu \circ g^\nu \circ f \circ (T^\nu)^{-1}$$

where B^ν, T^ν were linear maps and $f(0) = 0$. So $F^\nu(0) = (B^\nu \circ g^\nu \circ f)(0) = (B^\nu \circ g^\nu)(0)$. Now let us recall the explicit form of the map $g^\nu = \Phi_{w''}$ from [43], which is a polynomial automorphism of \mathbb{C}^n that reduces the defining function to a certain normal form as stated in section 2.1. Let us recall this reduction procedure for any given domain Ω in \mathbb{C}^n that is smooth pseudoconvex of finite type $2m$ and of Levi rank (at least) $n-2$ on a (small) open boundary piece Σ in $\partial\Omega$ and containing the origin, say. Let r be a smooth defining function for Σ with $\partial r/\partial z_n(z) \neq 0$ for all z in a small neighbourhood U in \mathbb{C}^n of Σ such that the vector fields

$$L_n = \partial/\partial z_n, \quad L_j = \partial/\partial z_j + b_j(z, \bar{z})\partial/\partial z_n$$

where $b_j = (\partial r/\partial z_n)^{-1} \partial r/\partial z_j$, form a basis of $\mathbb{C}T^{(1,0)}(U)$ and satisfy $L_j r \equiv 0$ for $1 \leq j \leq n-1$ and $\partial \bar{\partial} r(z)(L_i, \bar{L}_j)_{2 \leq i, j \leq n-1}$ has all its eigenvalues positive for each $z \in U$.

The reduction now consists of five steps – for $\zeta \in U$, the map $\Phi_\zeta = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ where each ϕ_j is described below.

ϕ_1 is the affine map given by

$$\phi_1(z_1, \dots, z_n) = (z_1 - \zeta_1, \dots, z_{n-1} - \zeta_{n-1}, z_n + \sum_{j=1}^{n-1} b_j^\zeta z_j - (\zeta_n + \sum_{j=1}^{n-1} b_j^\zeta \zeta_j)).$$

where the coefficients $b_j^\zeta = b_j(\zeta, \bar{\zeta})$ are clearly smooth functions of ζ on U . Therefore, ϕ_1 translates ζ to the origin and

$$r(\phi^{-1}(z)) = r(\zeta) + \Re z_n + \text{terms of higher order.}$$

where the constant term disappears when $\zeta \in \Sigma$.

The remaining reductions remove the occurrence of harmonic (not just pluriharmonic) monomials in the variables z_1, \dots, z_{n-1} of weights upto one in the weighted homogeneous expansion of the defining function with respect to the weights given by the multitype for Σ , i.e., the variable z_1 is assigned a weight of $1/2m$, z_n a weight of 1 while the others are assigned $1/2$ each. Now, since the Levi form restricted to the subspace

$$L_* = \text{span}_{\mathbb{C}^n} \langle L_2, \dots, L_{n-1} \rangle$$

of $T_\zeta^{(1,0)}(\partial\Omega)$ is positive definite, we may diagonalize it via a unitary transform ϕ_2 and a dilation ϕ_3 will then ensure that the quadratic part involving only z_2, z_3, \dots, z_{n-2} in the Taylor expansion of r is $|z_2|^2 + |z_3|^2 + \dots + |z_{n-2}|^2$. The entries of the matrix that represents the composite of the last two linear transformations are smooth functions of ζ and in the new coordinates still denoted by z_1, \dots, z_n , the defining function is in the form

$$(2.3) \quad r(z) = r(\zeta) + \Re z_n + \sum_{\alpha=2}^{n-1} \sum_{j=1}^m \Re((a_j^\alpha z_1^j + b_j^\alpha \bar{z}_1^j) z_\alpha) + \Re \sum_{\alpha=2}^{n-1} c_\alpha z_\alpha^2 \\ + \sum_{2 \leq j+k \leq 2m} a_{j,k} z_1^j \bar{z}_1^k + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq m \\ j,k > 0}} \Re(b_{j,k}^\alpha z_1^j \bar{z}_1^k z_\alpha) \\ + O(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1})$$

A change in the normal variable z_n to absorb the pluriharmonic terms here i.e., $z_1^k, \bar{z}_1^k, z_\alpha^2$ as well as $z_1^k z_\alpha, \bar{z}_1^k \bar{z}_\alpha$, can be done according to the following standard change of coordinates ϕ_4 given by

$$z_j = t_j \quad (1 \leq j \leq n-1), \\ z_n = t_n - P_1(t_1, \dots, t_{n-1})$$

where

$$P_1(t_1, \dots, t_{n-1}) = \sum_{k=2}^{2m} a_{k0} t_1^k - \sum_{\alpha=2}^{n-1} \sum_{k=1}^m a_k^\alpha t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha t_\alpha^2$$

with coefficients that are smooth functions of ζ .

Finally, just as we absorbed into the simplest pure term $\Re z_n$ that occurs in the Taylor expansion, other pure terms not divisible by this variable in the last step, we may absorb into (some among) the simplest of non-pluriharmonic (and non-harmonic) monomials occurring there namely, $|z_\alpha|^2$ where $2 \leq \alpha \leq n-1$ other harmonic (but non-pluriharmonic) terms of degree at least two which are not divisible by them. Let us do this to those of

weight at most one, remaining in (2.3) rewritten in the t -coordinates, which are of the form $\bar{t}_1^j t_\alpha$ by applying the transform ϕ_5 given by

$$\begin{aligned} t_1 &= w_1, \quad t_n = w_n, \\ t_\alpha &= w_\alpha - P_2(w_1) \quad (2 \leq \alpha \leq n-1) \end{aligned}$$

where

$$P_2(w_1) = \sum_{k=1}^m b_k^\alpha w_1^k$$

with coefficients smooth in ζ , as before (since all these coefficients are simply the derivatives of some order of the smooth defining function r evaluated at ζ). We then have the sought for simplification of the Taylor expansion.

Now, suppose that we already have the reduced form holding at one boundary point say the origin, then (since no further normalization would be required) Φ_0 may be taken to be the identity. Note that the normalizing map Φ_ζ is not uniquely determined – even among the class of all maps of the same form – owing to the (only) ambiguity in the choice of the diagonalizing map ϕ_2 ; however, this choice can certainly be done in a manner such that the coefficients of that unitary matrix are smooth in ζ and satisfying the ‘initial condition’ that ϕ_2 for the origin is the identity. Thus in all, the map Φ_ζ is smooth in the parameter ζ with Φ_0 being the identity map. This implies that the family $\Phi_\zeta(\cdot)$ is uniformly Lipschitz (where $\zeta \in U$) and converges uniformly on compact subsets of \mathbb{C}^n to the identity as ζ approaches the origin as mentioned in section 2.1. Next, the consequence of the simpler fact that the map $\Phi(\zeta, 0) = \Phi_\zeta(0)$ is Lipschitz in a neighbourhood of the origin, to our setting is that

$$|g^\nu(0)| \leq C_1 |w^\nu|$$

with C_1 independent of ν . Now

$$F^\nu(0) = B^\nu \circ g^\nu(0) = \left((\tau_1^\nu)^{-1}(g^\nu(0))_1, \dots, (\tau_{n-1}^\nu)^{-1}(g^\nu(0))_{n-1}, (\tau_n^\nu)^{-1}(g^\nu(0))_n \right).$$

From the fact that $\tau_1^\nu \gtrsim \gamma_\nu^{1/2}$ and $\tau_j^\nu = \gamma_\nu^{1/2}$ for $2 \leq j \leq n-1$ while $\tau_n^\nu = \gamma_\nu$, we get that

$$\left| (B^\nu \circ g^\nu(0)) \right| \leq C_2 |w^\nu| / \gamma_\nu$$

with C_2 again independent of ν . Now, since f is Lipschitz upto M and $\delta_\nu \lesssim \gamma_\nu$ we have

$$\begin{aligned} |w^\nu| &\leq \text{dist}(w^\nu, p^\nu) + \text{dist}(p^\nu, 0') \\ &= \text{dist}(p^\nu, \partial D') + |f(p^\nu) - f(0)| \\ &\leq C_3(\gamma_\nu + |p^\nu|) \\ &= C_3(\gamma_\nu + \delta_\nu) \\ &\leq C_4 \gamma_\nu \end{aligned}$$

with constants independent of ν as before, implying that $\{F^\nu(0)\}$ is bounded.

2.3. Stability of the Kobayashi metric. For Ω, Σ, U with $0 \in \Sigma$, as in the previous section, recall from [43] (see also [12]), the M -metric defined for $\zeta \in U \cap \Omega$ by

$$M_\Omega(\zeta, X) = \sum_{k=1}^n |(D\Phi_\zeta(\zeta)X)_k|/\tau_k(\zeta, \epsilon(\zeta)) = |D(B_\zeta \circ \Phi_\zeta)(\zeta)(X)|_{\mu}$$

with $B_\zeta = B_\zeta^{\epsilon(\zeta)}$ where $\epsilon(\zeta) > 0$ is such that $\zeta + (0, \dots, 0, \epsilon(\zeta))$ lies on Σ and

$$B_\zeta^\delta(z_1, \dots, z_n) = ((\tau_1)^{-1}z_1, \dots, (\tau_n)^{-1}z_n)$$

where $\tau_1 = \tau(\zeta, \delta)$, $\tau_j = \delta^{1/2}$ for $2 \leq j \leq n-1$ and $\tau_n = \delta$. Now let us get to our scaled domains $D'_\nu = B^\nu \circ g^\nu(D')$, where we recall $B^\nu = B_{p^\nu}$ and $g^\nu = \Phi_{w^\nu}$ and denote by ${}^\nu\Phi_\zeta$ and ${}^\nu B_\zeta$ the normalizing map and the dilation respectively, associated to (D'_ν, ζ) ; we shall drop the left superscript ν for the domain D' . Since all the coefficients of the polynomial automorphisms ${}^\nu\Phi_\zeta(\cdot)$ depend smoothly on the derivatives of the defining functions r_ν , which converge in the C^∞ -topology on U , there exists $L > 1$ such that

$$(2.4) \quad 1/L < |\det(D({}^\nu\Phi_\zeta(z)))| < L$$

for all ν and $\zeta \in U$. The main result of this section is the following stability theorem for the Kobayashi metric.

Theorem 2.3. *There exists a neighbourhood U of the origin such that*

$$K_{D'_\nu}(z, X) \geq CM_{D'}(z, X)$$

for all $z \in U \cap D'_\nu$, $\nu \gg 1$ where C is a positive constant independent of ν .

The proof of this requires two steps and follows the lines of the argument presented in [43]. The first step consists in uniformly localizing the Kobayashi metric of the scaled domains near the origin which translates to verifying the following uniform version of the attraction of analytic discs near a local plurisubharmonic peak point.

Lemma 2.4. *There exists a neighbourhood $V \subseteq U$ of the origin and $\delta > 0$ independent of j such that for any j large enough and any analytic disc $f : \Delta \rightarrow D'_j$ with $f(0) = \zeta \in V \cap D'^j$, we have*

$$f(\Delta_\delta) \subseteq U \cap D'_j$$

Proof. If the lemma were false, then for any neighbourhood $V \subset U$ and any $\delta > 0$, there exists a sequence of analytic discs $f_j : \Delta \rightarrow D'_j$ with $f_j(0) = \zeta^j \in V \cap D'^j$, converging to the origin but $f_j(\Delta_\delta) \not\subseteq U$. Consider

$$\tilde{f}_j = S_j^{-1} \circ f_j : \Delta \rightarrow D'_j$$

where $S_j = B^j \circ \Phi_{w^j}$ are the scaling maps. Then $\tilde{\zeta}^j = \tilde{f}_j(0) = S_j^{-1}(\zeta^j)$ converges to the origin. Indeed, the family $S_j^{-1} = \Phi_{w^j}^{-1} \circ (B^j)^{-1}$ is equicontinuous at the origin since their derivatives are

$$D(\Phi_{w^j}^{-1}) \circ (B^j)^{-1}$$

with $D(\Phi_{w^j}^{-1})$ being bounded in a neighbourhood of the origin and $(B^j)^{-1}$ converging to the zero map. Now, $0 \in \partial D'$ being a plurisubharmonic peak point by [17], the simplest version of the attraction property of analytic discs (see for instance lemma 2.1.1 in [33]), gives for all $j \gg 1$ that

$$\tilde{f}_j(\Delta_\gamma) \subseteq U \cap D'$$

for some $\gamma \in (0, 1)$. Rescale the \tilde{f}_j 's, so that we may take $\gamma = 1$. Now recall the sharper version of the attraction property given by lemma 3.6 of [43] namely,

Lemma 2.5. *Let $W \Subset U$ be a neighbourhood of the origin. There exist constants $\alpha, A \in [0, 1]$ and $K \geq 1$ such that for any analytic disc $f : \Delta_N \rightarrow U$ that satisfies $M_{D'}(f(t), f'(t)) \leq A$, $f(0) \in W$ and $K^{N-1}\epsilon(f(0)) < \alpha$ also satisfies,*

$$\overline{f(\Delta_N)} \subset Q\left(f(0), K^N \epsilon(f(0))\right)$$

We intend to apply this lemma to the analytic discs $\tilde{g}_j(t) = \tilde{f}_j(r_0 t)$ for which we only need to verify

$$M_{D'}(\tilde{g}_j(t), \tilde{g}'_j(t)) \leq A$$

since $\tilde{g}_j(0) = \tilde{f}_j(0)$ converges to the origin. Then

$$\begin{aligned} M_{D'}(\tilde{g}_j(t), \tilde{g}'_j(t)) &= M_{D'}(\tilde{f}_j(r_0 t), r_0 \tilde{f}'_j(r_0 t)) \\ &= r_0 M_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \\ &\leq r_0 C_5 K_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \end{aligned}$$

by theorem 3.10 of [43], where C_5 is a positive constant independent of j . At the last step, we use the fact that \tilde{f}_j 's map Δ into so small a neighbourhood of the origin where $K_{D'} \approx M_{D'}$. As the Kobayashi metric decreases under holomorphic mappings, we have

$$r_0 C_5 K_{D'}(\tilde{f}_j(r_0 t), \tilde{f}'_j(r_0 t)) \leq r_0 C_5 K_\Delta(t, \partial/\partial t)$$

So, if we choose r_0 such that

$$r_0 C_5 \left(\sup_{|t| \leq r_0} K_\Delta(t, \partial/\partial t) \right) \leq A,$$

we will have completed the required verification of the hypothesis for the discs $g_j : \Delta \rightarrow D' \cap U$ and the aforementioned lemma gives

$$(2.5) \quad \tilde{f}_j(\Delta_{r_0}) = \tilde{g}_j(\Delta) \subset Q(\tilde{g}_j(0), \epsilon(\tilde{g}_j(0))) = Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0))).$$

At this point observe also that r_0 does not depend on δ . Next, fix any compact subdisc $K \subset \Delta_{r_0}$ and note by (2.5) that,

$$\tilde{f}_j(K) \subset Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0)))$$

and subsequently,

$$\begin{aligned} (S_j \circ \tilde{f}_j)(K) &\subset S_j(Q(\tilde{f}_j(0), \epsilon(\tilde{f}_j(0)))) \\ &= (B^j \circ \Phi_{w^j}) \circ (B_{\tilde{\zeta}^j} \circ \Phi_{\tilde{\zeta}^j})^{-1}(\Delta^n) \\ (2.6) \quad &= \left(B_{p^j} \circ (\Phi_{w^j} \circ \Phi_{\tilde{\zeta}^j}^{-1}) \circ B_{\tilde{\zeta}^j}^{-1} \right)(\Delta^n) \end{aligned}$$

where $\tilde{\zeta}^j = \tilde{\zeta}^j + (0, \dots, 0, \epsilon(\tilde{\zeta}^j))$. Now the fact that $\epsilon(\tilde{\zeta}^j) \approx \epsilon(p^j)$, as both p^j and $\tilde{\zeta}^j$ converge to the origin and corollary 2.8 of [18] – the fact that the distinguished radius τ at any two points in U are comparable – together imply the boundedness of the sequence $\tau(\tilde{\zeta}^j, \epsilon(\tilde{\zeta}^j))/\tau(w^j, \epsilon(p^j))$, by C_6 say. This when combined with (2.4) and (2.6), yields the

stability of our discs f_j on compact subdiscs in Δ i.e., we have for a positive constant $C_K > C_6 L^2$ (which may depend on K but not on j), that

$$f_j(K) = (S_j \circ \tilde{f}_j)(K) \subseteq \Delta_{\sqrt{C_K}} \times \dots \times \Delta_{\sqrt{C_K}} \times \Delta_{C_K}$$

Hence by Montel's theorem, f_j 's converge to a map $f_\infty : \Delta_{r_0} \rightarrow \overline{D'}_\infty$, where $f_\infty(0) = 0 \in \partial D'_\infty$. If ϕ is a local plurisubharmonic peak function at the origin, then the maximum principle applied to the subharmonic function $\psi = \phi \circ f_\infty$ implies that $f_\infty \equiv \text{constant}$, since ϕ peaks precisely at one point. This contradicts our assumption that $f_j(\Delta_\delta) \not\subseteq U$ (as soon as δ is taken smaller than r_0) and completes the proof of the lemma. \square

Remark 2.6. The same line of argument gives proposition 2.1 by replacing the discs by balls in \mathbb{C}^m , covering any given compact $K \subset \omega$ by balls and using the engulfing property of the special polydiscs Q .

The second and more technical step, is a quantitative form of the Schwarz lemma at the boundary which generalizes lemma 3.6 of [43] mentioned earlier. To do this we require a stable version of the engulfing property for the distinguished polydiscs Q^ν associated to the scaled domain D'_ν . Fix a pair of neighbourhoods $0 \in W \Subset V \Subset U$.

Lemma 2.7. *There exist constants $\alpha, A \in [0, 1]$ and $K > 1$ such that for every analytic disc*

$$f : \Delta_N \rightarrow V \cap D'_\nu$$

that satisfies $M_{D^\nu}(f(t), f'(t)) < A$, $f(0) \in W$ and $K^{N-1}\epsilon(f(0)) < \alpha$ also satisfies,

$$\overline{f(\Delta_N)} \subset Q^\nu(f(0), K^N \epsilon(f(0))).$$

Recall the definition of $A_l(z)$ and $\tau(z, \delta)$ from section 2.1. Let $A_l^\nu(z)$ and $\tau^\nu(z, \delta)$ denote the corresponding quantities for the domain D'_ν and $A_l^\infty(z)$ and $\tau^\infty(z, \delta)$ that for D'_∞ . Then it is clear that $A_l^\nu(z) \rightarrow A_l^\infty(z)$ and consequently $\tau^\nu(z, \delta) \rightarrow \tau^\infty(z, \delta)$, both convergences being uniform for z in V . Let $\delta > 0$ be given. Let

$$M = \sup \{ (\tau^\nu(z, \delta))^l : z \in V, 2 \leq l \leq 2m', \nu \geq 1 \}$$

and $0 < \epsilon < \delta/M$. Then there exists by the uniform convergence of the A 's, an N_δ independent of $z \in V$ such that for all $2 \leq l \leq 2m$ and all $\nu > N_\delta$,

$$(2.7) \quad A_l^\infty(z) - \epsilon < A_l^\nu(z) < A_l^\infty(z) + \epsilon$$

The right inequality gives for all $2 \leq l \leq 2m$ that

$$(\tau^\infty(z, \delta))^l A_l^\nu(z) < (\tau^\infty(z, \delta))^l A_l^\infty(z) + (\tau^\infty(z, \delta))^l \epsilon < \delta + \delta$$

by the definition of τ^∞ and ϵ . The definition of τ^ν now makes this read as

$$(2.8) \quad \tau^\infty(z, \delta) < 2^{1/2} \tau^\nu(z, \delta)$$

for all $\nu > N_\delta$ and $z \in V$. Meanwhile the left inequality at (2.7) similarly gives,

$$(\tau^\nu(z, \delta))^l A_l^\infty(z) < (\tau^\nu(z, \delta))^l A_l^\nu(z) + (\tau^\nu(z, \delta))^l \epsilon < 2\delta$$

which implies $\tau^\nu(z, \delta) < 2^{1/2} \tau^\infty(z, \delta)$. Combining this with corollary 2.8 of [18] applied to τ^∞ , we get for $z \in Q^\infty(z', \delta)$ and all ν large that

$$(2.9) \quad 1/C \tau^\nu(z', \delta) < \tau^\nu(z, \delta) < C \tau^\nu(z', \delta)$$

for some $C > 0$, independent of z' and ν . This will lead to the following uniform engulfing property of these polydiscs.

Lemma 2.8. *There exists a positive constant C such that for all ν large, $z' \in U$ and $z'' \in Q^\nu(z', \delta)$ we have*

$$\begin{aligned} Q^\nu(z'', \delta) &\subset Q^\nu(z', C\delta) \\ Q^\nu(z', \delta) &\subset Q^\nu(z'', C\delta) \end{aligned}$$

Proof. Let us recall what Q^ν is and rewrite for instance the first statement as

$$\left({}^\nu B_{z''}^\delta \circ {}^\nu \Phi_{z''} \right)^{-1}(\Delta^n) \subset \left({}^\nu B_{z'}^{C\delta} \circ {}^\nu \Phi_{z'} \right)^{-1}(\Delta^n)$$

which is equivalent to saying that the map

$$(2.10) \quad {}^\nu B_{z'}^{C\delta} \circ {}^\nu \Phi_{z'} \circ \left({}^\nu \Phi_{z''} \right)^{-1} \circ \left({}^\nu B_{z''}^\delta \right)^{-1}$$

maps the unit polydisc into itself. Now recall from (2.4), that the sequence of Jacobians of ${}^\nu \Phi_{z'} \circ {}^\nu \Phi_{z''}^{-1}$ is uniformly bounded above by L^2 . Unravelling the definition of the scalings ${}^\nu B_{z'}^{C\delta}$ and ${}^\nu B_{z''}^\delta$, then shows that the map defined in (2.10) carries the unit polydisc into the polydisc of polyradius given by

$$(2.11) \quad \left(L^2 \tau^\nu(z'', \delta) / \tau^\nu(z', C\delta), L^2 \delta^{1/2} / (C\delta)^{1/2}, \dots, L^2 \delta^{1/2} / (C\delta)^{1/2}, L^2 \delta / C\delta \right)$$

Now the uniform comparability of the τ 's obtained in (2.9) gives $C_7 > 0$ independent of ν and z' , such that

$$\tau^\nu(z'', \delta) < C_7 \tau^\nu(z', \delta)$$

whereas looking at the definition of the τ directly gives

$$\tau^\nu(z', C\delta) > C^{1/2m} \tau^\nu(z', \delta)$$

Combining these two we see that the first component at (2.11) is bounded above by $L^2 C_7 / C^{1/2m}$. Therefore, choosing C to be such that $C^{1/2} > L^2$ and $C^{1/2m} > L^2 C_7$ ensures that the map at (2.10) leaves the unit polydisc invariant, completing the verification of the lemma. \square

Before passing, let us record one more stable estimate concerning the τ 's to be of use later, namely

$$(2.12) \quad \tau^\nu(z, \delta) \leq C_8 \delta^{1/2m'}$$

for $C_8 > 0$ (independent of ν), which follows as in section 2.1 from the fact that the $A_{2m'}^\nu(z)$'s are uniformly bounded below by a positive constant on a neighbourhood of the origin (which as usual we may assume to be U).

Next, observing that all the constants in the calculations in the proof of lemma 3.6 of [43] are stable owing to (2.9), lemma 2.8 and the stability of the estimates on the various coefficients of ${}^\nu \Phi$'s given in lemma 3.4 of [43] – because all these coefficients are nothing but the derivatives of some order (less than $2m'$) of the defining functions of the scaled domains, which converge in the C^∞ -topology on U – lemma 2.7 follows as well.

Proof of theorem 2.3. If the theorem were false, there would exist a subsequence ν_j of \mathbb{N} and $\zeta^j \in D^{\nu_j}$ converging to the origin such that

$$K_{D^{\nu_j}} \left(\zeta^j, X / M_{D^{\nu_j}}(\zeta^j, X) \right) < 1/j^2$$

which after re-indexing the D^{ν_j} 's as D^j and using lemma 2.4 reads as :

There exists a sequence $\zeta^j \in U \cap D^j$ with ζ^j converging to the origin and a sequence of analytic discs

$$f_j : \Delta \rightarrow U \cap D'_j$$

such that

$$f_j(0) = \zeta^j \text{ and } f'_j(0) = R_j \left(X/M_{D'_j}(\zeta^j, X) \right)$$

with $R_j \geq j^2$.

The idea as in [12] and [43] now consists of scaling these analytic discs appropriately to enlarge their domains to discs of growing radius while ensuring their normality also, so that taking their limit produces an entire curve lying inside D'_∞ to contradict its Brody hyperbolicity. To work this out, define $M_j(t) : \overline{\Delta}_{1/2} \rightarrow \mathbb{R}^+$ by

$$M_j(t) = M_{D'_j}(f_j(t), f'_j(t))$$

Then

$$\begin{aligned} M_j(0) &= M_{D'_j}(f_j(0), f'_j(0)) = M_{D'_j}(\zeta^j, R_j X/M_{D'_j}(\zeta^j, X)) \\ &= R_j \geq j^2 \end{aligned}$$

Recall the following lemma from [12].

Lemma 2.9. *Let (X, d) be a complete metric space and let $M : X \rightarrow \mathbb{R}^+$ be a locally bounded function. Then for all $\sigma > 0$ and for all $u \in X$ satisfying $M(u) > 0$, there exists $v \in X$ such that*

- (i) $d(u, v) \leq 2/\sigma M(u)$
- (ii) $M(v) \geq M(u)$
- (iii) $M(x) \leq 2M(v)$ if $d(x, v) \leq 1/\sigma M(v)$

Apply this lemma to $M_j(t)$ on $\overline{\Delta}_{1/2}$ with $u = 0$ and $\sigma = 1/j$, to get $a_j \in \overline{\Delta}_{1/2}$ such that $|a_j| \leq 2j/M_j(0)$ and

$$(2.13) \quad M_j(a_j) \geq M_j(0) \geq j^2$$

and furthermore,

$$(2.14) \quad M_j(t) \leq 2M_j(a_j) \text{ on } \Delta(a_j, j/M_j(a_j))$$

(for all j big enough so that the discs above lie inside $\Delta_{1/2}$). Now, scaling the unit disc with respect to the points a_j which approach the origin (since the contention is that the derivatives of the f_j 's near the origin, in the M -metric, are blowing up rapidly), we get the discs $\Delta_j = s_j(\Delta)$ where $s_j = b^j \circ \phi_{a_j}$ with ϕ_{a_j} the translation that transfers a_j to the origin and b^j being the map that dilates by a factor of $2A^{-1}M_j(a_j)$ with A the constant of lemma 2.7. The scaled analytic discs $g_j : \Delta_j \rightarrow U \cap D'_j$ are then given by

$$g_j(t) = (f_j \circ s_j^{-1})(t) = f_j(a_j + c_j t),$$

where $c_j = A/2M_j(a_j)$. Note that

$$g'_j(t) = c_j f'_j(a_j + c_j t)$$

which implies

$$\begin{aligned} M_{D^j}(g_j(t), g'_j(t)) &= c_j M_j\left(f_j(a_j + c_j t), f'_j(a_j + c_j t)\right) \\ &= c_j M_j(a_j + c_j t) \end{aligned}$$

Now note that $a_j + c_j t$ lies in $\Delta(a_j, j/M_j(a_j))$ for all $|t| < j$ as $0 < A < 1$ and therefore by (2.13), (2.14) and the definition of c_j , we have for $t \in \Delta_j$ that

$$\begin{aligned} M_{D^j}(g_j(t), g'_j(t)) &= c_j M_j(a_j + c_j t) \\ &< c_j (2M_j(a_j)) \\ (2.15) \qquad \qquad \qquad &= A \end{aligned}$$

We wish to apply lemma 2.7 to these g_j 's. First we must that verify their centres lie close enough to the origin; to that end write

$$|g_j(0)| \leq |f_j(a_j) - f_j(0)| + |f_j(0)|$$

and note that the first term on the right can be made arbitrarily small by the equicontinuity of the f_j 's which map into the bounded neighbourhood U . After passing to a subsequence if necessary to ensure that $\epsilon(g_j(0)) \leq \alpha/K^{j-1}$, we have with (2.15), verified all the criteria of lemma 2.7 and therefore

$$(2.16) \qquad g_j(\Delta_N) \subset Q^j\left(g_j(0), K^N \epsilon(g_j(0))\right)$$

for all $j \geq N$. Let $\eta_j = g_j(0)$ and $\eta'_j = \eta_j + (0, \dots, 0, \epsilon_j)$ with $\epsilon_j > 0$ such that $r_j(\eta'_j) = 0$ and note that

$$\eta_j \in Q^j(\eta'_j, C\epsilon_j)$$

for a uniform constant $C \geq 1$ since $\epsilon_j \approx r_j(\eta_j)$, uniformly in j . Consequently using lemma 2.8, (2.16) becomes

$$(2.17) \qquad g_j(\Delta_N) \subset Q^j\left(\eta'_j, CK^N \epsilon_j\right)$$

for all $j \geq N$. Now if we let

$$h_j = {}^j B_{\eta_j} \circ {}^j \Phi_{\eta'_j} \circ g_j : \Delta_j \rightarrow \tilde{S}_j(U \cap D')$$

where \tilde{S}_j is the map

$${}^j B_{\eta_j} \circ {}^j \Phi_{\eta'_j} \circ B_{p^j} \circ \Phi_{w^j}$$

then what (2.17) translates to, for the map h_j is that

$$h_j(\Delta_N) \subset \Delta_{(CK^N)^{1/2}} \times \dots \times \Delta_{(CK^N)^{1/2}} \times \Delta_{CK^N}$$

for all $N \leq j$, exactly as in the proof of lemma 2.4. Also, note that the domains $\tilde{S}_j(U \cap D')$ after passing to a subsequence if necessary, converge to a domain \tilde{D}'_∞ of the same form as D'_∞ , by the same argument as in section 2.1 together with (2.12). Thus Montel's theorem, a diagonal sequence argument and an application of the maximum principle to the limit, gives rise to an entire curve

$$h : \mathbb{C} \rightarrow \tilde{D}'_\infty.$$

Indeed, note that

$$h_j(0) = ({}^j B_{\eta_j} \circ {}^j \Phi_{\eta'_j})(\eta_j) = (0, \dots, 0, -1)$$

for all j and so $h(0) = (0, \dots, 0, -1)$, which lies in \tilde{D}'_∞ all of whose boundary points – including the point at infinity – are peak points. Now, to check that h is nonconstant, we examine the sequence of their derivatives at the origin. We write X_j for $g'_j(0) = c_j f'_j(a_j)$ (which tend to 0 but are bounded below in the M -metric).

$$\begin{aligned}
(2.18) \quad |h'_j(0)|_{l^1} &= |({}^j B_{\eta'_j} \circ D({}^j \Phi_{\eta'_j}))(\eta_j)(X_j)|_{l^1} \\
&\geq |({}^j B_{\eta_j} \circ D({}^j \Phi_{\eta_j}))(\eta_j)(X_j)|_{l^1} \\
&= M_{D^j}(g_j(0), X_j) \\
&= c_j M_j(a_j) = A/2
\end{aligned}$$

where the lower bound here is a consequence of (2.9) applied to τ^j for the points η_j and η'_j and the fact that the family

$$D({}^\nu \Phi_{z'}) \circ D({}^\mu \Phi_{z''}^{-1})$$

is uniformly bounded below in norm. Passing to the limit in (2.18), now gives

$$|h'(0)|_{l^1} \geq C(A/2) > 0$$

and we reach the contradiction mentioned earlier namely to the Brody hyperbolicity of $\partial \tilde{D}'_\infty$ (see for instance, lemma 3.8 of [43]). \square

Finally, let us note the consequence of theorem (2.3) in the form that we shall make use of

Corollary 2.10. *There is neighbourhood U of the origin and a positive constant C such that*

$$K_{D^\nu}(z, X) \geq C |X| / (\text{dist}(z, \partial D^\nu))^{1/2m'}$$

for all $z \in U \cap D'_\nu$ and all $\nu \gg 1$.

This comes from the facts that $\epsilon_\nu(z) \approx \text{dist}(z, \partial D^\nu)$ and for some $C_9 > 0$ we have for all ν that

$$M_{D^\nu}(z, X) \geq C_9 |X|_{l^1} / (\epsilon^\nu(z))^{1/2m'}$$

which in turn follows from (2.12).

2.4. Uniform Hölder continuity of the scaled maps near the origin. In this section, we recall from [20], the arguments which show that the family of scaled maps is uniformly Hölder continuous upto the boundary – we already know that each scaled map is Hölder continuous upto the origin.

Theorem 2.11. *There exist positive constants r, C such that*

$$|F^\nu(z') - F^\nu(z'')| \leq C |z' - z''|^{1/2m'}$$

for any $z', z'' \in \bar{D}_\nu \cap B(0, r)$ and all large ν .

By the previous section we may assume that $\{F^\nu(0)\}$ converges to a (finite) boundary point $q \in \partial D'_\infty$. Recall that the origin (respectively the $\Re z_n$ axis) is a common boundary point (respectively, the common normal at the origin) for all the scaled domains. At this special boundary point, we first show that the family of scaled mappings is equicontinuous – we already know that this holds on compact subsets of D_∞ – or equivalently that, given any neighbourhood of q , there exists a small ball about the origin, such that every scaled map carries the piece of its domain intercepted by this ball into that given neighbourhood.

Then, the uniform lower bound on the Kobayashi metric near q will sharpen the uniform boundary distance decreasing property to uniform Hölder continuity, of the scaled maps. In particular then, the family $\{F^\nu\} \cup \{F\}$ is equicontinuous near the origin, upto the respective boundary.

Lemma 2.12. *For any $\epsilon > 0$ there exists $\delta > 0$, such that $|F^\nu(z) - q| < \epsilon$ for any $z \in D_\nu \cap B(0, \delta)$ and all large ν .*

Proof. Suppose to obtain a contradiction, that the assertion were false. Then there exists $\epsilon_0 > 0$ and a sequence $a_\nu \in D_\nu$ such that $a_\nu \rightarrow 0$ and $|F^\nu(a_\nu) - q| \geq \epsilon_0$. Since every F^ν is continuous upto the boundary ∂D_ν , we can also choose a sequence of points $b_\nu \in D_\nu$ lying on the common inner normal to D_ν at the origin i.e., $b_\nu = (0, -\beta_\nu)$ with $\beta_\nu > 0$, such that $b_\nu \rightarrow 0$ and $|F^\nu(b_\nu) - q| \rightarrow 0$. Let $s_\nu = |a_\nu - b_\nu|$. It is not difficult to see that there exists a constant $C > 0$ such that for all ν , there is a smooth path $\gamma_\nu : [0, 3s_\nu] \rightarrow D_\nu$ with the following properties:

- (i) $\gamma_\nu(0) = a_\nu, \gamma_\nu(3s_\nu) = b_\nu$
- (ii) $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq Ct$, for $t \in [0, s_\nu]$,
 $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq Cs_\nu$, for $t \in [s_\nu, 2s_\nu]$,
 $\text{dist}(\gamma_\nu(t), \partial D_\nu) \geq C(3s_\nu - t)$, for $t \in [2s_\nu, 3s_\nu]$,
- (iii) $|d\gamma_\nu(t)/dt| \leq C$, for $t \in [0, s_\nu]$.

By corollary (2.10) there is a positive constant C such that for any $w \in D'_\nu \cap B(q, 2\alpha)$ for some $\alpha > 0$, and any vector $X \in \mathbb{C}^n$ the lower bound

$$(2.19) \quad K_{D'_\nu}(w, X) \geq C \text{dist}(w, \partial D'_\nu)^{-1/2m'} |X|$$

holds for all large ν . Let $\eta = \min\{\alpha/4, \epsilon_0/4\}$. Since $F^\nu(b_\nu)$ lies in $B(q, \alpha)$ for ν large enough, we can choose $t_\nu \in [0, 3s_\nu]$ such that $F^\nu \circ \gamma_\nu(t_\nu) \in \partial B(q, 2\eta)$ and $F^\nu \circ \gamma_\nu((t_\nu, 3s_\nu))$ is contained in $B(q, 2\eta)$. From section 2.1 we know that

$$(2.20) \quad \text{dist}(F^\nu(z), \partial D'_\nu) \leq C(R) \text{dist}(z, \partial D_\nu)$$

for any $R > 0$ and $z \in D_\nu \cap B(0, R)$ with $F^\nu(z) \in D'_\nu \cap (B(q, 2\alpha))$. Fix $r > 0$ and let $z \in D_\nu \cap B(0, r)$ be such that $F^\nu(z) \in D'_\nu \cap B(q, 2\tau)$. Since the Kobayashi metric is decreasing under holomorphic mappings, we get from (2.19) and (2.20) that

$$\begin{aligned} K_{D_\nu}(z, X) &\geq K_{D'_\nu}(F^\nu(z), dF^\nu(z)X) \\ &\geq C \text{dist}(F^\nu(z), \partial D'_\nu)^{-1/2m'} |dF^\nu(z)X| \\ &\geq C \text{dist}(z, \partial D_\nu)^{-1/2m'} |dF^\nu(z)X|. \end{aligned}$$

On the other hand it can be seen that for all ν ,

$$K_{D_\nu}(z, X) \leq |X|/\text{dist}(z, \partial D_\nu)$$

and this implies the uniform estimate

$$(2.21) \quad |dF^\nu(z)| \leq C \text{dist}(z, \partial D_\nu)^{-1+1/2m'}$$

for all large ν and $z \in D_\nu \cap B(0, r)$ such that $F^\nu(z) \in D_\nu \cap B(p, 2\alpha)$. Therefore,

$$\begin{aligned} |F^\nu(\gamma^\nu(t_\nu)) - F^\nu(b_\nu)| &\leq \int_{t_\nu}^{3s_\nu} |dF^\nu(\gamma^\nu(t))| |d\gamma_\nu/dt| dt \\ &\leq C \int_{t_\nu}^{3s_\nu} \text{dist}(\gamma_\nu(t), \partial D_\nu)^{-1+1/2m'} dt \\ &\leq C s_\nu^{1/2m'} \rightarrow 0 \end{aligned}$$

as $\nu \rightarrow \infty$, which is a contradiction.

Proof of theorem 2.11. It was just shown that $F^\nu(z)$ lies in $B(q, 2\alpha)$ for any $z \in \bar{D}_\nu \cap B(0, \delta)$ if $\delta > 0$ is chosen small enough. Hence (2.21) holds for any $w = F^\nu(z)$ and a similar integration argument as above, then gives the estimate asserted in the theorem with a uniform constant. \square

2.5. Compactness of $f^{-1}(0)$. Recall that to establish theorem 1.1, we were to prove that $f^{-1}(0)$ is compact in M . Suppose to obtain a contradiction that this were false. Then the intersection

$$f^{-1}(0) \cap M \cap \partial B(0, \epsilon) \neq \emptyset.$$

for all $\epsilon > 0$ small. Since the scaled mappings differ from f by a biholomorphic change of coordinates on the domain and the target, the same holds for them as well, i.e.,

$$(F^\nu)^{-1}(F^\nu(0)) \cap \partial D_\nu \cap \partial B(0, \epsilon) \neq \emptyset.$$

Let us show that this property passes to the limit as well, i.e.,

$$F^{-1}(q) \cap \partial D_\infty \cap \partial B(0, \epsilon) \neq \emptyset$$

for all $\epsilon > 0$ small – so small that Theorem 2.11 holds for $r = \epsilon$. Let a^ν be in $(F^\nu)^{-1}(F^\nu(0)) \cap \partial D_\nu \cap \partial B(0, \epsilon)$ and $a^\nu \rightarrow a \in \partial D_\infty \cap \partial B(0, \epsilon)$. Let b be a point in D_∞ near a . Since $D_\nu \rightarrow D_\infty$, there exists an integer N such that for all $\nu > N$, b lies in D^ν and

$$|F(a) - q| \leq |F(a) - F(b)| + |F(b) - F^\nu(b)| + |F^\nu(b) - F^\nu(a^\nu)| + |F^\nu(a^\nu) - q|$$

By taking ν large enough, the second term can be made arbitrarily small by the convergence of F^ν at b and the last term as well by the convergence of $F^\nu(a^\nu) = F^\nu(0)$ to q . The equicontinuity of the family $\{F^\nu\} \cup \{F\}$ given by theorem 2.11, then assures that the same can be done with the third and the first terms, by taking b sufficiently close to a and ν larger if necessary. Thus, $F^{-1}(q)$ is not compact in any neighbourhood of the origin in ∂D_∞ .

On the other hand, by starting from the standpoint of F being a holomorphic mapping between algebraic domains that extends continuously upto a boundary piece Σ of ∂D_∞ , we may apply the theorem of Webster to establish the algebraicity of F . Thus F extends as an analytic set and by the invariance property of Segre varieties (see [19]) as a locally finite to one holomorphic map near $0 \in \partial D_\infty$. Contradiction.

To make this work, we only have to show that F does not map an open piece of ∂D_∞ into the weakly pseudoconvex points on $\partial D'_\infty$. Let us denote by $w(\Gamma)$ the set of all weakly pseudoconvex points of a given smooth hypersurface Γ . Since

$$\partial D_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + P(z, \bar{z}) = 0\}$$

it follows that

$$w(\partial D_\infty) = (Z \times \mathbb{C}) \cap \partial D_\infty$$

where Z is the real algebraic variety in \mathbb{C}^{n-1} defined by the vanishing of the determinant of the complex Hessian of P in \mathbb{C}^{n-1} . The finite type assumption on ∂D_∞ implies that $w(\partial D_\infty)$ is a real algebraic set of dimension at most $2n - 2$. Recall that

$$F(\Sigma) \subset \partial D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 = 0\},$$

and denote $Q_{2m'}$ by Q for brevity.

Proposition 2.13. *The map F extends to an algebraic map in \mathbb{C}^n .*

Proof. Suppose now that there exists a strictly pseudoconvex point $a \in \partial D_\infty$ such that $F(a)$ is also a strictly pseudoconvex point in $\partial D'_\infty$. Then by [41], F is a smooth CR-diffeomorphism near a and extends locally biholomorphically across a by the reflection principle in [39]. Then Webster's theorem [46] assures us that F is algebraic. So, we may as well let Σ be the set of strictly pseudoconvex points, assume that F maps it into $w(\partial D'_\infty)$ and argue only to obtain a contradiction. As before note that

$$w(\partial D'_\infty) = \{z \in \mathbb{C}^n : \Delta Q(z_1, \bar{z}_1) = 0\} \cap \partial D'_\infty.$$

Then $w(\partial D'_\infty)$ admits a semi-analytic stratification by real analytic manifolds of dimension $2n - 2$ and $2n - 3$. Using the specific form of $\partial D'_\infty$, this can be explicitly described as follows. Let us denote by $V = V(\Delta Q)$ the real algebraic variety in \mathbb{C} defined by the polynomial ΔQ . The set of singular points $\text{Sng}(V)$, near which V may fail to be a smooth curve is finite. Let $a \in w(\partial D'_\infty)$ be such that $\pi_1(a) \in \text{Reg}(V)$ where π_1 is the natural surjection onto the z_1 -axis. Then V is a smooth curve near $\pi_1(a)$ and after a biholomorphic change of coordinates, we may assume that in its vicinity V coincides with $\{\Im z_1 = 0\}$. Note that the fibre of π_1 over a in $w(\partial D'_\infty)$ namely $\pi_1^{-1}(\pi_1(a)) \cap w(\partial D'_\infty)$, is given by

$$\left\{ z \in \mathbb{C}^n : 2\Re z_n + |z_2|^2 + \dots + |z_{n-1}|^2 = -Q(\pi_1(a), \overline{\pi_1(a)}) \right\} \cap \left\{ z \in \mathbb{C}^n : z_1 = \pi_1(a) \right\}$$

which is evidently equivalent to $\partial \mathbb{B}^{n-1} \subset \mathbb{C}^{n-1}$. This description of the fibres of π_1 restricted to $w(\partial D'_\infty)$ persists in a neighbourhood of a and then the real analytic strata $S_{2n-2} \subset w(\partial D'_\infty)$ of dimension $2n - 2$ is locally biholomorphic to $\partial \mathbb{B}^{n-1} \times \{\Im z_1 = 0\}$. The complement of S_{2n-2} in $w(\partial D'_\infty)$, which we will denote by S_{2n-3} is locally biholomorphic to $\partial \mathbb{B}^{n-1} \times \text{Sng}(V)$ and this evidently has dimension $2n - 3$. Observe also that S_{2n-2} is generic since its complex tangent space has dimension $n - 2$ as a complex vector space. Now, if

$$F(\Sigma) \cap S_{2n-2} \neq \emptyset$$

then by the continuity of F , there is an open piece of Σ that is mapped by F into S_{2n-2} . Denote this piece by $\tilde{\Sigma}$. We may assume that $0 \in \tilde{\Sigma}$ and that $F(0) = 0 \in S_{2n-2}$. Using an idea from Lemma 3.2 in [28], let L be a 2 dimensional complex plane that intersects $w(\partial D'_\infty)$ in a totally real submanifold of real dimension 2 – this is possible by the genericity of S_{2n-2} – and this also holds for all translates $L_{a'}$ of L passing through a' in a sufficiently small neighbourhood U' of the origin. We may therefore find a non-negative, strictly plurisubharmonic function $\phi_{a'}$ on U' that vanishes on

$$S_{a'} = L'_{a'} \cap w(\partial D'_\infty) \cap U'.$$

Indeed by a change of coordinates, we may assume that

$$L = \text{span}_{\mathbb{C}} \langle \partial/\partial z_1, \partial/\partial z_n \rangle = \{z_2 = \dots = z_{n-1} = 0\}$$

and then

$$\phi_{a'} = |z_2 - a'_2|^2 + \dots + |z_{n-1} - a'_{n-1}|^2 + |\Im z_1|^2 + (r'_\infty(z, \bar{z}))^2$$

furnishes an example. By the continuity of F and the openness of $\tilde{\Sigma}$ in ∂D_∞ , we can pick $b \in D_\infty$ so near the origin that $F(b) \in U'$ and $\partial A_b \subset \tilde{\Sigma}$ where

$$A_b = \{z \in \mathbb{C}^n : z_n = b_n\} \cap F^{-1}(L_{F(b)})$$

Since the pull-back of an analytic set under a holomorphic map is again analytic of no lesser dimension, A_b is a positive dimensional analytic set. Also, $b \in A_b$ and $F(\partial A_b) \subset S_{F(b)}$. Therefore, $\psi_b = \phi_{F(b)} \circ F$ is a non-negative, plurisubharmonic function on A_b that vanishes on ∂A_b . By the maximum principle $\psi_b \equiv 0$ on A_b which implies that F maps all of A_b into $S_{F(b)}$. Since F maps D_∞ into D'_∞ , this is a contradiction.

To finish, note that the remaining possibility is $F(\Sigma) \subset S_{2n-3}$. The above argument can be repeated in this case as well – we will only need to replace $|\Im z_1|^2$ as a subharmonic function vanishing on a curve-segment of $\text{Reg}(V)$ by $|p(z_1)|^2$ where p is a holomorphic polynomial that vanishes on the finite set $\text{Sng}(V)$. \square

3. PROOF OF THEOREM 1.3

We begin with the scaling template for (D, D', f) in section 2.1, maintaining as far as possible the notations therein. For clarity and completeness, let us briefly describe the scaling of D' which is simpler this time as M' is strongly pseudoconvex. As before, let $p^\nu = (0, -\delta_\nu) \in D$ and note that $p^\nu = f(p^\nu)$ converges to the origin which is a strongly pseudoconvex point on $\partial D'$. Let $w^\nu \in \partial D'$ be such that

$$|w^\nu - p^\nu| = \text{dist}(p^\nu, \partial D') = \gamma_\nu$$

Furthermore, since $\partial D'$ is strongly pseudoconvex near the origin, we may choose a strongly plurisubharmonic function in a neighbourhood of the origin that serves as a defining function for D' . Arguing as in Section 2.1, it follows that f preserves the distance to the boundary, i.e.,

$$\delta_\nu \approx \gamma_\nu$$

for $\nu \gg 1$. For each w^ν , lemma 2.2 in [40] provides a degree two polynomial automorphism of \mathbb{C}^n that firstly, transfers w^ν and the normal to $\partial D'$ there, to the origin and the $\Re z_n$ -axis respectively and secondly, ensures that the second order terms in the Taylor expansion of the defining function $r' \circ (g^\nu)^{-1}$ of the domain $g^\nu(D')$ constitute a hermitian form that coincides with the standard one i.e., $|z_1|^2 + \dots + |z_{n-1}|^2$, upon restriction to the complex tangent space. Define the dilations

$$B^\nu(z, z_n) = (\gamma_\nu^{-1/2} z, \gamma_\nu^{-1} z_n)$$

and note that the scaled domains $D'_\nu = (B^\nu \circ g^\nu)(D')$ are defined by

$$\gamma_\nu^{-1} r'_\nu(z) = 2\Re(g^\nu(z))_n + |(g^\nu(z))_1|^2 + \dots + |(g^\nu(z))_{n-1}|^2 + O(\gamma_\nu^{1/2}).$$

These converge in the Hausdorff metric to

$$D'_\infty = \mathbb{H} = \{z \in \mathbb{C}^n : 2\Re z_n + |z_1|^2 + \dots + |z_{n-1}|^2 < 0\},$$

which is the unbounded manifestation of the ball, in view of the fact that the g^ν 's converge uniformly on compact subsets of \mathbb{C}^n to the identity map.

Standard arguments as in [40] show that the scaled maps

$$F^\nu = B^\nu \circ g^\nu \circ f \circ (T^\nu)^{-1}$$

converge to a map $F : D_\infty \rightarrow \overline{\mathbb{H}}$. If some point z_0 of D_∞ is sent by F to $w_0 \in \partial\mathbb{H} \cup \{\infty\}$, then composing F with a local peak function at w_0 , we get a function holomorphic on a neighbourhood of z_0 and peaking precisely at z_0 . By the maximum principle, $F(z) \equiv w_0$. However, $F^\nu('0, -1) = B^\nu \circ g^\nu \circ f \circ T^\nu('0, -1) = B^\nu \circ g^\nu(f(p^\nu)) = B^\nu('0, -\gamma_\nu) = ('0, -1)$. Hence, $F('0, -1) = ('0, -1) \neq w_0$. Thus, F maps D_∞ into D'_∞ and is again as in section 2, a non-degenerate, locally proper map extending continuously upto the boundary in a neighbourhood of the origin. By theorem 2.1 in [19], F extends holomorphically across the origin. By composing with a suitable automorphism of the ball, we may also assume $F(0) = 0$. Then, $\Re(F_n(z))$ is a pluriharmonic function that is negative on D_∞ and attains a maximum at the origin. So, by the Hopf lemma, we must have $\alpha = \partial(\Re F_n)/\partial x_n(0) > 0$ which combined with the fact that DF preserves the complex tangent space and thereby the complex normal at the origin (to the hypersurfaces ∂D_∞ and $\partial D'_\infty$, which themselves correspond under F near the origin), implies that

$$F_n(z) = \alpha z_n + g(z)$$

for some holomorphic function g with $g(z) = o(|z|)$. Now, let us compare the two defining functions for D_∞ , near the origin:

$$2\Re(F_n(z)) + |F_1(z)|^2 + \dots + |F_{n-1}(z)|^2 = h(z, \bar{z})(2\Re z_n + P('z, ' \bar{z}))$$

for a non-vanishing real analytic function $h(z, \bar{z})$. Contemplate a weighted homogeneous expansion of the above equation with respect to the weight $(1/m_n, \dots, 1/m_2)$ given by the multitype of ∂D_∞ . Note firstly that on the left, pluriharmonic terms arise precisely from $\Re(F_n(z))$. Next, note that the lowest possible weight for any term on the right is one and the non-pluriharmonic component of this weight is $h(0)P('z, ' \bar{z})$. What this means for the left, is that each F_j must expand as

$$F_j(z) = P_j('z) + (\text{terms of weight } > 1/2)$$

where each P_j is either weighted homogeneous of weight $1/2$ or identically zero and

$$(3.1) \quad h(0)P('z, ' \bar{z}) = |P_1('z)|^2 + \dots + |P_{n-1}('z)|^2$$

Clearly, all the P_j 's cannot be zero as P is non-zero. In fact, the finite type character of ∂D_∞ forces all of them to be non-zero, as follows. After a rearrangement if necessary, assume that $P_j \in \mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_{n-1}]$ is non-zero precisely when $1 \leq j \leq m \leq n-1$. Then the common zero set $V \subset \mathbb{C}^{n-1}$, of these P_j 's gives rise to the complex analytic variety $i(V)$ in ∂D_∞ where $i : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^n$ is the natural inclusion. The finite type constraint compels this variety and needless to say V , to be discrete. Furthermore, the weighted homogeneity reduces it to $\{0\}$: If (z_1, \dots, z_{n-1}) is a non-trivial zero of the P_j 's and $t \in \mathbb{C}$, then

$$P_j(e^{t/m_n} z_1, \dots, e^{t/m_2} z_{n-1}) = e^{t/2} P_j(z_1, \dots, z_{n-1}) = 0$$

for all $1 \leq j \leq m$ and so the entire curve defined by

$$\gamma(t) = (e^{t/m_n} z_1, \dots, e^{t/m_2} z_{n-1})$$

lies inside V . Now, consider the ideal I generated by these polynomials P_1, \dots, P_m , which is a \mathbb{C} -algebra whose transcendence degree cannot exceed $m \leq n - 1$. On the other hand, by the Nullstellensatz, for a large integer N the algebraically independent monomials, z_1^N, \dots, z_{n-1}^N must all lie in I , forcing its transcendence degree over \mathbb{C} and hence m to equal $n - 1$. The upshot therefore, is that P is the squared norm of a weighted homogeneous polynomial endomorphism \hat{P} of \mathbb{C}^{n-1} with $\hat{P}^{-1}(0) = 0$. Now, put $z_2 = \dots = z_{n-1} = 0$ in (3.1). This cannot reduce the left hand side there to zero, otherwise the z_1 -axis will lie inside the zero set of P and hence in ∂D_∞ . What this means for the right hand side of (3.1), is that terms involving z_1 (and \bar{z}_1) must occur. Since all polynomials therein are homogeneous of same weight, we conclude that terms involving z_1, \bar{z}_1 there, must only be of the form $c|z_1|^{m_n}$ for some $c > 0$. Needless to say, the same holds for all the other variables z_j for $2 \leq j \leq n - 1$, as well and we have the last statement of theorem 1.3, namely

$$P(z', \bar{z}) = c_1|z_1|^{m_n} + c_2|z_2|^{m_{n-1}} + \dots + c_{n-1}|z_{n-1}|^{m_2} + \text{mixed terms}$$

with all c_j 's being positive and the mixed terms comprising of weight one monomials in $\{z', \bar{z}\}$ each of which is annihilated by at least one of the natural quotient maps $\mathbb{C}[z', \bar{z}] \rightarrow \mathbb{C}[z', \bar{z}]/(z_j \bar{z}_k)$, $1 \leq j, k \leq n - 1$ where $j \neq k$.

4. PROOF OF THEOREM 1.4

Proof. As before by the lower semi-continuity of rank, $\partial D'$ is of rank at least $n - 2$ in a neighbourhood $\Gamma' \subset \partial D'$ of p' which we may assume also to be of finite type and pseudoconvex, consequently regular. Let us apply Theorem A of [13] to the proper holomorphic correspondence $f^{-1} : D' \rightarrow D$. The hypothesis on $cl_{f^{-1}}(\Gamma')$ there holds, since ∂D is globally regular. Therefore by that theorem, f^{-1} extends continuously upto Γ' as a proper correspondence. However, we are not certain of the splitting of f^{-1} near p into branches – for that p we will have to lie away from the branch locus of f^{-1} . Nevertheless, there exist neighbourhoods U' of p' and U of p and a (local) correspondence

$$f_{loc}^{-1} : U' \cap D' \rightarrow U \cap D,$$

extending continuously upto the boundary such that the graph of f_{loc}^{-1} is contained in that of f^{-1} and $cl_{f_{loc}^{-1}}(p') = \{p\}$ where the last condition comes from the fact that f is finite to one upto the boundary, by that theorem again. Assume that both $p = 0$ and $p' = 0$ and choose a sequence $p'' = (0, -\delta_\nu) \in D' \cap U'$ on the inner normal approaching the origin. Since $f : D \rightarrow D'$ is proper and $0 \in cl_f(0)$ there exists a sequence $p'' \in D$ with $p'' \rightarrow 0$ such that $f(p'') = p''$. Moreover, by the continuity of f_{loc}^{-1} upto the boundary and the condition $cl_{f_{loc}^{-1}}(0) = \{0\}$, we may assume after shrinking U, U' if necessary, that $p'' \in D \cap U$ with $f_{loc}^{-1}(p'') = \{p''\}$. Now scale D with respect to $\{p''\}$ and D' with respect to $\{p''\}$ – to scale D' , we only consider the dilations

$$T^\nu(z_1, \dots, z_n) = ((\tau_1^\nu)^{-1}z_1, (\tau_2^\nu)^{-1}, \dots, (\tau_n^\nu)^{-1}z_n)$$

where $\tau_1^\nu = \tau(0, \delta_\nu)$, $\tau_j^\nu = \delta_\nu^{1/2}$ for $2 \leq j \leq n - 1$ and $\tau_n^\nu = \delta_\nu$, while for D we use the composition $B^\nu \circ g^\nu$ where g^ν and B^ν are as in section 3. As before, the limiting domains for $D_\nu = (B^\nu \circ g^\nu)(U \cap D)$ and $D'_\nu = T^\nu(U' \cap D')$ are the ball \mathbb{B}^n and

$$D'_\infty = \{z \in \mathbb{C}^n : 2\Re z_n + Q_{2m'}(z_1, \bar{z}_1) + |z_2|^2 + \dots + |z_{n-1}|^2 < 0\},$$

respectively, where this time $Q_{2m'}$ is homogeneous of degree $2m'$ and coincides with the polynomial of this degree in the (homogeneous) Taylor expansion of the defining function r' for $\partial D'$ near the origin. Normality of the scaled mappings

$$F^\nu = T^\nu \circ f \circ (B^\nu \circ g^\nu)^{-1}$$

follows as before by proposition 2.1. But section 2 can no longer guarantee nondegeneracy of a limit map, owing to the lack of a clear cut boundary distance conservation property of f . However, the existence of the inverse as a proper correspondence paves the way for a different approach. To begin with, recall from [35], the notion of normality for correspondences and theorem 3 therein, the version of Montel's theorem for proper holomorphic correspondences with varying domains and ranges. Let K_μ be an exhaustion of D'_∞ by compact subsets containing $(0, -1)$. To establish the normality of scaled correspondences

$$(f_{loc}^{-1})^\nu = (B^\nu \circ g^\nu) \circ f_{loc}^{-1} \circ (T^\nu)^{-1},$$

it suffices by theorem 3 of [35] to show that $(f_{loc}^{-1})^\nu(K_\mu) \Subset D_\infty$ for each $\mu \in \mathbb{N}$. To this end, fix a K_μ , let $(\tilde{f}_{loc}^{-1})^\nu$ denote the correspondence $f_{loc}^{-1} \circ (T^\nu)^{-1}$ and note that $(\tilde{f}_{loc}^{-1})^\nu(K_\mu)$ is connected (since any of its components must contain p^ν). Now (viewing the origin in ∂D as the sequence p^ν that approaches it from the interior of D), we recall the Schwarz lemma for correspondences given by theorem 1.2 of [45], which assures us that the images of K_μ under these correspondences $(\tilde{f}_{loc}^{-1})^\nu$, will be contained in Kobayashi balls about p^ν of a fixed size, i.e., there exists $R > 0$ independent of ν such that

$$(\tilde{f}_{loc}^{-1})^\nu(K_\mu) \subset B_D^K(p^\nu, R),$$

where $B_\Omega^K(p, R)$ denotes the Kobayashi ball centered at $p \in \Omega$ of radius R , for any given domain Ω . Then, since biholomorphisms preserve Kobayashi balls,

$$(f_{loc}^{-1})^\nu(K_\mu) \subset (B^\nu \circ g^\nu)((\tilde{f}_{loc}^{-1})^\nu(K_\mu)) \subset (B^\nu \circ g^\nu)(B_D^K(p^\nu, R)) = B_{D_\nu}^K((0, -1), R)$$

while $B_{D_\nu}^K((0, -1), R) \subset B_{\mathbb{B}^n}^K((0, -1), R+1)$ for all ν large by lemma 4.4 of [37], giving the stability of the images of K_μ under scaling. Noting that $((0, -1), (0, -1)) \in \text{Graph}(f_{loc}^{-1})^\nu$ for all ν , we conclude that $\{(f_{loc}^{-1})^\nu\}$ must admit a subsequence that converges to a correspondence that is inverse to F , where $F : \mathbb{B}^n \rightarrow D'_\infty$ is a limit of $\{F^\nu\}$ (which means that the composition of these correspondences **contains** the graph of the identity). By [13], both F and the inverse correspondence F^{-1} extend continuously upto the respective boundaries and in particular F is finite to one on the boundary. Then, F is smooth upto the boundary by [9], preserves boundaries and then is algebraic again as in section 2.6. and thereafter by [5] extends holomorphically past the boundary. Now recall that the determinant of the Levi forms are related as

$$\lambda_{\mathbb{B}^n}(z) = \lambda_{D'_\infty}(F(z)) |J_F(z)|^2$$

for all $z \in \partial \mathbb{B}^n$, which readily gives the strict pseudoconvexity of $\partial D'_\infty$. In particular therefore, $Q_{2m'}(z_1, \bar{z}_1)$ must be $|z_1|^2$ which gives the strict pseudoconvexity of p' in $\partial D'$. \square

Proof of corollary 1.5. By the proof of Theorem 1.1 in section 2, we have that $f^{-1}(f(p))$ is compact in M and as described in [9], it is possible to choose neighbourhoods U of p and U' of p' in \mathbb{C}^n such that if D and D' are the pseudoconvex sides of $U \cap M$ and $U' \cap M'$ respectively then f extends to be a proper map from D into D' , putting us in the situation of theorem 1.4. \square

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