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Unified a priori analysis of four second-order FEM for fourth-order quadratic semilinear problems

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Abstract
A unified framework for fourth-order semilinear problems with trilinear nonlinearity and general sources allows for quasi-best approximation with lowest-order finite element methods. This paper establishes the stability and a priori error control in the piecewise energy and weaker Sobolev norms under minimal hypotheses. Applications include the stream function vorticity formulation of the incompressible 2D Navier-Stokes equations and the von Kármán equations with Morley, discontinuous Galerkin, C^0 interior penalty, and weakly over-penalized symmetric interior penalty schemes. The proposed new discretizations consider quasi-optimal smoothers for the source term and smoother-type modifications inside the nonlinear terms.

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1 Introduction

The abstract framework for fourth-order semilinear elliptic problems with trilinear nonlinearity in this paper allows a source term $F \in H^{-2}(\Omega)$ in a bounded polygonal Lipschitz domain Ω. It simultaneously applies to the Morley finite element method (FEM) [8, 15], the discontinuous Galerkin (dG) FEM [18], the C^0 interior penalty (C^0IP) method [3], and the weakly over-penalized symmetric interior penalty (WOPSIP) scheme [1] for the approximation of a regular solution to a fourth-order semilinear problem with the biharmonic operator as the leading term. In comparison to [8], this article includes dG/C^0IP/WOPSIP schemes and more general source terms that allow single forces. It thereby continues [11] for the linear biharmonic equation to semilinear problems and, for the *first* time, establishes quasi-best approximation results for a discretisation by the Morley/dG/C^0IP schemes with smoother-type modifications in the nonlinearities.

A general source term $F \in H^{-2}(\Omega)$ cannot be immediately evaluated at a possibly discontinuous test function $v_h \in V_h \not\subset H_0^2(\Omega)$ for the nonconforming FEMs of this paper. The post-processing procedure in [3] enables a new C^0IP method for right-hand sides in $H^{-2}(\Omega)$. The articles [25–27] employ a map Q, referred to as a smoother, that transforms a nonsmooth function y_h to a smooth version Qy_h. The discrete schemes are modified by replacing F with $F \circ Q$ and the quasi-best approximation follows for Morley and C^0IP schemes for linear problems in the energy norm. The quasi-optimal smoother $Q = JI_{\mathrm{M}}$ in [11] for dG schemes is based on a (generalised) Morley interpolation operator I_{M} and a companion operator J from [12, 19].

In addition to the smoother Q in the right-hand side, this article introduces operators $R, S \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ in the trilinear form $\Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Sv_h)$ that lead to *nine* new discretizations for each of the four discretization schemes (Morley/dG/C^0IP/WOPSIP) in two applications. Here $R, S = \mathrm{id}$ means no smoother, I_{M} is averaging in the Morley finite element space, while JI_{M} is the quasi-optimal smoother. The simultaneous analysis applies to the stream function vorticity formulation of the 2D Navier-Stokes equations [6, 13, 14] and von Kármán equations [16, 23] defined on a bounded polygonal Lipschitz domain Ω in the plane. For $S = JI_{\mathrm{M}}$ and all $R \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$, the Morley/dG/$C^0$IP schemes allow for the quasi-best approximation

$$\|u - u_h\|_{\widehat{X}} \leq C_{\mathrm{qo}} \min_{x_h \in X_h} \|u - x_h\|_{\widehat{X}}. \tag{1.1}$$

Duality arguments lead to optimal convergence rates in weaker Sobolev norm estimates for the discrete schemes with specific choices of R in the trilinear form summarised in Table 1. The comparison results suggest that, amongst the lowest-order methods for fourth-order semilinear problems with trilinear nonlinearity, the attractive Morley FEM is the *simplest* discretization scheme with optimal error estimates in (piecewise) energy and weaker Sobolev norms.

For $F \in H^{-r}(\Omega)$ with $2 - \sigma \leq r \leq 2$ (with the index of elliptic regularity $\sigma_{\mathrm{reg}} > 0$ and $\sigma := \min\{\sigma_{\mathrm{reg}}, 1\} > 0$) and for the biharmonic, the 2D Navier-Stokes, and the von Kármán equations with homogeneous Dirichlet boundary conditions, it is known that the exact solution belongs to $H_0^2(\Omega) \cap H^{4-r}(\Omega)$.

Table 1 Summary for Navier-Stokes and von Kármán eqn from Sects. 8 and 9 with $F \in H^{-r}(\Omega)$ for $2-\sigma \le r, s \le 2$ and $R, S \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ arbitrary unless otherwise specified

Method	Results	
	Quasi-best for $S = JI_{\mathrm{M}}$	$\Vert u-u_h\Vert_{H^s(\mathcal{T})}$
Morley dG/C^0IP	(1.1)	$O(h_{\max}^{\min\{4-2r,4-r-s\}})$
WOPSIP	Perturbed Theorem 8.11.a & 9.4.a	$O(h_{\max}^{2-r})$ for $R = \mathrm{id}$, $O(h_{\max}^{\min\{4-2r,4-r-s\}})$ for $R \in \{I_{\mathrm{M}}, JI_{\mathrm{M}}\}$

Organisation. The remaining parts are organised as follows. Section 2 discusses an abstract discrete inf-sup condition for linearised problems. Section 3 introduces the main results **(A)-(C)** of this article. Section 4 discusses the quadratic convergence of Newton's scheme and the unique existence of a local discrete solution u_h that approximates a regular root $u \in H_0^2(\Omega)$ for data $F \in H^{-2}(\Omega)$. Section 5 presents an abstract a priori error control in the piecewise energy norm with a quasi-best approximation for $S = JI_{\mathrm{M}}$ in (1.1). Section 6 discusses the goal-oriented error control and derives an a priori error estimate in weaker Sobolev norms. There are at least two reasons for this abstract framework enfolded in Sects. 2–6. First it minimizes the repetition of mathematical arguments in two important applications and four popular discrete schemes. Second, it provides a platform for further generalizations to more general smooth semilinear problems as it derives all the necessities for the leading terms in the Taylor expansion of a smooth semilinearity. Section 7 presents preliminiaries, triangulations, discrete spaces, the conforming companion, discrete norms and some auxiliary results on I_{M} and J. Sections 8 and 9 apply the abstract results to the stream function vorticity formulation of the 2D Navier-Stokes and the von Kármán equations for the Morley/dG/C^0IP/WOPSIP approximations. They contain comparison results and convergence rates displayed in Table 1.

2 Stability

This section establishes an abstract discrete inf-sup condition under the assumptions (2.1)–(2.3), (2.5), (2.8) and **(H1)-(H3)** stated below. This is a key step and has consequences for second-order elliptic problems (as in [8, Section 2]) and in this paper for the well-posedness of the discretization. In comparison to [8] that merely addresses nonconforming FEM, the proof of the stability in this section applies to all the discrete schemes. Let $\widehat{X}$ (resp. $\widehat{Y}$) be a real Banach space with norm $\Vert \bullet \Vert_{\widehat{X}}$ (resp. $\Vert \bullet \Vert_{\widehat{Y}}$) and suppose X and X_h (resp. Y and Y_h) are two complete linear subspaces of $\widehat{X}$ (resp. $\widehat{Y}$) with inherited norms $\Vert \bullet \Vert_X := \big(\Vert \bullet \Vert_{\widehat{X}}\big)|_X$ and $\Vert \bullet \Vert_{X_h} := \big(\Vert \bullet \Vert_{\widehat{X}}\big)|_{X_h}$ (resp. $\Vert \bullet \Vert_Y := \big(\Vert \bullet \Vert_{\widehat{Y}}\big)|_Y$ and $\Vert \bullet \Vert_{Y_h} := \big(\Vert \bullet \Vert_{\widehat{Y}}\big)|_{Y_h}$); $X + X_h \subseteq \widehat{X}$ and $Y + Y_h \subseteq \widehat{Y}$.

Table 2 summarizes the bounded bilinear forms and associated operators with norms. Let the linear operators $A \in L(X; Y^*)$ and $A + B \in L(X; Y^*)$ be associated to the bilinear forms a and $a + b$ and suppose A and $A + B$ are invertible so that the inf-sup conditions

Table 2 Bilinear forms, operators, and norms

Bilinear form	Domain	Associated operator	Operator norm
a_{pw}	$\widehat{X}\times\widehat{Y}$	–	–
$a:=a_{\mathrm{pw}}\|_{X\times Y}$	$X\times Y$	$A\in L(X;Y^*)$ $Ax=a(x,\bullet)\in Y^*$	$\\|A\\|:=\\|A\\|_{L(X;Y^*)}$
a_h	$X_h\times Y_h$	$A_h\in L(X_h;Y_h^*)$ $A_hx_h=a_h(x_h,\bullet)\in Y_h^*$	–
$\widehat{b}$	$\widehat{X}\times\widehat{Y}$	–	$\\|\widehat{b}\\|:=\\|\widehat{b}\\|_{\widehat{X}\times\widehat{Y}}$
$b:=\widehat{b}\|_{X\times Y}$	$X\times Y$	$B\in L(X;Y^*)$ $Bx=b(x,\bullet)\in Y^*$	$\\|b\\|:=\\|b\\|_{X\times Y}$

$$0<\alpha:=\inf_{\substack{x\in X\\ \|x\|_X=1}}\sup_{\substack{y\in Y\\ \|y\|_Y=1}} a(x,y) \text{ and } 0<\beta:=\inf_{\substack{x\in X\\ \|x\|_X=1}}\sup_{\substack{y\in Y\\ \|y\|_Y=1}}(a+b)(x,y) \tag{2.1}$$

hold. Assume that the linear operator $A_h : X_h \to Y_h^*$ is invertible and

$$0<\alpha_0\le\alpha_h:=\inf_{\substack{x_h\in X_h\\ \|x_h\|_{X_h}=1}}\sup_{\substack{y_h\in Y_h\\ \|y_h\|_{Y_h}=1}} a_h(x_h,y_h) \tag{2.2}$$

holds for some universal constant α_0. Let the linear operators $P\in L(X_h;X)$, $Q\in L(Y_h;Y)$, $R\in L(X_h;\widehat{X})$, $S\in L(Y_h;\widehat{Y})$ and the constants $\Lambda_{\mathrm{P}},\Lambda_{\mathrm{Q}},\Lambda_{\mathrm{R}},\Lambda_{\mathrm{S}}\ge 0$ satisfy

$$\|(1-P)x_h\|_{\widehat{X}}\le\Lambda_{\mathrm{P}}\|x-x_h\|_{\widehat{X}} \quad\text{for all } x_h\in X_h \text{ and } x\in X, \tag{2.3}$$
$$\|(1-Q)y_h\|_{\widehat{Y}}\le\Lambda_{\mathrm{Q}}\|y-y_h\|_{\widehat{Y}} \quad\text{for all } y_h\in Y_h \text{ and } y\in Y, \tag{2.4}$$
$$\|(1-R)x_h\|_{\widehat{X}}\le\Lambda_{\mathrm{R}}\|x-x_h\|_{\widehat{X}} \quad\text{for all } x_h\in X_h \text{ and } x\in X, \tag{2.5}$$
$$\|(1-S)y_h\|_{\widehat{Y}}\le\Lambda_{\mathrm{S}}\|y-y_h\|_{\widehat{Y}} \quad\text{for all } y_h\in Y_h \text{ and } y\in Y. \tag{2.6}$$

Suppose the operator $I_{\mathrm{X}_h}\in L(X;X_h)$, the constants $\Lambda_1,\delta_2,\delta_3\ge 0$, the above bilinear forms a, a_h, $\widehat{b}$, and the linear operator A from Table 2 satisfy, for all $x_h\in X_h$, $y_h\in Y_h$, $x\in X$, and $y\in Y$, that

(H1) $a_h(x_h,y_h)-a(Px_h,Qy_h)\le\Lambda_1\|x_h-Px_h\|_{\widehat{X}}\|y_h\|_{Y_h}$,

(H2) $\delta_2:=\sup_{\substack{x_h\in X_h\\ \|x_h\|_{X_h}=1}}\|(1-I_{\mathrm{X}_h})A^{-1}(\widehat{b}(Rx_h,\bullet)|_Y)\|_{\widehat{X}}$,

(H3) $\delta_3:=\sup_{\substack{x_h\in X_h\\ \|x_h\|_{X_h}=1}}\|\widehat{b}(Rx_h,(Q-S)\bullet)\|_{Y_h^*}$.

In applications, we establish that δ_2 and δ_3 are sufficiently small. Given $\alpha,\beta,\alpha_h,\Lambda_{\mathrm{P}}$, $\Lambda_1,\Lambda_{\mathrm{R}},\delta_2,\delta_3$ from above and the norms $\|A\|$ and $\|\widehat{b}\|$ from Table 2, define

$$\widehat{\beta}:=\frac{\beta}{\Lambda_{\mathrm{P}}\beta+\|A\|\left(1+\Lambda_{\mathrm{P}}\left(1+\alpha^{-1}\|\widehat{b}\|(1+\Lambda_{\mathrm{R}})\right)\right)}, \tag{2.7}$$

$$\beta_0 := \alpha_h \widehat{\beta} - \delta_2(\|Q^*A\|(1+\Lambda_{\mathrm{P}}) + \alpha_h + \Lambda_1\Lambda_{\mathrm{P}}) - \delta_3 \tag{2.8}$$

with the adjoint Q^* of Q. In all applications of this article, $1/\alpha$, $1/\beta$, $1/\alpha_h$, Λ_{P}, Λ_{Q}, Λ_{R}, Λ_{S}, Λ_1, and $\|Q^*A\|$ are bounded from above by generic constants, while δ_2 and δ_3 are controlled in terms of the maximal mesh-size $h_{\max}$ of an underlying triangulation and tend to zero as $h_{\max} \to 0$. Hence, $\beta_0 > 0$ is positive for sufficiently fine triangulations and even bounded away from zero, $\beta_0 \gtrsim 1$. (Here $\beta_0 \gtrsim 1$ means $\beta_0 \geq C$ for some positive generic constant C.) This enables the following discrete inf-sup condition.

Theorem 2.1 (discrete inf-sup condition) *Under the aforementioned notation, (2.1)–(2.3), (2.5), (2.8) and* **(H1)**–**(H3)** *imply the stability condition*

$$\beta_h := \inf_{\substack{x_h \in X_h \\ \|x_h\|_{X_h}=1}} \sup_{\substack{y_h \in Y_h \\ \|y_h\|_{Y_h}=1}} (a_h(x_h, y_h) + \widehat{b}(Rx_h, Sy_h)) \geq \beta_0. \tag{2.9}$$

Before the proof of Theorem 2.1 completes this section, some remarks on the particular choices of R and S are in order to motivate the general description.

Example 2.2 (*quasi-optimal smoother JI_{M}*) This paper follows [11] in the definition of the quasi-optimal smoother $P = Q = JI_{\mathrm{M}}$ in the applications with $X = Y = V =: H_0^2(\Omega)$ for the biharmonic operator A and the linearisation B of the trilinear form. Then (2.3)–(2.4) follow in Sect. 7.3 below; cf. Definition 7.2 (resp. Lemma 7.4) for the definition of the Morley interpolation I_{M} (resp. the companion operator J).

Example 2.3 (*no smoother in nonlinearity*) The natural choice in the setting of Example 2.2 reads $R = \mathrm{id} = S$ [8]. Then $\Lambda_{\mathrm{R}} = 0 = \Lambda_{\mathrm{S}}$ in (2.5)–(2.6) and a priori error estimates will be available for the respective discrete energy norms. However, only a few optimal convergence results shall follow for the error in the piecewise weaker Sobolev norms, e.g., for the Morley scheme for the Navier-Stokes (Theorem 8.5.c) and for the von Kármán equations (Theorem 9.3.b).

Example 2.4 (*smoother in nonlinearity*) The choices $R = P$ and $S = Q$ lead to $\Lambda_{\mathrm{R}} = \Lambda_{\mathrm{P}}$ and $\Lambda_{\mathrm{S}} = \Lambda_{\mathrm{Q}}$ in (2.5)–(2.6), while $\delta_3 = 0$ in **(H3)**. This allows for optimal a priori error estimates in the piecewise energy and in weaker Sobolev norms and this is more than an academic exercise for a richer picture on the respective convergence properties; cf. [10] for exact convergence rates for the Morley FEM. This is important for the analysis of quasi-orthogonality in the proof of optimal convergence rates of adaptive mesh-refining algorithms in [9].

Example 2.5 (*simpler smoother in nonlinearity*) The realisation of $R = S = P = JI_{\mathrm{M}}$ in the setting of Example 2.2 may lead to cumbersome implementations in the nonlinear terms and so the much cheaper choice $R = S = I_{\mathrm{M}}$ shall also be discussed in the applications below.

Remark 2.6 (*on* **(H1)**) The paper [11] adopts [25]-[27] and extends those results to the dG scheme as a preliminary work on linear problems for this paper. The resulting abstract condition (**H1**) therein is a key property to analyze the linear terms simultaneously.

Remark 2.7 (*comparison with* [8]) The set of hypotheses for the discrete inf-sup condition in this article differs from those in [8]. This paper allows smoothers in the nonlinear terms and also applies to dG/C^0IP/WOPSIP schemes.

Remark 2.8 (*consequences of* (2.3)–(2.6)) The estimates in (2.3)–(2.6) give rise to a typical estimate utilised throughout the analysis in this paper. For instance, (2.3) (resp. (2.5)) and a triangle inequality show, for all $x \in X$ and $x_h \in X_h$, that

$$\|x - Px_h\|_X \leq (1 + \Lambda_{\rm P})\|x - x_h\|_{\widehat{X}} (\text{resp. } \|x - Rx_h\|_{\widehat{X}} \leq (1 + \Lambda_{\rm R})\|x - x_h\|_{\widehat{X}}). \tag{2.10}$$

The analog (2.4) (resp. (2.6)) leads, for all $y \in Y$ and $y_h \in Y_h$, to

$$\|y - Qy_h\|_Y \leq (1 + \Lambda_{\rm Q})\|y - y_h\|_{\widehat{Y}} (\text{resp. } \|y - Sy_h\|_{\widehat{Y}} \leq (1 + \Lambda_{\rm S})\|y - y_h\|_{\widehat{Y}}). \tag{2.11}$$

Proof of Theorem 2.1. The proof of Theorem 2.1 departs as in [8, Theorem 2.1] for nonconforming schemes for any given $x_h \in X_h$ with $\|x_h\|_{X_h} = 1$. Define

$$x := Px_h,\ \eta := A^{-1}(Bx),\ \xi := A^{-1}(\widehat{b}(Rx_h, \bullet)|_Y) \in X, \text{ and } \xi_h := I_{X_h}\xi \in X_h.$$

The definitions of $\xi \in X$ and $\xi_h \in X_h$ lead in **(H2)** to

$$\|\xi - \xi_h\|_{\widehat{X}} \leq \delta_2. \tag{2.12}$$

The second inf-sup condition in (2.1) and $A\eta = Bx \in Y^*$ result in

$$\beta\|x\|_X \leq \|Ax + Bx\|_{Y^*} = \|A(x + \eta)\|_{Y^*} \leq \|A\|\|x + \eta\|_X$$

with the operator norm of A in the last step. This and triangle inequalities imply

$$(\beta/\|A\|)\, \|x\|_X \leq \|x + \eta\|_X \leq \|x - x_h\|_{\widehat{X}} + \|x_h + \xi\|_{\widehat{X}} + \|\xi - \eta\|_X. \tag{2.13}$$

The above definitions of ξ and η guarantee $a(\xi - \eta, \bullet) = \widehat{b}(Rx_h - x, \bullet)|_Y \in Y^*$. This, (2.1), and the norm $\|\widehat{b}\|$ of the bilinear form $\widehat{b}$ show

$$\alpha\|\xi - \eta\|_X \leq \|\widehat{b}(x - Rx_h, \bullet)\|_{Y^*} \leq \|\widehat{b}\|\|x - Rx_h\|_{\widehat{X}} \leq \|\widehat{b}\|(1 + \Lambda_{\rm R})\|x - x_h\|_{\widehat{X}}$$

with (2.10) in the last step. Note that the definition $x = Px_h$ and (2.3) imply

$$\|x - x_h\|_{\widehat{X}} \leq \Lambda_{\rm P}\|x_h + \xi\|_{\widehat{X}}. \tag{2.14}$$

The combination of (2.13)–(2.14) results in

$$\|x\|_X \leq \|x_h + \xi\|_{\widehat{X}}(1 + \Lambda_{\rm P}(1 + \alpha^{-1}\|\widehat{b}\|(1 + \Lambda_{\rm R})))\|A\|/\beta. \tag{2.15}$$

A triangle inequality, (2.14)–(2.15), and the definition of $\widehat{\beta}$ in (2.7) lead to

$$1 = \|x_h\|_{X_h} \leq \|x - x_h\|_{\widehat{X}} + \|x\|_X \leq \widehat{\beta}^{-1}\|x_h + \xi\|_{\widehat{X}}.$$

This in the first inequality below and a triangle inequality plus (2.12) show

$$\widehat{\beta} \leq \|x_h + \xi\|_{\widehat{X}} \leq \|x_h + \xi_h\|_{X_h} + \|\xi - \xi_h\|_{\widehat{X}} \leq \|x_h + \xi_h\|_{X_h} + \delta_2. \tag{2.16}$$

The condition (2.2) implies for $x_h + \xi_h \in X_h$ and for any $\epsilon > 0$, the existence of some $\phi_h \in Y_h$ such that $\|\phi_h\|_{Y_h} \leq 1+\epsilon$ and $\alpha_h\|x_h+\xi_h\|_{X_h} = a_h(x_h+\xi_h, \phi_h)$. Elementary algebra shows

$$\begin{aligned}\alpha_h\|x_h + \xi_h\|_{X_h} = a_h(x_h, \phi_h) + a_h(\xi_h, \phi_h) - a(P\xi_h, Q\phi_h) + a(P\xi_h - \xi, Q\phi_h)\\ + a(\xi, Q\phi_h)\end{aligned} \tag{2.17}$$

and motivates the control of the terms below.
Hypothesis **(H1)** and (2.3) imply

$$a_h(\xi_h, \phi_h) - a(P\xi_h, Q\phi_h) \leq \Lambda_1\Lambda_{\mathrm{P}}\|\xi - \xi_h\|_{\widehat{X}}\|\phi_h\|_{Y_h} \leq \Lambda_1\Lambda_{\mathrm{P}}\delta_2(1+\epsilon) \tag{2.18}$$

with (2.12) and $\|\phi_h\|_{Y_h} \leq 1 + \epsilon$ in the last step above. The boundedness of $Q^*A \in L(X; Y_h^*)$, $\|\phi_h\|_{Y_h} \leq 1+\epsilon$, (2.10), and (2.12) for $\|\xi - P\xi_h\|_X \leq (1+\Lambda_{\mathrm{P}})\|\xi - \xi_h\|_{\widehat{X}} \leq (1 + \Lambda_{\mathrm{P}})\delta_2$ reveal

$$a(P\xi_h - \xi, Q\phi_h) \leq \|Q^*A\|(1 + \Lambda_{\mathrm{P}})\delta_2(1+\epsilon). \tag{2.19}$$

The definition of ξ shows that $a(\xi, Q\phi_h) = \widehat{b}(Rx_h, Q\phi_h)$. This, $\|\phi_h\|_{Y_h} \leq 1+\epsilon$, and **(H3)** imply

$$a(\xi, Q\phi_h) \leq \widehat{b}(Rx_h, S\phi_h) + \delta_3(1+\epsilon). \tag{2.20}$$

The combination of (2.17)- (2.20) reads

$$\begin{aligned}\alpha_h\|x_h + \xi_h\|_{X_h} \leq a_h(x_h, \phi_h) + \widehat{b}(Rx_h, S\phi_h) + \big((\|Q^*A\|(1 + \Lambda_{\mathrm{P}})\\ + \Lambda_1\Lambda_{\mathrm{P}})\delta_2 + \delta_3\big)(1+\epsilon).\end{aligned} \tag{2.21}$$

This, (2.16), and $\|\phi_h\|_{Y_h} \leq 1 + \epsilon$ imply $\alpha_h\widehat{\beta} \leq (\|a_h(x_h, \bullet) + \widehat{b}(Rx_h, S\bullet)\|_{Y_h^*} + (\|Q^*A\|(1 + \Lambda_{\mathrm{P}}) + \Lambda_1\Lambda_{\mathrm{P}})\delta_2 + \delta_3\big)(1+\epsilon) + \alpha_h\delta_2$. This and (2.8) demonstrate

$$\alpha_h\widehat{\beta} \leq (\|a_h(x_h, \bullet) + \widehat{b}(Rx_h, S\bullet)\|_{Y_h^*} + \alpha_h\widehat{\beta} - \beta_0)(1+\epsilon) - \epsilon\alpha_h\delta.$$

At this point, we may choose $\epsilon \searrow 0$ and obtain

$$\beta_0 \leq \|a_h(x_h, \bullet) + \widehat{b}(Rx_h, S\bullet)\|_{Y_h^*}.$$

Since $x_h \in X_h$ is arbitrary with $\|x_h\|_{X_h} = 1$, this proves the discrete inf-sup condition (2.9). (In this section Y_h is a closed subspace of the Banach space $\widehat{Y}$ and not necessarily reflexive. In the sections below, Y_h is finite-dimensional and the above arguments apply immediately to $\epsilon = 0$.) □

3 Main results

This section introduces the continuous and discrete nonlinear problems, associated notations, and states the main results of this article in **(A)-(C)** below. The paper has two parts written in abstract results of Sects. 2, 4–6 and their applications in Sects. 8-9. In the first part, the hypotheses **(H1)-(H3)** in the setting of Sect. 2 and the hypothesis **(H4)** stated below guarantee the existence and uniqueness of an approximate solution for the discrete problem, feasibility of an iterated Newton scheme, and an a priori energy norm estimate in **(A)-(B)**. An additional hypothesis (**$\widehat{\mathbf{H1}}$**) enables a priori error estimates in weaker Sobolev norms stated in **(C)**. The second part in Sects. 8-9 verifies the abstract results for the 2D Navier-Stokes equations in the stream function vorticity formulation and for the von Kármán equations.

Adopt the notation on the Banach spaces X and Y (with $X_h, \widehat{X}$ and $Y_h, \widehat{Y}$) of the previous section and suppose that the quadratic function $N : X \to Y^*$ is

$$N(x) := Ax + \Gamma(x, x, \bullet) - F(\bullet) \quad \text{for all } x \in X \tag{3.1}$$

with a bounded linear operator $A \in L(X; Y^*)$, a bounded trilinear form $\Gamma : X \times X \times Y \to \mathbb{R}$, and a linear form $F \in Y^*$. Suppose there exists a bounded trilinear form $\widehat{\Gamma} : \widehat{X} \times \widehat{X} \times \widehat{Y} \to \mathbb{R}$ with $\Gamma = \widehat{\Gamma}|_{X\times X\times Y}$, $\Gamma_h = \widehat{\Gamma}|_{X_h\times X_h\times Y_h}$, and let

$$\|\widehat{\Gamma}\| := \|\widehat{\Gamma}\|_{\widehat{X}\times\widehat{X}\times\widehat{Y}} := \sup_{\substack{\widehat{x}\in\widehat{X} \\ \|\widehat{x}\|_{\widehat{X}}=1}} \sup_{\substack{\widehat{\xi}\in\widehat{X} \\ \|\widehat{\xi}\|_{\widehat{X}}=1}} \sup_{\substack{\widehat{y}\in\widehat{Y} \\ \|\widehat{y}\|_{\widehat{Y}}=1}} \widehat{\Gamma}(\widehat{x}, \widehat{\xi}, \widehat{y}) < \infty.$$

The linearisation of $\widehat{\Gamma}$ at $u \in X$ defines the bilinear form $\widehat{b} : \widehat{X} \times \widehat{Y} \to \mathbb{R}$,

$$\widehat{b}(\bullet, \bullet) := \widehat{\Gamma}(u, \bullet, \bullet) + \widehat{\Gamma}(\bullet, u, \bullet). \tag{3.2}$$

The boundedness of $\widehat{\Gamma}(\bullet, \bullet, \bullet)$ applies to (3.2) and provides $\|\widehat{b}\| \leq 2\|\widehat{\Gamma}\|\,\|u\|_X$.

Definition 3.1 (*regular root*) A function $u \in X$ is a regular root to (3.1), if u solves

$$N(u; y) = a(u, y) + \Gamma(u, u, y) - F(y) = 0 \quad \text{for all} y \in Y \tag{3.3}$$

and the Frechét derivative $DN(u) =: (a + b)(\bullet, \bullet)$ defines an isomorphism $A + B$ and in particular satisfies the inf-sup condition (2.1) for $b := \widehat{b}|_{X\times Y}$ and $\widehat{b}$ from (3.2). □

Abbreviate $(a + b)(x, y) := a(x, y) + b(x, y)$ etc. Several *discrete problems* in this article are defined for different choices of R and S with (2.5)–(2.6) to approximate the regular root u to N. In the applications of Sects. 8-9, $R, S \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ lead to *eight* new discrete nonlinearities. Let X_h and Y_h be finite-dimensional spaces and let

$$N_h(x_h):=a_h(x_h,\bullet)+\widehat{\Gamma}(Rx_h,Rx_h,S\bullet)-F(Q\bullet)\in Y_h^*. \tag{3.4}$$

The discrete problem seeks a root $u_h \in X_h$ to N_h; in other words it seeks $u_h \in X_h$ that satisfies

$$N_h(u_h;y_h):=a_h(u_h,y_h)+\widehat{\Gamma}(Ru_h,Ru_h,Sy_h)-F(Qy_h)=0 \text{ for all } y_h\in Y_h. \tag{3.5}$$

The local discrete solution $u_h \in X_h$ depends on R and S (suppressed in the notation).

Suppose

(H4) $\exists x_h \in X_h$ such that $\delta_4:=\|u-x_h\|_{\widehat{X}} < \beta_0/2(1+\Lambda_{\mathrm{R}})\|\widehat{\Gamma}\|\|R\|\|S\|$,

so that, in particular,

$$\beta_1:=\beta_0-2(1+\Lambda_{\mathrm{R}})\|\widehat{\Gamma}\|\|R\|\|S\|\delta_4>0. \tag{3.6}$$

The non-negative parameters $\Lambda_1, \delta_2, \delta_3, \delta_4, \beta$, and $\|\widehat{b}\|$ depend on the regular root u to N (suppressed in the notation).

The hypotheses **(H1)-(H4)** with sufficiently small $\delta_2, \delta_3, \delta_4$ imply the results stated in **(A)-(B)** below for parameters $\epsilon_1, \epsilon_2, \delta, \rho, C_{\mathrm{qo}} > 0$ and $0 < \kappa < 1$, such that **(A)-(B)** hold for any underlying triangulation $\mathcal{T}$ with maximum mesh-size $h_{\max} \le \delta$ in the applications of this article.

(A) *local existence of a discrete solution.* There exists a unique discrete solution $u_h \in X_h$ to $N_h(u_h)=0$ in (3.5) with $\|u-u_h\|_{\widehat{X}} \le \epsilon_1$. For any initial iterate $v_h \in X_h$ with $\|u_h-v_h\|_{X_h} \le \rho$, the Newton scheme converges quadratically to u_h.

(B) *a priori error control in energy norm.* The continuous (resp. discrete) solution $u \in X$ (resp. $u_h \in X_h$) with $\|u-u_h\|_{\widehat{X}} \le \epsilon_2 := \min\left\{\epsilon_1, \frac{\kappa\beta_1}{(1+\Lambda_{\mathrm{R}})^2\|S\|\|\widehat{\Gamma}\|}\right\}$ satisfies

$$\|u-u_h\|_{\widehat{X}} \le C_{\mathrm{qo}} \min_{x_h\in X_h} \|u-x_h\|_{\widehat{X}} + \beta_1^{-1}(1-\kappa)^{-1}\|\widehat{\Gamma}(u,u,(S-Q)\bullet)\|_{Y_h^*}$$

with a lower bound β_1 of β_h defined in (3.6). The quasi-best approximation result (1.1) holds for $S=Q$.

(C) *a priori error control in weaker Sobolev norms.* In addition to **(H1)–(H4)**, suppose the existence of $\Lambda_5 > 0$ such that, for all $x_h \in X_h$, $y_h \in Y_h$, $x \in X$, and $y \in Y$,

($\widehat{\mathbf{H1}}$) $a_h(x_h,y_h)-a(Px_h,Qy_h) \le \Lambda_5\|x-x_h\|_{\widehat{X}}\|y-y_h\|_{\widehat{Y}}$.

For any $G \in X^*$, if $z \in Y$ solves the dual linearised problem $a(\bullet,z)+b(\bullet,z)=G(\bullet)$ in X^*, then any $z_h \in Y_h$ satisfies

$$\begin{aligned}\|u-u_h\|_{X_s} &\le \omega_1(\|u\|_X,\|u_h\|_{X_h})\|z-z_h\|_{\widehat{Y}}\|u-u_h\|_{\widehat{X}}\\ &\quad+\omega_2(\|z_h\|_{Y_h})\|u-u_h\|_{\widehat{X}}^2\end{aligned}$$

$$+ \|u_h - Pu_h\|_{X_s} + \widehat{\Gamma}(u, u, (S-Q)z_h)$$
$$+ \widehat{\Gamma}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h)$$

with appropriate weights defined in (6.2) below. Here X_s is a Hilbert space with $X + X_h \subset X_s$.

The abstract results **(A)-(C)** are established in Theorems 4.1, 5.1, and 6.2. A summary of their consequences in the applications in Sects. 8-9 for a triangulation with sufficiently small maximal mesh-size $h_{\max}$ is displayed in Table 1.

4 Existence and uniqueness of discrete solution

This section applies the Newton-Kantorovich convergence theorem to establish **(A)**. Let $u \in X$ be a regular root to N. Let (2.3), (2.5), and **(H1)-(H4)** hold with parameters $\Lambda_P,\ \Lambda_R,\ \Lambda_1,\ \delta_2,\ \delta_3,\ \delta_4 \geq 0$. Define $L{:=}2\|\widehat{\Gamma}\|\|R\|^2\|S\|$, $m{:=}L/\beta_1$, and

$$\epsilon_0 := \beta_1^{-1}\big((\Lambda_1\Lambda_P + \|Q^*A\|(1+\Lambda_P) + (1+\Lambda_R)(\|R\|\|S\|\|x_h\|_{X_h} + \|Q\|\|u\|_X)\|\widehat{\Gamma}\|)\delta_4 + \|x_h\|_{X_h}\delta_3/2\big). \tag{4.1}$$

In this section (and in Sect. 5 below), $Q \in L(Y_h; Y)$ (resp. $S \in L(Y_h; \widehat{Y})$) is bounded, but (2.4) (resp. (2.6)) is not employed.

Theorem 4.1 (existence and uniqueness of a discrete solution) *(i) If $\epsilon_0 m \leq 1/2$, then there exists a root $u_h \in X_h$ of N_h with $\|u - u_h\|_{\widehat{X}} \leq \epsilon_1{:=}\delta_4 + (1-\sqrt{1-2\epsilon_0 m})/m$.*
(ii) If $\epsilon_0 m < 1/2$, then given any $v_h \in X_h$ with $\|u_h - v_h\|_{X_h} \leq \rho{:=}(1+\sqrt{1-2\epsilon_0 m})/m > 0$, the Newton scheme with initial iterate v_h converges quadratically to the root u_h to N_h in (i).
(iii) If $\epsilon_1 m \leq 1/2$, then there exists at most one root u_h to N_h with $\|u - u_h\|_{\widehat{X}} \leq \epsilon_1$.

The proof of Theorem 4.1 applies the well-known Newton-Kantorovich convergence theorem found, e.g., in [21, Subsection 5.5] for $X = Y = \mathbb{R}^n$ and in [28, Subsection 5.2] for Banach spaces. The notation is adapted to the present situation.

Theorem 4.2 (Kantorovich (1948)) *Assume the Frechét derivative $DN_h(x_h)$ of N_h at some $x_h \in X_h$ satisfies*

$$\|DN_h(x_h)^{-1}\|_{L(Y_h^*;X_h)} \leq 1/\beta_1 \quad \textit{and} \quad \|DN_h(x_h)^{-1}N_h(x_h)\|_{X_h} \leq \epsilon_0. \tag{4.2}$$

Suppose that DN_h is Lipschitz continuous with Lipschitz constant L and that $2\epsilon_0 L \leq \beta_1$. Then there exists a root $u_h \in \overline{B(x_1, r_-)}$ of N_h in the closed ball around the first iterate $x_1{:=}x_h - DN_h(x_h)^{-1}N_h(x_h)$ of radius $r_-{:=}(1-\sqrt{1-2\epsilon_0 m})/m - \epsilon_0$ and this is the only root of N_h in $\overline{B(x_h, \rho)}$ with $\rho{:=}(1+\sqrt{1-2\epsilon_0 m})/m$. If $2\epsilon_0 L < \beta_1$, then the Newton scheme with initial iterate x_h leads to a sequence in $B(x_h, \rho)$ that converges R-quadratically to u_h. □

Proof of Theorem 4.1. *Step 1 establishes* (4.2). The bounded trilinear form $\widehat{\Gamma}$ leads to the Frechét derivative $DN_h(x_h) \in L(X_h; Y_h^*)$ of N_h from (3.4) evaluated at any $x_h \in X_h$ for all $\xi_h \in X_h$, $\eta_h \in Y_h$ with

$$DN_h(x_h; \xi_h, \eta_h) = a_h(\xi_h, \eta_h) + \widehat{\Gamma}(Rx_h, R\xi_h, S\eta_h) + \widehat{\Gamma}(R\xi_h, Rx_h, S\eta_h). \quad (4.3)$$

For any $x_h^1, x_h^2, \xi_h \in X_h$ and $\eta_h \in Y_h$, (4.3) implies the global Lipschitz continuity of DN_h with Lipschitz constant $L:=2\|\widehat{\Gamma}\|\|R\|^2\|S\|$, and so

$$|DN_h(x_h^1; \xi_h, \eta_h) - DN_h(x_h^2; \xi_h, \eta_h)| \leq L\|x_h^1 - x_h^2\|_{X_h}\|\xi_h\|_{X_h}\|\eta_h\|_{Y_h}.$$

Recall x_h from **(H4)** with $\delta_4 = \|u - x_h\|_{\widehat{X}}$. For this $x_h \in X_h$, (2.10) leads to $\|u - Rx_h\|_{\widehat{X}} \leq (1 + \Lambda_{\mathrm{R}})\delta_4$. This and the boundedness of $\widehat{\Gamma}(\bullet, \bullet, \bullet)$ show

$$\begin{aligned}&\widehat{\Gamma}(u - Rx_h, R\xi_h, S\eta_h) + \widehat{\Gamma}(R\xi_h, u - Rx_h, S\eta_h)\\ &\quad \leq 2\delta_4(1 + \Lambda_{\mathrm{R}})\|\widehat{\Gamma}\|\|R\|\|S\|\|\xi_h\|_{X_h}\|\eta_h\|_{Y_h}.\end{aligned}$$

The discrete inf-sup condition in Theorem 2.1, elementary algebra, and the above displayed estimate establish a positive inf-sup constant

$$0 < \beta_1 = \beta_0 - 2(1 + \Lambda_{\mathrm{R}})\|\widehat{\Gamma}\|\|R\|\|S\|\delta_4 \leq \inf_{\substack{\xi_h \in X_h\\ \|\xi_h\|_{X_h}=1}} \sup_{\substack{\eta_h \in Y_h\\ \|\eta_h\|_{Y_h}=1}} DN_h(x_h; \xi_h, \eta_h) \quad (4.4)$$

for the discrete bilinear form (4.3). The inf-sup constant $\beta_1 > 0$ in (4.4) is known to be (an upper bound of the) reciprocal of the operator norm of $DN_h(x_h)$ and that provides the first estimate in (4.2). It also leads to

$$\|DN_h(x_h)^{-1}N_h(x_h)\|_{X_h} \leq \beta_1^{-1}\|N_h(x_h)\|_{Y_h^*}. \quad (4.5)$$

To establish the second inequality in (4.2), for any $y_h \in Y_h$ with $\|y_h\|_{Y_h} = 1$, set $y:=Qy_h \in Y$. Since $N(u; y) = 0$, (3.3)-(3.4) reveal

$$\begin{aligned}N_h(x_h; y_h) = N_h(x_h; y_h) - N(u; y) &= a_h(x_h, y_h) - a(u, y)\\ &\quad + \widehat{\Gamma}(Rx_h, Rx_h, Sy_h) - \Gamma(u, u, y).\end{aligned} \quad (4.6)$$

The combination of **(H1)** and (2.3) results in

$$\begin{aligned}a_h(x_h, y_h) - a(u, Qy_h) &= a_h(x_h, y_h) - a(Px_h, Qy_h) - a(u - Px_h, Qy_h)\\ &\leq \Lambda_1\Lambda_{\mathrm{P}}\|u - x_h\|_{\widehat{X}} + \|Q^*A\|\|u - Px_h\|_X\end{aligned}$$

with the operator norm $\|Q^*A\|$ of Q^*A in $L(X; Y_h^*)$ in the last step. Utilize (2.10) and **(H4)** to establish $\|u - Px_h\|_X \leq (1 + \Lambda_{\mathrm{P}})\delta_4$. This and the previous estimates imply

$$a_h(x_h, y_h) - a(u, Qy_h) \leq (\Lambda_1\Lambda_{\mathrm{P}} + \|Q^*A\|(1 + \Lambda_{\mathrm{P}}))\delta_4.$$

Elementary algebra and the boundedness of $\widehat{\Gamma}(\bullet,\bullet,\bullet)$, (2.5), and **(H3)-(H4)** show

$$
\begin{aligned}
&2(\widehat{\Gamma}(Rx_h, Rx_h, Sy_h) - \widehat{\Gamma}(u,u,y))\\
&\quad = \widehat{\Gamma}(Rx_h - u, Rx_h, Sy_h) + \widehat{\Gamma}(Rx_h, Rx_h - u, Sy_h)\\
&\qquad + \widehat{\Gamma}(u, Rx_h - u, y) + \widehat{\Gamma}(Rx_h - u, u, y) - \widehat{b}(Rx_h, (Q-S)y_h)\\
&\quad \le 2\delta_4(1+\Lambda_{\mathrm{R}})\left(\|R\|\|S\|\|x_h\|_{X_h} + \|Q\|\|u\|_X\right)\|\widehat{\Gamma}\| + \delta_3\|x_h\|_{X_h}.
\end{aligned}
$$

A combination of the two above displayed estimates in (4.6) reveals

$$
\begin{aligned}
|N_h(x_h; y_h)| \le &(\Lambda_1\Lambda_{\mathrm{P}} + \|Q^*A\|(1+\Lambda_{\mathrm{P}})\\
&+(1+\Lambda_{\mathrm{R}})(\|R\|\|S\|\|x_h\|_{X_h} + \|Q\|\|u\|_X)\|\widehat{\Gamma}\|)\delta_4 + \frac{1}{2}\|x_h\|_{X_h}\delta_3.
\end{aligned}
$$

This implies $\|N_h(x_h)\|_{Y_h^*} \le \beta_1\epsilon_0$ with $\epsilon_0 \ge 0$ from (4.1). The latter bound leads in (4.5) to the second condition in (4.2).

Step 2 establishes the assertion (i) and (ii). Since $\epsilon_0 m \le 1/2$, $r_-, \rho \ge 0$ is well-defined, $2\epsilon_0 L \le \beta_1$, and hence Theorem 4.2 applies.

We digress to discuss the degenerate case $\epsilon_0 = 0$ where (4.1) implies $\delta_4 = 0$. An immediate consequence is that **(H4)** results in $u = x_h \in X_h$. The proof of Step 1 remains valid and $N_h(x_h) = 0$ (since $\epsilon_0 = 0$) provides that $x_h = u$ is the discrete solution u_h. Observe that in this particular case, the Newton iterates form the constant sequence $u = x_h = x_1 = x_2 = \cdots$ and Theorem 4.2 holds for the trivial choice $r_- = 0$.

Suppose $\epsilon_0 > 0$. For $\epsilon_0 m \le 1/2$, Theorem 4.2 shows the existence of a root u_h to N_h in $\overline{B(x_1, r_-)}$ that is the only root in $\overline{B(x_h, \rho)}$. This, $\|x_1 - x_h\|_{X_h} \le \epsilon_0$, with ϵ_0 from (4.1), for the Newton correction $x_1 - x_h$ in the second inequality of (4.2), and triangle inequalities result in

$$
\begin{aligned}
\|u - u_h\|_{\widehat{X}} &\le \|u - x_h\|_{\widehat{X}} + \|x_1 - x_h\|_{X_h} + \|x_1 - u_h\|_{X_h}\\
&\le \delta_4 + (1 - \sqrt{1 - 2\epsilon_0 m})/m = \epsilon_1.
\end{aligned}
\tag{4.7}
$$

This proves the existence of a discrete solution u_h in $X_h \cap \overline{B(u, \epsilon_1)}$ as asserted in (i). Theorem 4.2 implies (ii).

Step 3 establishes the assertion (iii). Recall from Theorem 4.2 that the limit $u_h \in \overline{B(x_1, r_-)}$ in (i)-(ii) is the only discrete solution in $\overline{B(x_h, \rho)}$. Suppose there exists a second solution $\widetilde{u}_h \in X_h \cap \overline{B(u, \epsilon_1)}$ to $N_h(\widetilde{u}_h) = 0$. Since u_h is unique in $\overline{B(x_h, \rho)}$, $\widetilde{u}_h$ lies outside $\overline{B(x_h, \rho)}$. This and a triangle inequality show

$$
\begin{aligned}
\frac{1}{m} \le (1 + \sqrt{1 - 2\epsilon_0 m})/m = \rho < \|x_h - \widetilde{u}_h\|_{\widehat{X}} &\le \|u - \widetilde{u}_h\|_{\widehat{X}} + \|u - x_h\|_{\widehat{X}}\\
&\le \epsilon_1 + \delta_4 \le 2\epsilon_1 \le \frac{1}{m}
\end{aligned}
$$

with $2m\epsilon_1 \le 1$ in the last step. This contradiction concludes the proof of (iii). □

Remark 4.3 (*error estimate*) Recall δ_4 from **(H4)** and ϵ_0 from (4.1). An algebraic manipulation in (4.7) reveals, for $\epsilon_0 m \le 1/2$, that

$$\|u - u_h\|_{\widehat{X}} \le \delta_4 + \frac{2\epsilon_0}{1 + \sqrt{1 - 2\epsilon_0 m}} \le \delta_4 + 2\epsilon_0.$$

In the applications of Sects. 8-9, this leads to the energy norm estimate.

Remark 4.4 (*estimate on* ϵ_1) In the applications, (4.1) leads to $\epsilon_0 \lesssim \delta_3 + \delta_4$. This, the definition of ϵ_1 in Theorem 4.1, (4.7), and Remark 4.3 provide $\epsilon_1 \lesssim \delta_3 + \delta_4$.

5 A priori error control

This section is devoted to a quasi-best approximation up to perturbations **(B)**. Recall that the bounded bilinear form $a : X \times Y \to \mathbb{R}$ satisfies (2.1), the trilinear form $\Gamma : X \times X \times Y \to \mathbb{R}$ is bounded, and $F \in Y^*$. The assumptions on the discretization with $a_h : X_h \times Y_h \to \mathbb{R}$ with non-trivial finite-dimensional spaces X_h and Y_h of the same dimension $\dim(X_h) = \dim(Y_h) \in \mathbb{N}$ are encoded in the stability and quasi-optimality. The stability of a_h and (2.2) mean $\alpha_0 > 0$ and the quasi-optimality assumes $P \in L(X_h; X)$ with (2.3), $R \in L(X_h; \widehat{X})$ with (2.5), $S \in L(Y_h; \widehat{Y})$, and $Q \in L(Y_h; Y)$ (in this section, (2.4) and (2.6) are not employed). Recall β_1 and ϵ_1 from (3.6) and Theorem 4.1.

Theorem 5.1 (a priori error control) *Let $u \in X$ be a regular root to (3.3), let $u_h \in X_h$ solve (3.5), and suppose* **(H1)***, (2.2)-(2.3), (2.5), $\|u - u_h\|_{\widehat{X}} \le \epsilon_2 := \min\left\{\epsilon_1, \frac{\kappa\beta_1}{(1+\Lambda_\mathrm{R})^2\|S\|\|\widehat{\Gamma}\|}\right\}$, and $0 < \kappa < 1$. Then*

$$\|u - u_h\|_{\widehat{X}} \le C_{\mathrm{qo}} \min_{x_h \in X_h} \|u - x_h\|_{\widehat{X}} + \beta_1^{-1}(1-\kappa)^{-1}\|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{Y_h^*}$$

holds for $C_{\mathrm{qo}} = C'_{\mathrm{qo}}\beta_1^{-1}(1-\kappa)^{-1}(\beta_1 + 2(1 + \Lambda_\mathrm{R})\|S\|\|\widehat{\Gamma}\|\|u\|_X)$ with $C'_{\mathrm{qo}} := 1 + \alpha_0^{-1}(\Lambda_1\Lambda_\mathrm{P} + \|Q^* A\|(1 + \Lambda_\mathrm{P}))$.

The theorem establishes a quasi-best approximation result (1.1) for $S = Q$. The proof utilizes a quasi-best approximation result from [11] for linear problems.

Lemma 5.2 (quasi-best approximation for linear problem *[11]*) *If $u^* \in X$ and $G(\bullet) = a(u^*, \bullet) \in Y^*$, $u_h^* \in X_h$ and $a_h(u_h^*, \bullet) = G(Q\bullet) \in Y_h^*$, then (2.2)-(2.3) and* **(H1)** *imply*

$$\textbf{\textit{(QO)}} \quad \|u^* - u_h^*\|_{\widehat{X}} \le C'_{\mathrm{qo}} \inf_{x_h \in X_h} \|u^* - x_h\|_{\widehat{X}}. \tag{5.1}$$

Proof This is indicated in [11, Theorem 5.4.a] for Hilbert spaces and we give the proof for completeness. For any $x_h \in X_h$, the inf-sup condition (2.2) leads for $e_h := x_h - u_h^* \in X_h$ to some $\|y_h\|_{Y_h} \le 1$ such that

$$\alpha_0 \|e_h\|_{X_h} \le a_h(x_h, y_h) - a_h(u_h^*, y_h).$$

Since $a_h(u_h^*, y_h) = G(Qy_h) = a(u^*, Qy_h)$, this implies

$$\begin{aligned}\alpha_0 \|e_h\|_{X_h} &\le a_h(x_h, y_h) - a(Px_h, Qy_h) + a(Px_h - u^*, Qy_h)\\ &\le \Lambda_1 \|x_h - Px_h\|_{\widehat{X}} + \|Q^*A\| \|u^* - Px_h\|_X\end{aligned}$$

with **(H1)**, the operator norm $\|Q^*A\|$ of $Q^*A = a(\bullet, Q\bullet)$, and $\|y_h\|_{Y_h} \le 1$ in the last step. Recall (2.3) and $\|u^* - Px_h\|_X \le (1 + \Lambda_P)\|u^* - x_h\|_{\widehat{X}}$ from (2.10) to deduce

$$\alpha_0 \|e_h\|_{X_h} \le (\Lambda_1 \Lambda_P + (1 + \Lambda_P)\|Q^*A\|)\|u^* - x_h\|_{\widehat{X}}.$$

This and a triangle inequality $\|u^* - u_h^*\|_{\widehat{X}} \le \|e_h\|_{X_h} + \|u^* - x_h\|_{\widehat{X}}$ conclude the proof. □

Proof of Theorem 5.1. Given a regular root $u \in X$ to (3.3), $G(\bullet) := F(\bullet) - \Gamma(u, u, \bullet) \in Y^*$ is an appropriate right-hand side in the problem $a(u, \bullet) = G(\bullet)$ with a discrete solution $u_h^* \in X_h$ to $a_h(u_h^*, \bullet) = G(Q\bullet)$ in Y_h. Lemma 5.2 implies (5.1) with u^* substituted by u, namely

$$\|u - u_h^*\|_{\widehat{X}} \le C'_{\mathrm{qo}} \inf_{x_h \in X_h} \|u - x_h\|_{\widehat{X}}. \tag{5.2}$$

Given the discrete solution $u_h \in X_h$ to (3.5) and the approximation $u_h^* \in X_h$ from above, let $e_h := u_h^* - u_h \in X_h$. The stability of the discrete problem from Theorem 2.1 leads to the existence of some $y_h \in Y_h$ with norm $\|y_h\|_{Y_h} \le 1/\beta_h$ for $\beta_h \ge \beta_0$ from (2.9) and

$$\begin{aligned}\|e_h\|_{X_h} &= a_h(e_h, y_h) + \widehat{b}(Re_h, Sy_h)\\ &= a_h(e_h, y_h) + \widehat{\Gamma}(u, Re_h, Sy_h) + \widehat{\Gamma}(Re_h, u, Sy_h)\end{aligned}$$

with (3.2) in the last step. The definition of u_h^*, G, and (3.5) show

$$\begin{aligned}a_h(u_h^*, y_h) &= F(Qy_h) - \Gamma(u, u, Qy_h)\\ &= a_h(u_h, y_h) + \widehat{\Gamma}(Ru_h, Ru_h, Sy_h) - \Gamma(u, u, Qy_h).\end{aligned}$$

The combination of the two previous displayed identities and elementary algebra show that

$$\begin{aligned}\|e_h\|_{X_h} = &\,\widehat{\Gamma}(Ru_h, Ru_h, Sy_h) - \widehat{\Gamma}(u, u, Sy_h) + \widehat{\Gamma}(u, Re_h, Sy_h)\\ &+ \widehat{\Gamma}(Re_h, u, Sy_h) + \widehat{\Gamma}(u, u, (S - Q)y_h)\end{aligned}$$

$$
\begin{aligned}
&= \widehat{\Gamma}(u - Ru_h, u - Ru_h, Sy_h) + \widehat{\Gamma}(u, Ru_h^* - u, Sy_h) \\
&\quad + \widehat{\Gamma}(Ru_h^* - u, u, Sy_h) + \widehat{\Gamma}(u, u, (S - Q)y_h) \\
&\le (\|S\|\|\widehat{\Gamma}\|\|u - Ru_h\|_{\widehat{X}}^2 + 2\|u\|_X\|S\|\|\widehat{\Gamma}\|\|u - Ru_h^*\|_{\widehat{X}} \\
&\quad + \|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{Y_{h*}})/\beta_h
\end{aligned}
$$

with the boundedness of $\widehat{\Gamma}(\bullet, \bullet, \bullet)$ and $\|y_h\|_{Y_h} \le 1/\beta_h$ in the last step. This, $\|u - Ru_h\|_{\widehat{X}} \le (1+\Lambda_{\mathrm{R}})\|u-u_h\|_{\widehat{X}}$ (resp. $\|u-Ru_h^*\|_{\widehat{X}} \le (1+\Lambda_{\mathrm{R}})\|u-u_h^*\|_{\widehat{X}}$) from (2.10), $\beta_1 \le \beta_h$, and a triangle inequality show

$$
\begin{aligned}
\beta_1\|u - u_h\|_{\widehat{X}} \le{}& \left(\beta_1 + 2(1+\Lambda_{\mathrm{R}})\|S\|\|\widehat{\Gamma}\|\|u\|_{\widehat{X}}\right)\|u-u_h^*\|_{\widehat{X}} + \|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{Y_h^*} \\
&+ (1 + \Lambda_{\mathrm{R}})^2\|S\|\|\widehat{\Gamma}\|\|u - u_h\|_{\widehat{X}}^2.
\end{aligned}
$$

Recall the assumption on $\|u - u_h\|_{\widehat{X}} \le \epsilon_2$ to absorb the last term and obtain

$$
\|u - u_h\|_{\widehat{X}} \le \frac{(\beta_1 + 2(1 + \Lambda_{\mathrm{R}})\|S\|\|\widehat{\Gamma}\|\|u\|_X)\|u - u_h^*\|_{\widehat{X}} + \|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{Y_h^*}}{\beta_1 - \epsilon_2(1 + \Lambda_{\mathrm{R}})^2\|S\|\|\widehat{\Gamma}\|}.
$$

This, the definition of ϵ_2, and (5.2) conclude the proof. □

Remark 5.3 (*estimate on* ϵ_2) The assumption of Theorem 5.1 and Remark 4.4 reveal $\epsilon_2 \le \epsilon_1 \lesssim \delta_3 + \delta_4$ for the applications of Sects. 8, 9.

6 Goal-oriented error control

This section proves an a priori error estimate in weaker Sobolev norms based on a duality argument. Suppose Y is reflexive throughout this section so that, given any $G \in X^*$, there exists a unique solution $z \in Y$ to the dual linearised problem

$$
a(\bullet, z) + b(\bullet, z) = G(\bullet) \text{ in } X^*. \tag{6.1}
$$

Recall N from (3.1), A and B from Table 2 with (3.2), P, Q, R, and S with (2.3)–(2.6), and ($\widehat{\mathbf{H1}}$) from Sect. 3. Since $u \in X$ is a regular root, the derivative $A + B \in L(X; Y^*)$ of N evaluated at u is a bijection and so is its dual operator $A^* + B^* \in L(Y; X^*)$.

Theorem 6.1 (goal-oriented error control) *Let $u \in X$ be a regular root to (3.3) and let $u_h \in X_h$ (resp. $z \in Y$) solve (3.5) (resp. (6.1)). Suppose* ($\widehat{\mathbf{H1}}$) *and (2.3)–(2.6). Then, any $G \in X^*$ and any $z_h \in Y_h$ satisfy*

$$
\begin{aligned}
G(u - Pu_h) \le{}& \omega_1(||u||_X, ||u_h||_{X_h})\|u - u_h\|_{\widehat{X}}\|z - z_h\|_{\widehat{Y}} + \omega_2(\|z_h\|_{Y_h})\|u - u_h\|_{\widehat{X}}^2 \\
&+ \widehat{\Gamma}(u, u, (S - Q)z_h) + \widehat{\Gamma}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h)
\end{aligned}
$$

with the weights

$$\omega_1(\|u_X\|, \|u_h\|_{X_h}) := (1+\Lambda_{\mathrm{P}})(1+\Lambda_{\mathrm{Q}})(\|A\|+2\|\Gamma\|\|u\|_X)+\Lambda_5+(1+\Lambda_{\mathrm{R}})(\Lambda_{\mathrm{S}}+\Lambda_{\mathrm{Q}}) \\ \times \|\widehat{\Gamma}\|(\|Ru_h\|_{\widehat{X}}+\|u\|_X), \quad \omega_2(\|z_h\|_{Y_h}) := \|\Gamma\|(1+\Lambda_{\mathrm{P}})^2\|Qz_h\|_Y. \tag{6.2}$$

Proof Since $z \in Y$ solves (6.1), elementary algebra with (3.3), (3.5), and any $z_h \in Y_h$ lead to

$$G(u-Pu_h) = (a+b)(u-Pu_h, z) = (a+b)(u-Pu_h, z-Qz_h) \\ + b(u-Pu_h, Qz_h) + \big(a_h(u_h, z_h) - a(Pu_h, Qz_h)\big) \\ + \widehat{\Gamma}(Ru_h, Ru_h, Sz_h) - \Gamma(u, u, Qz_h). \tag{6.3}$$

The first term $(a+b)(u-Pu_h, z-Qz_h)$ on the right-hand side of (6.3) is bounded by

$$(\|A\|+2\|\Gamma\|\|u\|_X)\|u-Pu_h\|_X\|z-Qz_h\|_Y \\ \le (\|A\|+2\|\Gamma\|\|u\|_X)(1+\Lambda_{\mathrm{P}})(1+\Lambda_{\mathrm{Q}})\|u-u_h\|_{\widehat{X}}\|z-z_h\|_{\widehat{Y}}$$

with (2.10)–(2.11) in the last step. The hypothesis ($\widehat{\mathbf{H1}}$) controls the third term on the right-hand side of (6.3), namely

$$a_h(u_h, z_h) - a(Pu_h, Qz_h) \le \Lambda_5\|u-u_h\|_{\widehat{X}}\|z-z_h\|_{\widehat{Y}}.$$

Elementary algebra with (3.2) shows that the remaining terms $\widehat{\Gamma}(Ru_h, Ru_h, Sz_h) - \Gamma(u, u, Qz_h) + b(u-Pu_h, Qz_h)$ on the right-hand side of (6.3) can be re-written as

$$\widehat{\Gamma}(Ru_h, Ru_h, (S-Q)z_h) + \widehat{\Gamma}(Ru_h, Ru_h, Qz_h) \\ - \Gamma(Pu_h, Pu_h, Qz_h) + \Gamma(u-Pu_h, u-Pu_h, Qz_h). \tag{6.4}$$

Elementary algebra with the first term on the right-hand side of (6.4) reveals

$$\widehat{\Gamma}(Ru_h, Ru_h, (S-Q)z_h) = \widehat{\Gamma}(Ru_h-u, Ru_h, (S-Q)z_h) + \widehat{\Gamma}(u, Ru_h-u, (S-Q)z_h) \\ + \widehat{\Gamma}(u, u, (S-Q)z_h).$$

The boundedness of $\widehat{\Gamma}(\bullet, \bullet, \bullet)$, (2.4), (2.6), and (2.10) show

$$\widehat{\Gamma}(Ru_h-u, Ru_h, (S-Q)z_h) = \widehat{\Gamma}(Ru_h-u, Ru_h, (S-I)z_h) \\ + \widehat{\Gamma}(Ru_h-u, Ru_h, (I-Q)z_h) \\ \le (\Lambda_{\mathrm{S}}+\Lambda_{\mathrm{Q}})\|\widehat{\Gamma}\|(1+\Lambda_{\mathrm{R}})\|Ru_h\|_{\widehat{X}}\|u \\ - u_h\|_{\widehat{X}}\|z-z_h\|_{\widehat{Y}}.$$

$$\widehat{\Gamma}(u, Ru_h-u, (S-Q)z_h) \le (\Lambda_{\mathrm{S}}+\Lambda_{\mathrm{Q}})\|\widehat{\Gamma}\|(1+\Lambda_{\mathrm{R}})\|u\|_X\|u-u_h\|_{\widehat{X}}\|z-z_h\|_{\widehat{Y}}.$$

The boundedness of $\Gamma(\bullet, \bullet, \bullet)$ and (2.10) lead to

$$\Gamma(u - Pu_h, u - Pu_h, Qz_h) \leq \|\Gamma\|(1 + \Lambda_{\mathrm{P}})^2 \|u - u_h\|_{\widehat{X}}^2 \|Qz_h\|_Y.$$

A combination of the above estimates of the terms in (6.3) concludes the proof. □

An abstract a priori estimate for error control in weaker Sobolev norms concludes this section.

Theorem 6.2 (a priori error estimate in weaker Sobolev norms) *Let X_{s} be a Hilbert space with $X + X_h \subset X_{\mathrm{s}}$. Under the assumptions of Theorem 6.1, any $z_h \in Y_h$ satisfies*

$$\begin{aligned}\|u - u_h\|_{X_{\mathrm{s}}} \leq{}& \omega_1(\|u_X\|, \|u_h\|_{X_h}) \|u - u_h\|_{\widehat{X}} \|z - z_h\|_{\widehat{Y}} \\ &+ \omega_2(\|z_h\|_{Y_h}) \|u - u_h\|_{\widehat{X}}^2 + \|u_h - Pu_h\|_{X_{\mathrm{s}}} \\ &+ \widehat{\Gamma}(u, u, (S - Q)z_h) + \widehat{\Gamma}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h).\end{aligned}$$

Proof Given $u - Pu_h \in X \subset X_{\mathrm{s}}$, a corollary of the Hahn-Banach extension theorem leads to some $G \in X_{\mathrm{s}}^* \subset X^*$ with norm $\|G\|_{X_{\mathrm{s}}^*} \leq 1$ in X_{s}^* and $G(u - Pu_h) = \|u - Pu_h\|_{X_{\mathrm{s}}}$ [4]. This, a triangle inequality, and Theorem 6.1 conclude the proof. □

7 Auxiliary results for applications

7.1 General notation

Standard notation of Lebesgue and Sobolev spaces, their norms, and L^2 scalar products applies throughout the paper such as the abbreviation $\| \bullet \|$ for $\| \bullet \|_{L^2(\Omega)}$. For real s, $H^s(\Omega)$ denotes the Sobolev space endowed with the Sobolev-Slobodeckii semi-norm (resp. norm) $| \bullet |_{H^s(\Omega)}$ (resp. $\| \bullet \|_{H^s(\Omega)}$) [20]; $H^s(K){:=}H^s(\mathrm{int}(K))$ abbreviates the Sobolev space with respect to the interior $\mathrm{int}(K) \neq \emptyset$ of a triangle K. The closure of $D(\Omega)$ in $H^s(\Omega)$ is denoted by $H_0^s(\Omega)$ and $H^{-s}(\Omega)$ is the dual of $H_0^s(\Omega)$. The semi-norm and norm in $W^{s,p}(\Omega)$, $1 \leq p \leq \infty$, are denoted by $| \bullet |_{W^{s,p}(\Omega)}$ and $\| \bullet \|_{W^{s,p}(\Omega)}$. The Hilbert space $V{:=}H_0^2(\Omega)$ is endowed with the energy norm $\vert\!\vert\!\vert \bullet \vert\!\vert\!\vert{:=}| \bullet |_{H^2(\Omega)}$. The product space $H^s(\Omega) \times H^s(\Omega)$ (resp. $L^p(\Omega) \times L^p(\Omega)$) is denoted by $\mathbf{H}^s(\Omega)$ (resp. $\mathbf{L}^p(\Omega)$) and $\mathbf{V} =: V \times V$. The energy norm in the product space $\mathbf{H}^2(\Omega)$ is also denoted by $\vert\!\vert\!\vert \bullet \vert\!\vert\!\vert$ and is $(\vert\!\vert\!\vert \varphi_1 \vert\!\vert\!\vert^2 + \vert\!\vert\!\vert \varphi_2 \vert\!\vert\!\vert^2)^{1/2}$ for all $\Phi = (\varphi_1, \varphi_2) \in \mathbf{H}^2(\Omega)$. The norm on $\mathbf{W}^{s,p}(\Omega)$ is denoted by $\| \bullet \|_{\mathbf{W}^{s,p}(\Omega)}$. Given any function $v \in L^2(\omega)$, define the integral mean $⨍_\omega v\,\mathrm{dx}{:=}1/|\omega| \int_\omega v\,\mathrm{dx}$; where $|\omega|$ denotes the area of ω. The notation $A \lesssim B$ (resp. $A \gtrsim B$) abbreviates $A \leq CB$ (resp. $A \geq CB$) for some positive generic constant C, which depends exclusively on Ω and the shape regularity of a triangulation $\mathcal{T}$; $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

Triangulation. Let $\mathcal{T}$ denote a shape regular triangulation of the polygonal Lipschitz domain Ω with boundary $\partial\Omega$ into compact triangles and $\mathbb{T}(\delta)$ be a set of uniformly shape-regular triangulations $\mathcal{T}$ with maximal mesh-size smaller than or equal to $\delta > 0$. Given $\mathcal{T} \in \mathbb{T}$, define the piecewise constant mesh function $h_{\mathcal{T}}(x) = h_K = \mathrm{diam}(K)$

for all $x \in K \in \mathcal{T}$, and set $h_{\max}:=\max_{K\in\mathcal{T}} h_K$. The set of all interior vertices (resp. boundary vertices) of the triangulation $\mathcal{T}$ is denoted by $\mathcal{V}(\Omega)$ (resp. $\mathcal{V}(\partial\Omega)$) and $\mathcal{V}:=\mathcal{V}(\Omega)\cup\mathcal{V}(\partial\Omega)$. Let $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}(\partial\Omega)$) denote the set of all interior edges (resp. boundary edges) in $\mathcal{T}$. Define a piecewise constant edge-function on $\mathcal{E}:=\mathcal{E}(\Omega)\cup\mathcal{E}(\partial\Omega)$ by $h_{\mathcal{E}}|_E = h_E = \operatorname{diam}(E)$ for any $E \in \mathcal{E}$. For a positive integer m, define the Hilbert (resp. Banach) space $H^m(\mathcal{T}) \equiv \prod_{K\in\mathcal{T}} H^m(K)$ (resp. $W^{m,p}(\mathcal{T}) \equiv \prod_{K\in\mathcal{T}} W^{m,p}(K)$). The triple norm $\|\!|\bullet\|\!|:=|\bullet|_{H^m(\Omega)}$ is the energy norm and $\|\!|\bullet\|\!|_{\mathrm{pw}}:=|\bullet|_{H^m(\mathcal{T})}:=\|D^m_{\mathrm{pw}}\bullet\|$ is its piecewise version with the piecewise partial derivatives D^m_{pw} of order $m\in\mathbb{N}$. For $1<s<2$, the piecewise Sobolev space $H^s(\mathcal{T})$ is the product space $\prod_{T\in\mathcal{T}} H^s(T)$ defined as $\{v_{\mathrm{pw}}\in L^2(\Omega) : \forall T\in\mathcal{T}, v_{\mathrm{pw}}|_T \in H^s(T)\}$ and is equipped with the Euclid norm of those contributions $\|\bullet\|_{H^s(T)}$ for all $T\in\mathcal{T}$. For $s=1+\nu$ with $0<\nu<1$, the 2D Sobolev-Slobodeckii norm [20] of $f\in H^s(\Omega)$ reads $\|f\|^2_{H^s(\Omega)}:=\|f\|^2_{H^1(\Omega)}+|f|^2_{H^\nu(\Omega)}$ and

$$|f|_{H^s(\Omega)}:=\left(\sum_{|\beta|=1}\int_\Omega\int_\Omega \frac{|\partial^\beta f(x)-\partial^\beta f(y)|^2}{|x-y|^{2+2\nu}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

The piecewise version of the energy norm in $H^2(\mathcal{T})$ reads $\|\!|\bullet\|\!|_{\mathrm{pw}}:=|\bullet|_{H^2(\mathcal{T})}:=\|D^2_{\mathrm{pw}}\bullet\|$ with the piecewise Hessian D^2_{pw}. The curl of a scalar function v is defined by $\operatorname{Curl} v = \big(-\partial v/\partial y, -\partial v/\partial x\big)^T$ and its piecewise version is denoted by $\operatorname{Curl}_{\mathrm{pw}}$. The seminorm (resp. norm) in $W^{m,p}(\mathcal{T})$ is denoted by $|\bullet|_{W^{m,p}(\mathcal{T})}$ (resp. $\|\bullet\|_{W^{m,p}(\mathcal{T})}$). Define the jump $[\varphi]_E:=\varphi|_{K_+}-\varphi|_{K_-}$ and the average $\langle\varphi\rangle_E:=\frac{1}{2}\left(\varphi|_{K_+}+\varphi|_{K_-}\right)$ across the interior edge E of $\varphi\in H^1(\mathcal{T})$ of the adjacent triangles K_+ and K_-. Extend the definition of the jump and the average to an edge on boundary by $[\varphi]_E:=\varphi|_E$ and $\langle\varphi\rangle_E:=\varphi|_E$ for $E\in\mathcal{E}(\partial\Omega)$. For any vector function, the jump and the average are understood component-wise. Let Π_k denote the $L^2(\Omega)$ orthogonal projection onto the piecewise polynomials $P_k(\mathcal{T}):=\left\{v\in L^2(\Omega) : \forall\, K\in\mathcal{T}, v|_K\in P_k(K)\right\}$ of degree at most $k\in\mathbb{N}_0$. (The notation $\|\!|\bullet\|\!|_{\mathrm{pw}}$, Π_K, and V_h below hides the dependence on $\mathcal{T}\in\mathbb{T}$.)

7.2 Finite element function spaces and discrete norms

This section introduces the discrete spaces and norms for the Morley/dG/C^0IP/WOPSIP schemes. The Morley finite element space [15] reads

$$\mathrm{M}(\mathcal{T}):=\left\{ v_{\mathrm{M}}\in P_2(\mathcal{T}) \;\middle|\; \begin{array}{l} v_{\mathrm{M}} \text{ is continuous at the vertices and its normal}\\ \text{derivatives } \nu_E\cdot D_{\mathrm{pw}}v_{\mathrm{M}} \text{ are continuous at}\\ \text{the midpoints of interior edges, } v_{\mathrm{M}} \text{ vanishes}\\ \text{at the vertices of } \partial\Omega \text{ and } \nu_E\cdot D_{\mathrm{pw}}v_{\mathrm{M}}\\ \text{vanishes at the midpoints of boundary edges}\end{array}\right\}.$$

The semi-scalar product a_{pw} is defined by the piecewise Hessian D^2_{pw}, for all $v_{\mathrm{pw}}, w_{\mathrm{pw}} \in H^2(\mathcal{T})$ as

$$a_{\mathrm{pw}}(v_{\mathrm{pw}}, w_{\mathrm{pw}}) := \int_\Omega D^2_{\mathrm{pw}} v_{\mathrm{pw}} : D^2_{\mathrm{pw}} w_{\mathrm{pw}}\, \mathrm{dx}. \tag{7.1}$$

The bilinear form $a_{\mathrm{pw}}(\bullet, \bullet)$ induces a piecewise H^2 seminorm $|||\bullet|||_{\mathrm{pw}} = a_{\mathrm{pw}}(\bullet, \bullet)^{1/2}$ that is a norm on $V + \mathrm{M}(\mathcal{T})$ [10]. The piecewise Hilbert space $H^2(\mathcal{T})$ is endowed with a norm $\|\bullet\|_h$ [7] defined by

$$\begin{aligned} \|v_{\mathrm{pw}}\|_h^2 &:= |||v_{\mathrm{pw}}|||_{\mathrm{pw}}^2 + j_h(v_{\mathrm{pw}})^2 \text{ for all } v_{\mathrm{pw}} \in H^2(\mathcal{T}),\\ j_h(v_{\mathrm{pw}})^2 &:= \sum_{E\in\mathcal{E}} \sum_{z\in\mathcal{V}(E)} h_E^{-2} |\left[v_{\mathrm{pw}}\right]_E(z)|^2 + \sum_{E\in\mathcal{E}} \left| ⨍_E \left[\partial v_{\mathrm{pw}}/\partial \nu_E\right]_E \mathrm{d}s\right|^2 \end{aligned} \tag{7.2}$$

with the jumps $\left[v_{\mathrm{pw}}\right]_E(z) = v_{\mathrm{pw}}|_{\omega(E)}(z)$ for $z \in \mathcal{V}(\partial\Omega)$; the edge-patch $\omega(E) := \mathrm{int}(K_+ \cup K_-)$ of the interior edge $E = \partial K_+ \cap \partial K_- \in \mathcal{E}(\Omega)$ is the interior of the union $K_+ \cup K_-$ of the neighboring triangles K_+ and K_-, and $\left[\partial v_{\mathrm{pw}}/\partial \nu_E\right]_E = \frac{\partial v_{\mathrm{pw}}}{\partial \nu_E}|_E$ for $E \in \mathcal{E}(\partial\Omega)$ at the boundary with jump partner zero owing to the homogeneous boundary conditions.

For all $v_{\mathrm{pw}}, w_{\mathrm{pw}} \in H^2(\mathcal{T})$ and parameters $\sigma_1, \sigma_2 > 0$ (that will be chosen sufficiently large but fixed in applications), define $c_{\mathrm{dG}}(\bullet, \bullet)$ and the mesh dependent dG norm $\|\bullet\|_{\mathrm{dG}}$ by

$$\begin{aligned} c_{\mathrm{dG}}(v_{\mathrm{pw}}, w_{\mathrm{pw}}) :=& \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E^3} \int_E \left[v_{\mathrm{pw}}\right]_E \left[w_{\mathrm{pw}}\right]_E \mathrm{ds}\\ &+ \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E} \int_E \left[\partial v_{\mathrm{pw}}/\partial \nu_E\right]_E \left[\partial w_{\mathrm{pw}}/\partial \nu_E\right]_E \mathrm{ds},\\ \|v_{\mathrm{pw}}\|_{\mathrm{dG}}^2 :=& |||v_{\mathrm{pw}}|||_{\mathrm{pw}}^2 + c_{\mathrm{dG}}(v_{\mathrm{pw}}, v_{\mathrm{pw}}). \end{aligned} \tag{7.3}$$

The discrete space for the C^0IP scheme is $S^2_0(\mathcal{T}) := P_2(\mathcal{T}) \cap H^1_0(\Omega)$. The restriction of $\|\bullet\|_{\mathrm{dG}}$ to $H^1_0(\Omega)$ with a stabilisation parameter $\sigma_{\mathrm{IP}} > 0$ defines the norm for the C^0IP scheme below,

$$\begin{aligned} c_{\mathrm{IP}}(v_{\mathrm{pw}}, w_{\mathrm{pw}}) :=& \sum_{E\in\mathcal{E}} \frac{\sigma_{\mathrm{IP}}}{h_E} \int_E \left[\partial v_{\mathrm{pw}}/\partial \nu_E\right] \left[\partial w_{\mathrm{pw}}/\partial \nu_E\right] \mathrm{ds},\\ \|v_{\mathrm{pw}}\|_{\mathrm{IP}}^2 :=& |||v_{\mathrm{pw}}|||_{\mathrm{pw}}^2 + c_{\mathrm{IP}}(v_{\mathrm{pw}}, v_{\mathrm{pw}}). \end{aligned} \tag{7.4}$$

For all $v_{\mathrm{pw}}, w_{\mathrm{pw}} \in H^2(\mathcal{T})$, the WOPSIP norm $\|\bullet\|_{\mathrm{P}}$ is defined by

$$c_{\mathrm{P}}(v_{\mathrm{pw}}, w_{\mathrm{pw}}) := \sum_{E\in\mathcal{E}} \sum_{z\in\mathcal{V}(E)} h_E^{-4} \left(\left[v_{\mathrm{pw}}\right]_E(z)\right)\left(\left[w_{\mathrm{pw}}\right]_E(z)\right)$$

$$+\sum_{E\in\mathcal{E}} h_E^{-2} \fint_E [\partial v_{pw}/\partial \nu_E]\, ds \fint_E [\partial w_{pw}/\partial \nu_E]\, ds, \tag{7.5}$$

$$\|v_{pw}\|_P^2 := |||v_{pw}|||_{pw}^2 + c_P(v_{pw}, v_{pw}). \tag{7.6}$$

The discrete space for dG/WOPSIP schemes is $P_2(\mathcal{T})$. The discrete norms $|||\bullet|||_{pw}$, $\|\bullet\|_{dG}$ and $\|\bullet\|_{IP}$ are all equivalent to $\|\bullet\|_h$ on $V+V_h$ for $V_h \in \{M(\mathcal{T}), P_2(\mathcal{T}), S_0^2(\mathcal{T})\}$. In comparison to $j_h(\bullet)$, the jump contribution in $\|\bullet\|_P$ involves smaller negative powers of the mesh-size and so $j_h(v_{pw})^2 \lesssim c_P(v_{pw}, v_{pw})$ (with $h_E \le \mathrm{diam}(\Omega) \lesssim 1$); but there is no equivalence of $\|\bullet\|_h$ with $\|\bullet\|_P$ in $V + P_2(\mathcal{T})$.

Lemma 7.1 *(*Equivalence of norms *[11, Remark 9.2]) It holds* $\|\bullet\|_h = |||\bullet|||_{pw}$ *on* $V+M(\mathcal{T})$*,* $\|\bullet\|_h \approx \|\bullet\|_{dG} \lesssim \|\bullet\|_P$ *on* $V+P_2(\mathcal{T})$*, and* $\|\bullet\|_h \approx \|\bullet\|_{IP}$ *on* $V+S_0^2(\mathcal{T})$*.*

7.3 Interpolation and companion operators

The classical Morley interpolation operator I_M is generalized from $H_0^2(\Omega)$ to the piecewise H^2 functions by averaging in [11].

Definition 7.2 (*Morley interpolation* [11, Definition 3.5]) Given any $v_{pw} \in H^2(\mathcal{T})$, define $I_M v_{pw} := v_M \in M(\mathcal{T})$ by the degrees of freedom as follows. For any interior vertex $z \in \mathcal{V}(\mathcal{T})$ with the set of attached triangles $\mathcal{T}(z)$ of cardinality $|\mathcal{T}(z)| \in \mathbb{N}$ and for any interior edge $E \in \mathcal{E}(\Omega)$ with a mean value operator $\langle\bullet\rangle_E$ set

$$v_M(z) := |\mathcal{T}(z)|^{-1} \sum_{K\in\mathcal{T}(z)} (v_{pw}|_K)(z) \text{ and } \fint_E \frac{\partial v_M}{\partial \nu_E}\, ds := \fint_E \left\langle \frac{\partial v_{pw}}{\partial \nu_E} \right\rangle ds.$$

The remaining degrees of freedom at vertices and edges on the boundary are set zero owing to the homogeneous boundary conditions.

Lemma 7.3 *(*interpolation error estimates *[11, Lemma 3.2, Theorem 4.3]) Any* $v_{pw} \in H^2(\mathcal{T})$ *and its Morley interpolation* $I_M v_{pw} \in M(\mathcal{T})$ *satisfy*

(a) $\sum_{m=0}^2 |h_{\mathcal{T}}^{m-2}(v_{pw} - I_M v_{pw})|_{H^m(\mathcal{T})} \lesssim \|(1-\Pi_0)D_{pw}^2 v_{pw}\| + j_h(v_{pw}) \lesssim \|v_{pw}\|_h$;
(b) $\sum_{m=0}^2 |h_{\mathcal{T}}^{m-2}(v_{pw} - I_M v_{pw})|_{H^m(\mathcal{T})} \approx \min_{w_M\in M(\mathcal{T})} \|v_{pw} - w_M\|_h \approx \min_{w_M\in M(\mathcal{T})} \sum_{m=0}^2 |h_{\mathcal{T}}^{m-2}(v_{pw} - w_M)|_{H^m(\mathcal{T})}$;
(c) the integral mean property of the Hessian, $D_{pw}^2 I_M = \Pi_0 D^2$ *in* V;
(d) $|||v - I_M v|||_{pw} \lesssim h_{max}^{t-2} \|v\|_{H^t(\Omega)}$ *for all* $v \in H^t(\Omega)$ *with* $2 \le t \le 3$.

Let $HCT(\mathcal{T})$ denote the Hsieh-Clough-Tocher finite element space [15, Chapter 6].

Lemma 7.4 *(*right-inverse *[10, 11, 19]) There exists a linear map* $J : M(\mathcal{T}) \to (HCT(\mathcal{T}) + P_8(\mathcal{T})) \cap H_0^2(\Omega)$ *such that any* $v_M \in M(\mathcal{T})$ *and any* $v_2 \in P_2(\mathcal{T})$ *satisfy (a)–(h).*

(a) $J v_M(z) = v_M(z)$ *for any* $z \in \mathcal{V}$;
(b) $\nabla(J v_M)(z) = |\mathcal{T}(z)|^{-1} \sum_{K\in\mathcal{T}(z)} (\nabla v_M|_K)(z)$ *for* $z \in \mathcal{V}(\Omega)$;

(c) $\fint_E \partial J v_\mathrm{M}/\partial \nu_E ds = \fint_E \partial v_\mathrm{M}/\partial \nu_E ds$ *for any* $E \in \mathcal{E}$;
(d) $v_\mathrm{M} - J v_\mathrm{M} \perp P_2(\mathcal{T})$ *in* $L^2(\Omega)$;
(e) $\sum_{m=0}^{2} \|h_\mathcal{T}^{m-2} D_\mathrm{pw}^m (v_\mathrm{M} - J v_\mathrm{M})\| \lesssim \min_{v\in V} \|\!|v_\mathrm{M} - v|\!\|_\mathrm{pw}$;
(f) $\|v_2 - J I_\mathrm{M} v_2\|_{H^t(\mathcal{T})} \lesssim h_{\max}^{2-t} \min_{v\in V} \|v_2 - v\|_h$ *holds for* $0 \le t \le 2$;
(g) $\sum_{m=0}^{2} \|h_\mathcal{T}^{m-3} D_\mathrm{pw}^m ((1 - I_\mathrm{M}) v_2)\| + \sum_{m=0}^{2} \|h_\mathcal{T}^{m-2} D_\mathrm{pw}^m ((1 - J) I_\mathrm{M} v_2)\| \lesssim \min_{v\in V} \|v - v_2\|_\mathrm{P}$;
(h) $|v_2 - J I_\mathrm{M} v_2|_{W^{1,2/(1-t)}(\mathcal{T})} \lesssim h_{\max}^{1-t} \min_{v\in V} \|v - v_2\|_h$ *holds for* $0 < t < 1$.

Proof of (a)-(f). This is included in [10, 19], [11, Lemma 3.7, Theorem 4.5]. □

Proof of (g). The inequality $\sum_{m=0}^{2} \|h_\mathcal{T}^{m-3} D_\mathrm{pw}^m ((1 - I_\mathrm{M}) v_2)\| \lesssim \|v - v_2\|_\mathrm{P}$ follows as in the proof of Lemma 10.2 in [11]. Lemma 7.4.e and a triangle inequality show

$$\sum_{m=0}^{2} \|h_\mathcal{T}^{m-2} D_\mathrm{pw}^m (1 - J) I_\mathrm{M} v_2\| \lesssim \|\!|I_\mathrm{M} v_2 - v|\!\|_\mathrm{pw} \le \|\!|I_\mathrm{M} v_2 - v_2|\!\|_\mathrm{pw} + \|\!|v_2 - v|\!\|_\mathrm{pw}.$$

Since $\|\!|I_\mathrm{M} v_2 - v_2|\!\|_\mathrm{pw} \le h_{\max} \|\!|h_\mathcal{T}^{-1}(I_\mathrm{M} v_2 - v_2)|\!\|_\mathrm{pw} \lesssim h_{\max} \|v - v_2\|_\mathrm{P}$ from the first part of (g) with $m = 2$, the above displayed estimate, and $\|\!| \bullet |\!\|_\mathrm{pw} \le \| \bullet \|_\mathrm{P}$ conclude the proof of (g). □

Proof of (h). An inverse estimate [17, Lemma 12.1], [2, Lemma 4.5.3], [15, Theorem 3.2.6] on each triangle $\widehat{T}$ in the HCT subtriangulation $\widehat{\mathcal{T}}$ of $\mathcal{T}$ in each component of $g := \nabla_\mathrm{pw}(v_2 - J I_\mathrm{M} v_2)$ reads $\|g\|_{L^{2/(1-t)}(\widehat{T})} \le C_\mathrm{inv} h_{\widehat{T}}^{-t} \|g\|_{L^2(\widehat{T})}$. Consequently,

$$C_\mathrm{inv}^{-1} \|g\|_{L^{2/(1-t)}(\Omega)} \le \left(\sum_{\widehat{T}\in\widehat{\mathcal{T}}} \|h_{\widehat{T}}^{-t} g\|_{L^2(\widehat{T})}^{2/(1-t)} \right)^{(1-t)/2} \le \left(\sum_{\widehat{T}\in\widehat{\mathcal{T}}} \|h_{\widehat{T}}^{-t} g\|_{L^2(\widehat{T})}^{2} \right)^{1/2}$$

with $\| \bullet \|_{\ell^{2/(1-t)}} \le \| \bullet \|_{\ell^2}$ in the sequence space $\mathbb{R}^\mathbb{N}$ (ℓ^p is decreasing in $p \ge 1$) in the last step. With the shape regularity $h_{\widehat{\mathcal{T}}} \approx h_\mathcal{T}$, this reads

$$|v_2 - J I_\mathrm{M} v_2|_{W^{1,2/(1-t)}(\mathcal{T})} \lesssim |h_\mathcal{T}^{-t}(v_2 - J I_\mathrm{M} v_2)|_{H^1(\mathcal{T})}. \tag{7.7}$$

Since $I_\mathrm{M}(v_2 - J I_\mathrm{M} v_2) = 0$ by Lemma 7.4, Lemma 7.3.a provides

$$|h_\mathcal{T}^{-t}(v_2 - J I_\mathrm{M} v_2)|_{H^1(\mathcal{T})} \le h_{\max}^{1-t} |h_\mathcal{T}^{-1}(v_2 - J I_\mathrm{M} v_2)|_{H^1(\mathcal{T})} \lesssim h_{\max}^{1-t} \|v_2 - J I_\mathrm{M} v_2\|_h. \tag{7.8}$$

Since $j_h(J I_\mathrm{M} v_2) = 0 = j_h(v)$, the definition of $j_h(\bullet)$ shows $j_h(v_2 - J I_\mathrm{M} v_2) = j_h(v_2 - v)$. This, the definition of $\| \bullet \|_h$ in (7.2), and Lemma 7.4.f imply

$$\|v_2 - J I_\mathrm{M} v_2\|_h \lesssim \|v - v_2\|_h. \tag{7.9}$$

The combination of (7.7)–(7.9) implies the assertion. □

Remark 7.5 (*orthogonality of* J) Since J is a right-inverse of I_{M}, i.e., $I_{\mathrm{M}}J = \mathrm{id}$ in $\mathrm{M}(\mathcal{T})$ [11, (3.9)], the integral mean property of the Hessian from Lemma 7.3.c reveals $a_{\mathrm{pw}}(v_2, (1-J)v_{\mathrm{M}}) = a_{\mathrm{pw}}(v_2, (1-I_{\mathrm{M}})Jv_{\mathrm{M}}) = 0$ for any $v_2 \in P_2(\mathcal{T})$ and $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$.

Lemma 7.6 (an intermediate bound) *For* $1 < p < \infty$*, any* $(v_2, v) \in P_2(\mathcal{T}) \times V$ *satisfies* $|v + v_2|_{W^{1,p}(\mathcal{T})} \lesssim \|v + v_2\|_h$.

Proof The triangle inequality $|v + v_2|_{W^{1,p}(\mathcal{T})} \le |v + JI_{\mathrm{M}}v_2|_{W^{1,p}(\Omega)} + |v_2 - JI_{\mathrm{M}}v_2|_{W^{1,p}(\mathcal{T})}$ and the Sobolev embedding $H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ in 2D lead to

$$|v+JI_{\mathrm{M}}v_2|_{W^{1,p}(\Omega)} \lesssim \|\!|v+JI_{\mathrm{M}}v_2|\!\| \le \|\!|v+v_2|\!\|_{\mathrm{pw}} + \|\!|v_2 - JI_{\mathrm{M}}v_2|\!\|_{\mathrm{pw}} \lesssim \|v + v_2\|_h$$

with $\|\!|\bullet|\!\|_{\mathrm{pw}} \le \|\bullet\|_h$ and Lemma 7.4.f in the last step. The inequality $|v_2 - JI_{\mathrm{M}}v_2|_{W^{1,p}(\mathcal{T})} \le |\Omega|^{1/p}|v_2 - JI_{\mathrm{M}}v_2|_{W^{1,\infty}(\mathcal{T})}$ leads to some $K \in \mathcal{T}$ with $|v_2 - JI_{\mathrm{M}}v_2|_{W^{1,\infty}(\mathcal{T})} = |v_2 - JI_{\mathrm{M}}v_2|_{W^{1,\infty}(K)}$. The inverse estimate $|v_2 - JI_{\mathrm{M}}v_2|_{W^{1,\infty}(K)} \lesssim h_K^{-1}|v_2 - JI_{\mathrm{M}}v_2|_{H^1(K)}$ and Lemma 7.4.f reveal $|v_2 - JI_{\mathrm{M}}v_2|_{W^{1,\infty}(\mathcal{T})} \lesssim \|v + v_2\|_h$. The combination of the above inequalities concludes the proof. □

Lemma 7.7 (quasi-optimal smoother R) *Any* $R \in \{id, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ *and* $\widehat{V} = V + V_h$ *with*

$$V_h (\textit{resp. } \|\bullet\|_{\widehat{V}}) := \begin{cases} \mathrm{M}(\mathcal{T}) \textit{ for the Morley scheme } (\textit{resp. } \|\!|\bullet|\!\|_{\mathrm{pw}}), \\ P_2(\mathcal{T}) \textit{ for the dG scheme } (\textit{resp. } \|\bullet\|_{\mathrm{dG}}), \\ S_0^2(\mathcal{T}) \textit{ for the } C^0 \textit{IP scheme } (\textit{resp. } \|\bullet\|_{\mathrm{IP}}), \\ P_2(\mathcal{T}) \textit{ for the WOPSIP scheme } (\textit{resp. } \|\bullet\|_{\mathrm{P}}) \end{cases}$$

satisfy

$$\|(1-R)v_h\|_{\widehat{V}} \le \Lambda_{\mathrm{R}}\|v - v_h\|_{\widehat{V}} \textit{ for all } (v_h, v) \in V_h \times V.$$

The constant Λ_{R} *exclusively depends on the shape regularity of* $\mathcal{T}$.

Proof for $R = \mathrm{id}$. This holds with $\Lambda_{\mathrm{R}} = 0$. □

Proof for $R = I_{\mathrm{M}}$. Since $\|(1-\Pi_0)D_{\mathrm{pw}}^2 v_h\| = 0$ for $v_h \in V_h \subseteq P_2(\mathcal{T})$, Lemma 7.3.a leads to $\|\!|(1-I_{\mathrm{M}})v_h|\!\|_{\mathrm{pw}} \lesssim j_h(v_h)$. This, the definition of $\|\bullet\|_h$, and $j_h(I_{\mathrm{M}}v_h) = 0 = j_h(v)$ show

$$\|\!|(1-I_{\mathrm{M}})v_h|\!\|_{\mathrm{pw}} \le \|(1-I_{\mathrm{M}})v_h\|_h \lesssim j_h(v_h) = j_h(v - v_h) \le \|v - v_h\|_h \lesssim \|v - v_h\|_{\widehat{V}}$$

with Lemma 7.1 in the last step. Theorem 4.1 of [11] provides $\|(1-I_{\mathrm{M}})v_h\|_{\widehat{V}} \lesssim \|(1-I_{\mathrm{M}})v_h\|_h$ for the dG/C^0IP norm $\|\bullet\|_{\widehat{V}}$. The combination proves the assertion for Morley/dG/C^0IP.

For WOPSIP, the definition of $\|\bullet\|_{\mathrm{P}}$ in (7.6), $\|\!|(1-I_{\mathrm{M}})v_h|\!\|_{\mathrm{pw}} \lesssim \|v - v_h\|_{\mathrm{P}}$ from the displayed inequality above, and $c_{\mathrm{P}}(v, v) = c_{\mathrm{P}}(v, v_h) = 0$ reveal

$$\|(1-I_{\mathrm{M}})v_h\|_{\mathrm{P}} \le \|\!|(1-I_{\mathrm{M}})v_h|\!\|_{\mathrm{pw}} + c_{\mathrm{P}}(v_h, v_h)^{1/2} \lesssim \|v - v_h\|_{\mathrm{P}}. \qquad \square$$

Proof for $R = JI_{\mathrm{M}}$. Triangle inequalities and $\|\bullet\|_{\widehat{V}} = \|\!|\bullet|\!\|_{\mathrm{pw}}$ in V show

$$\|(1 - JI_{\mathrm{M}})v_h\|_{\widehat{V}} \le \|v - v_h\|_{\widehat{V}} + \|\!|v - JI_{\mathrm{M}}v_h|\!\|_{\mathrm{pw}} \le 2\|v - v_h\|_{\widehat{V}} + \|\!|(1 - JI_{\mathrm{M}})v_h|\!\|_{\mathrm{pw}}.$$

Lemma 7.4.f and Lemma 7.1 conclude the proof for $R = JI_{\mathrm{M}}$. □

The transfer from $\mathrm{M}(\mathcal{T})$ into V_h [11] is modeled by some linear map $I_h : \mathrm{M}(\mathcal{T}) \to V_h$ that is bounded in the sense that there exists some constant $\Lambda_h \ge 0$ such that $\|v_{\mathrm{M}} - I_h v_{\mathrm{M}}\|_h \le \Lambda_h \|\!|v_{\mathrm{M}} - v|\!\|_{\mathrm{pw}}$ holds for all $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ and all $v \in V$. A precise definition of $I_h = I_{\mathrm{C}} I_{\mathrm{M}}$ concludes this section.

Definition 7.8 (*transfer operator* [11, (8.4)]) For $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$, let $I_{\mathrm{C}} : \mathrm{M}(\mathcal{T}) \to \mathrm{S}^2_0(\mathcal{T})$ be defined by

$$(I_{\mathrm{C}} v_{\mathrm{M}})(z) = \begin{cases} v_{\mathrm{M}}(z) \text{ at } z \in \mathcal{V}, \\ \langle v_{\mathrm{M}} \rangle_E(z) \text{ at } z = \mathrm{mid}(E) \text{ for } E \in \mathcal{E}(\Omega), \\ 0 \text{ at } z = \mathrm{mid}(E) \text{ for } E \in \mathcal{E}(\partial\Omega) \end{cases}$$

followed by Lagrange interpolation in $P_2(K)$ for all $K \in \mathcal{T}$.

Remark 7.9 (*approximation*) A triangle inequality with $I_{\mathrm{M}} v$, Lemma 7.1, and $\|v_{\mathrm{M}} - I_{\mathrm{C}} v_{\mathrm{M}}\|_h \lesssim \|\!|v - v_{\mathrm{M}}|\!\|_{\mathrm{pw}}$ for any $v \in V$ and $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ from [11, (5.11)] show $\|v - I_{\mathrm{C}} I_{\mathrm{M}} v\|_h \lesssim \|\!|v - I_{\mathrm{M}} v|\!\|_{\mathrm{pw}}$. In particular, given any $v \in V$ and given any positive $\epsilon > 0$, there exists $\delta > 0$ such that for any triangulation $\mathcal{T} \in \mathbb{T}(\delta)$ with discrete space V_h, we have $\|v - v_h\|_{\widehat{V}} < \epsilon$ for some $v_h \in V_h$. (The proof utilizes the density of smooth functions in V, the preceding estimates, and Lemma 7.3.)

8 Application to Navier-Stokes equations

This section verifies the hypotheses **(H1)–(H4)** and **($\widehat{\mathbf{H1}}$)** and establishes **(A)-(C)** for the 2D Navier-Stokes equations in the stream function vorticity formulation. Sections 8.1 and 8.2 describe the problem and four quadratic discretizations. The a priori error control for the Morley/dG/C^0IP (resp. WOPSIP) schemes follows in Sects. 8.3–8.6 (resp. Sect. 8.7) .

8.1 Stream function vorticity formulation of Navier-Stokes equations

The stream function vorticity formulation of the incompressible 2D Navier–Stokes equations in a bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ seeks $u \in H^2_0(\Omega) =: V = X = Y$ such that

$$\Delta^2 u + \frac{\partial}{\partial x}\left((-\Delta u)\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left((-\Delta u)\frac{\partial u}{\partial x}\right) = F \tag{8.1}$$

for a given right-hand side $F \in V^*$. The biharmonic operator Δ^2 is defined by $\Delta^2\phi := \phi_{xxxx} + \phi_{yyyy} + 2\phi_{xxyy}$. The analysis of extreme viscosities lies beyond the scope of this article, and the viscosity in (8.1) is set one.

For all $\phi, \chi, \psi \in V$, define the bilinear and trilinear forms $a(\bullet, \bullet)$ and $\Gamma(\bullet, \bullet, \bullet)$ by

$$a(\phi, \chi) := \int_\Omega D^2\phi : D^2\chi \,\mathrm{d}x \text{ and } \Gamma(\phi, \chi, \psi) := \int_\Omega \Delta\phi \left(\frac{\partial \chi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial x} \frac{\partial \psi}{\partial y} \right) \mathrm{d}x. \tag{8.2}$$

The weak formulation that corresponds to (8.1) seeks $u \in V$ such that

$$a(u, v) + \Gamma(u, u, v) = F(v) \quad \text{for all } v \in V. \tag{8.3}$$

8.2 Four quadratic discretizations

This subsection presents four lowest-order discretizations, namely, the Morley/dG/C^0IP/WOPSIP schemes for (8.3). Define the discrete bilinear forms

$$a_h := a_{\mathrm{pw}} + \mathsf{b}_h + \mathsf{c}_h : (V_h + \mathrm{M}(\mathcal{T})) \times (V_h + \mathrm{M}(\mathcal{T})) \to \mathbb{R},$$

with a_{pw} from (7.1) and $\mathsf{b}_h, \mathsf{c}_h$ in Table 3 for the four discretizations. Let $\widehat{\Gamma}(\bullet, \bullet, \bullet) := \Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ be the piecewise trilinear form defined for all $\phi, \chi, \psi \in H^2(\mathcal{T})$ by

$$\Gamma_{\mathrm{pw}}(\phi, \chi, \psi) := \sum_{K \in \mathcal{T}} \int_K \Delta\phi \left(\frac{\partial \chi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial x} \frac{\partial \psi}{\partial y} \right) \mathrm{d}x. \tag{8.4}$$

For all the four discretizations of Table 3, recall $\widehat{b}(\bullet, \bullet) := \Gamma_{\mathrm{pw}}(u, \bullet, \bullet) + \Gamma_{\mathrm{pw}}(\bullet, u, \bullet) : (V + P_2(\mathcal{T})) \times (V + P_2(\mathcal{T})) \to \mathbb{R}$ from (3.2). Given $R, S \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$, the discrete schemes for (8.3) seek a solution $u_h \in V_h$ to

$$N_h(u_h; v_h) := a_h(u_h, v_h) + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Sv_h) - F(JI_{\mathrm{M}} v_h) = 0 \text{ for all } v_h \in V_h. \tag{8.5}$$

8.3 Main results

This subsection states the results on the a priori control for the discrete schemes of Sect. 8.2. Lemma 7.1 shows that $\| \bullet \|_{\widehat{V}} \approx \| \bullet \|_h$ for the Morley/dG/C^0IP schemes. The WOPSIP scheme is discussed in Sect. 8.7. Unless stated otherwise, $R \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ is arbitrary.

Theorem 8.1 (a priori energy norm error control) *Given a regular root $u \in V = H_0^2(\Omega)$ to (8.3) with $F \in H^{-2}(\Omega)$ and $0 < t < 1$, there exist $\epsilon, \delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $u_h \in V_h$ to (8.5) with $\|u - u_h\|_h \le \epsilon$ for the Morley/dG/C^0IP schemes satisfies*

Table 3 Spaces, operators, bilinear forms, and norms in Sect. 8

Scheme	Morley	dG	C^0IP	WOPSIP
$\widehat{X} = \widehat{Y} := \widehat{V} = V + V_h$	$V + \mathrm{M}(\mathcal{T})$	$V + P_2(\mathcal{T})$	$V + S^2_0(\mathcal{T})$	$V + P_2(\mathcal{T})$
$\\| \bullet \\|_{\widehat{V}}$	$\vert\vert\vert \bullet \vert\vert\vert_{\mathrm{pw}}$	$\\| \bullet \\|_{\mathrm{dG}}$	$\\| \bullet \\|_{\mathrm{IP}}$	$\\| \bullet \\|_{\mathrm{P}}$
$P = Q$	J	$J I_{\mathrm{M}}$	$J I_{\mathrm{M}}$	$J I_{\mathrm{M}}$
I_h	id	id	I_{C} from Definition 7.8	id
$I_{\mathrm{X}_h} = I_{\mathrm{V}_h} = I_h I_{\mathrm{M}}$	I_{M}	I_{M}	$I_{\mathrm{C}} I_{\mathrm{M}}$	I_{M}
$\mathcal{J}(\bullet, \bullet)$	–	$\sum_{E \in \mathcal{E}} \int_E \langle D^2 v_2 \nu_E \rangle_E \cdot [\![\nabla w_2]\!]_E \, \mathrm{d}s$		–
$\mathrm{b}_h(\bullet, \bullet)$	0	$-\theta \mathcal{J}(v_2, w_2) - \mathcal{J}(w_2, v_2), -1 \leq \theta \leq 1$		0
$\mathrm{c}_h(\bullet, \bullet)$	0	c_{dG} from (7.3)	c_{IP} from (7.4)	c_{P} from (7.5)

$$\|u-u_h\|_h \lesssim \min_{v_h\in V_h} \|u-v_h\|_h + \begin{cases} 0 \textit{ for } S = JI_{\mathrm{M}}, \\ h_{\max}^{1-t} \textit{ for } S = \mathrm{id} \textit{ or } I_{\mathrm{M}}. \end{cases} \tag{8.6}$$

If $F \in H^{-r}(\Omega)$ *for some* $r < 2$*, then (8.6) holds with* $t = 0$.

Remark 8.2 (*quasi best-approximation*) The best approximation result (1.1) holds for $S = Q = JI_{\mathrm{M}}$.

A comparison result follows as in [11, Theorem 9.1] and the proof is therefore omitted.

Theorem 8.3 *(*comparison for $R \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$and $S = Q = JI_{\mathrm{M}}$*) The regular root* $u \in V$ *to (8.3) and for* $h_{\max}$ *sufficiently small, the respective local discrete solution* $u_{\mathrm{M}}, u_{\mathrm{dG}}, u_{\mathrm{IP}} \in V_h$ *to (8.5) for the Morley/dG/C*^{0}*IP schemes with* $S = JI_{\mathrm{M}}$ *satisfy*

$$\|u-u_{\mathrm{M}}\|_h \approx \|u-u_{\mathrm{dG}}\|_h \approx \|u-u_{\mathrm{IP}}\|_h \approx \|(1-\Pi_0)D^2u\|_{L^2(\Omega)}.$$

A summary of the a priori error control in Theorem 8.5 below is

$$\|u-u_h\|_{H^s(\mathcal{T})} \lesssim \|u-u_h\|_h \left(h_{\max}^a + \|u-u_h\|_h\right) + C_b h_{\max}^b \tag{8.7}$$

with a, b, C_b as described in Table 4.

Remark 8.4 (*Table* 1 *vs* 4) Note that the parameter $t > 0$ appears in Table 4 and not in Table 1. For $r = 2$, (8.7) solely asserts $\|u-u_h\|_{H^s(\mathcal{T})} \lesssim \|u-u_h\|_h^2 \lesssim 1$ even though a and b depend on t.

Recall the index of elliptic regularity σ_{reg} and $\sigma := \min\{\sigma_{\mathrm{reg}}, 1\} > 0$ from Section 1.

Theorem 8.5 *(*a priori error control in weaker Sobolev norms*) Given a regular root* $u \in V$ *to (8.3) with* $F \in H^{-2}(\Omega)$, $2-\sigma \le s < 2$*, and* $0 < t < 1$*, there exist* $\epsilon, \delta > 0$ *such that, for any* $\mathcal{T} \in \mathbb{T}(\delta)$*, the unique discrete solution* $u_h \in V_h$ *to (8.5) with* $\|u-u_h\|_{\widehat{V}} \le \epsilon$ *satisfies (a)–(e).*

(a) For the Morley/dG/C^0*IP schemes with* $R := JI_{\mathrm{M}}$,

$$\|u-u_h\|_{H^s(\mathcal{T})} \lesssim \|u-u_h\|_h \left(h_{\max}^{2-s} + \|u-u_h\|_h\right) + \begin{cases} 0 \textit{ for } S = JI_{\mathrm{M}}, \\ h_{\max}^{3-t-s} \textit{ for } S = \mathrm{id} \textit{ or } I_{\mathrm{M}}. \end{cases}$$

(b) For the Morley/dG/C^0*IP schemes with* $R := I_{\mathrm{M}}$ *and (c) for the Morley scheme with* $R = \mathrm{id}$,

$$\|u-u_h\|_{H^s(\mathcal{T})} \lesssim \|u-u_h\|_h \left(h_{\max}^{\min\{2-s,1-t\}} + \|u-u_h\|_h\right) + \begin{cases} 0 \textit{ for } S = JI_{\mathrm{M}}, \\ h_{\max}^{3-t-s} \textit{ for } S = \mathrm{id} \textit{ or } I_{\mathrm{M}}. \end{cases}$$

Table 4 Summary of error control in (8.7) from Theorem 8.5

r	s	R		S	a	b	C_b
		Morley	dG/C^0IP	Morley/dG/ C^0IP			
$r<2$	$2-\sigma\leq s<2$	id, I_M, JI_M	I_M, JI_M	JI_M	$2-s$	–	0
				id, I_M		$3-s$	1
$r=2$	$1<s<2$	id, I_M, JI_M	I_M, JI_M	JI_M	$2-s$	–	0
				id, I_M		$4-2s$	1
	$s=\sigma=1$	JI_M		JI_M	1	–	0
				id, I_M		$2-t$	1
		id, I_M	I_M	JI_M	$1-t$	–	0
				id, I_M		$2-t$	1

(d) For $\sigma < 1$, whence $1 < s < 2$, for the Morley/dG/C^0IP schemes with $R \in \{I_M, JI_M\}$ and for the Morley scheme with $R = \mathrm{id}$,

$$\|u-u_h\|_{H^s(\mathcal{T})} \lesssim \|u-u_h\|_h \left(h_{\max}^{2-s} + \|u-u_h\|_h\right) + \begin{cases} 0 \textit{ for } S = JI_M, \\ h_{\max}^{4-2s} \textit{ for } S = \mathrm{id} \textit{ or } I_M. \end{cases}$$

(e) If $F \in H^{-r}(\Omega)$ for some $r < 2$, then (a)-(c) hold with $t = 0$.

Remark 8.6 (*constant dependency*) The constants hidden in the notation $\lesssim$ of Theorem 8.1 (resp. 8.5) exclusively depend on the exact solution u (resp. u and z) to (8.3) (resp. (8.3) and (6.1)), shape regularity of $\mathcal{T}$, t (resp. s, t), and on respective stabilisation parameters $\sigma_1, \sigma_2, \sigma_{\mathrm{IP}} \approx 1$.

Remark 8.7 (*scaling for WOPSIP*) The semi-scalar product $\mathsf{c}_h(\bullet, \bullet)$ in the WOPSIP scheme is an analog to the one in j_h from (7.2) with different powers of the mesh-size. It is a consequence of the different scaling of the norms that **(H1)** and $(\widehat{\mathbf{H1}})$ do not hold for the WOPSIP scheme.

8.4 Preliminaries

This section investigates the piecewise trilinear form $\Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ from (8.4) and its boundedness with a global parameter $0 < t < 1$ that may be small. Recall the energy norm $|||\bullet|||$, and the discrete norms $|||\bullet|||_{\mathrm{pw}}$, $\|\bullet\|_h$, and $\|\bullet\|_{\mathrm{P}}$ from Sect. 7.2. The constants hidden in the notation $\lesssim$ in Lemma 8.8 below exclusively depend on the shape regularity of $\mathcal{T}$ and on t.

Lemma 8.8 (boundedness of the trilinear form) *Any $\psi \in V$ and any $\widehat{\phi}, \widehat{\chi}, \widehat{\psi} \in V + P_2(\mathcal{T})$, satisfy*

$$(a)\Gamma_{\mathrm{pw}}(\widehat{\phi}, \widehat{\chi}, \widehat{\psi}) \lesssim |||\widehat{\phi}|||_{\mathrm{pw}} \|\widehat{\chi}\|_h \|\widehat{\psi}\|_h \textit{ and}$$
$$(b)\Gamma_{\mathrm{pw}}(\widehat{\phi}, \widehat{\chi}, \psi) \lesssim |||\widehat{\phi}|||_{\mathrm{pw}} \|\widehat{\chi}\|_h \|\psi\|_{H^{1+t}(\Omega)}.$$

Proof A general Hölder inequality reveals

$$\Gamma_{\mathrm{pw}}(\widehat{\phi},\widehat{\chi},\widehat{\psi}) \le \sqrt{2}|\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}} |\widehat{\chi}|_{W^{1,2/t}(\mathcal{T})} |\widehat{\psi}|_{W^{1,2/(1-t)}(\mathcal{T})} \tag{8.8}$$

(owing to $t/2 + (1-t)/2 = 1/2$ and $|\Delta_{\mathrm{pw}}\widehat{\phi}| \le \sqrt{2}|D^2_{\mathrm{pw}}\widehat{\phi}|$ a.e.). Lemma 7.6 provides $|\widehat{\chi}|_{W^{1,2/t}(\mathcal{T})} \lesssim \|\widehat{\chi}\|_h$ and $|\widehat{\psi}|_{W^{1,2/(1-t)}(\mathcal{T})} \lesssim \|\widehat{\psi}\|_h$. The combination with (8.8) concludes the proof of (a). For $\psi \in V$ (replacing $\widehat{\psi}$), the Sobolev embedding $H^t(\Omega) \hookrightarrow L^{2/(1-t)}(\Omega)$ [4, Corollary 9.15] provides

$$|\psi|_{W^{1,2/(1-t)}(\mathcal{T})} = |\psi|_{W^{1,2/(1-t)}(\Omega)} \lesssim \|\psi\|_{H^{1+t}(\Omega)}.$$

The combination with (8.8) concludes the proof of (b). □

Lemma 8.9 *(*approximation properties*) For all $t > 0$, there exists a constant $C(t) > 0$ such that any $\phi, \chi \in V \cap H^{2+t}(\Omega)$, $\widehat{\phi}, \widehat{\chi} \in V + P_2(\mathcal{T})$, and $(v, v_2, v_{\mathrm{M}}) \in V \times P_2(\mathcal{T}) \times \mathrm{M}(\mathcal{T})$ satisfy*

(a) $\Gamma_{\mathrm{pw}}(\widehat{\phi}, \widehat{\chi}, (1 - JI_{\mathrm{M}})v_2) \le C(t) h_{\max}^{1-t} |\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}} \|\widehat{\chi}\|_h \|v - v_2\|_h$,
(b) $\Gamma_{\mathrm{pw}}(\widehat{\phi}, \chi, (1 - JI_{\mathrm{M}})v_2) \le C(t) h_{\max} |\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}} \|\chi\|_{H^{2+t}(\Omega)} \|v - v_2\|_h$,
(c) $\Gamma_{\mathrm{pw}}((1-J)v_{\mathrm{M}}, \widehat{\phi}, \widehat{\chi}) \le C(t) h_{\max}^{1-t} |\!|\!|v - v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}} \|\widehat{\phi}\|_h \|\widehat{\chi}\|_h$.
(d) $\Gamma_{\mathrm{pw}}((1-J)v_{\mathrm{M}}, \phi, \chi) \le C(t) h_{\max} |\!|\!|v - v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}} \|\phi\|_{H^{2+t}(\Omega)} \|\chi\|_{H^{2+t}(\Omega)}$.

Proof of (a). Lemma 7.6 and 7.4.h establish $|\widehat{\chi}|_{W^{1,2/t}(\mathcal{T})} \lesssim \|\widehat{\chi}\|_h$ and $|(1 - JI_{\mathrm{M}})v_2|_{W^{1,2/(1-t)}(\mathcal{T})} \lesssim h_{\max}^{1-t} \|v - v_2\|_h$. The combination with (8.8) concludes the proof of (a). □
Proof of (b). A generalised Hölder inequality and the embedding $H^{2+t}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ [4, Corollary 9.15] provide

$$\begin{aligned}\Gamma_{\mathrm{pw}}(\widehat{\phi}, \chi, (1 - JI_{\mathrm{M}})v_2) &\le \sqrt{2}|\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}} |\chi|_{W^{1,\infty}(\mathcal{T})} |(1 - JI_{\mathrm{M}})v_2|_{H^1(\mathcal{T})} \\ &\lesssim |\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}} \|\chi\|_{H^{2+t}(\mathcal{T})} |(1 - JI_{\mathrm{M}})v_2|_{H^1(\mathcal{T})}.\end{aligned}$$

Lemma 7.4.f controls the last factor and concludes the proof of (b). □
Proof of (c). Lemma 7.3.c implies $\int_\Omega \Delta_{\mathrm{pw}}(v_{\mathrm{M}} - Jv_{\mathrm{M}}) \Pi_0 D_{\mathrm{pw}}\widehat{\phi} \cdot \Pi_0 \mathrm{Curl}_{\mathrm{pw}}\widehat{\chi}\, \mathrm{dx} = 0$ and so

$$\begin{aligned}\Gamma_{\mathrm{pw}}((1-J)v_{\mathrm{M}}, \widehat{\phi}, \widehat{\chi}) &= \int_\Omega \Delta_{\mathrm{pw}}((1-J)v_{\mathrm{M}})((1-\Pi_0)D_{\mathrm{pw}}\widehat{\phi}) \cdot \mathrm{Curl}_{\mathrm{pw}}\widehat{\chi}\, \mathrm{dx} \\ &\quad + \int_\Omega \Delta_{\mathrm{pw}}((1-J)v_{\mathrm{M}})\, \Pi_0 D_{\mathrm{pw}}\widehat{\phi} \cdot ((1-\Pi_0)\mathrm{Curl}_{\mathrm{pw}}\widehat{\chi})\, \mathrm{dx}.\end{aligned} \tag{8.9}$$

A generalised Hölder inequality shows

$$\int_\Omega \Delta_{\mathrm{pw}}((1-J)v_{\mathrm{M}})((1-\Pi_0)D_{\mathrm{pw}}\widehat{\phi}) \cdot \mathrm{Curl}_{\mathrm{pw}}\widehat{\chi}\, \mathrm{dx}$$

$$\leq \|h_{\mathcal{T}}\Delta_{\mathrm{pw}}(1-J)v_{\mathrm{M}}\|_{L^{2/(1-t)}(\Omega)}\|h_{\mathcal{T}}^{-1}(1-\Pi_0)D_{\mathrm{pw}}\widehat{\phi}\|_{L^2(\Omega)}|\widehat{\chi}|_{W^{1,2/t}(\mathcal{T})}. \tag{8.10}$$

Abbreviate $a_T := h_T^{2-t}\|\Delta(v_{\mathrm{M}} - Jv_{\mathrm{M}})\|_{L^\infty(T)}$ for a triangle $T \in \mathcal{T}$ with area $|T| \leq h_T^2$ to establish

$$\|h_{\mathcal{T}}\Delta_{\mathrm{pw}}(1-J)v_{\mathrm{M}}\|_{L^{2/(1-t)}(\Omega)} \leq \Big(\sum_{T\in\mathcal{T}} a_T^{2/(1-t)}\Big)^{(1-t)/2} \leq \Big(\sum_{T\in\mathcal{T}} a_T^2\Big)^{1/2}$$

with the monotone decreasing ℓ^p norm for $2 \leq 2/(1-t)$ in the last step. An inverse estimate (with respect to the HCT refinement $\widehat{\widehat{\mathcal{T}}}$ of $\mathcal{T}$) as in the proof of Lemma 7.4.h provides $\|\Delta((1-J)v_{\mathrm{M}})\|_{L^\infty(T)} \leq \sqrt{2}\|v_{\mathrm{M}} - Jv_{\mathrm{M}}\|_{W^{2,\infty}(\Omega)} \lesssim h_T^{-1}\|v_{\mathrm{M}} - Jv_{\mathrm{M}}\|_{H^2(T)}$. Hence $a_T \lesssim h_T^{1-t}\|v_{\mathrm{M}} - Jv_{\mathrm{M}}\|_{H^2(T)}$ and

$$\|h_{\mathcal{T}}\Delta_{\mathrm{pw}}(1-J)v_{\mathrm{M}}\|_{L^{2/(1-t)}(\Omega)} \lesssim |\!|\!|h_{\mathcal{T}}^{1-t}(v_{\mathrm{M}} - Jv_{\mathrm{M}})|\!|\!|_{\mathrm{pw}} \leq h_{\max}^{1-t}|\!|\!|v_{\mathrm{M}} - Jv_{\mathrm{M}}|\!|\!|_{\mathrm{pw}}.$$

A piecewise Poincaré inequality with Payne-Weinberger constant h_T/π [24] reads

$$\pi\|h_{\mathcal{T}}^{-1}(1-\Pi_0)D_{\mathrm{pw}}\widehat{\phi}\|_{L^2(\Omega)} \leq |\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}}.$$

Recall $|\widehat{\chi}|_{W^{1,2/t}(\mathcal{T})} \lesssim \|\widehat{\chi}\|_h$ from the proof of (a). The combination of the previous estimates of the three terms in (8.10) proves the asserted estimate for the first integral in the right-hand side of (8.9). The analysis for the second term is rather analogue (interchange the role of $\widehat{\phi}$ and $\widehat{\chi}$). Notice that (c) follows even in the form $\Gamma_{\mathrm{pw}}((1-J)v_{\mathrm{M}}, \widehat{\phi}, \widehat{\chi}) \leq C(t)h_{\max}^{1-t}|\!|\!|v - v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}}(|\!|\!|\widehat{\phi}|\!|\!|_{\mathrm{pw}}\|\widehat{\chi}\|_h + \|\widehat{\phi}\|_h|\!|\!|\widehat{\chi}|\!|\!|_{\mathrm{pw}})$. □

Proof of (d). Substitute $\phi \equiv \widehat{\phi}$, $\chi \equiv \widehat{\chi}$ in (8.9) (with $\phi, \chi \in V \cap H^{2+t}(\Omega)$) and employ a different generalised Hölder inequality for the first term to infer

$$\int_\Omega \Delta_{\mathrm{pw}}((1-J)v_{\mathrm{M}})((1-\Pi_0)D\phi)\cdot \mathrm{Curl}\chi \,\mathrm{dx}$$
$$\leq \|\Delta_{\mathrm{pw}}(1-J)v_{\mathrm{M}}\|_{L^2(\Omega)}\|(1-\Pi_0)D\phi\|_{L^2(\Omega)}|\chi|_{W^{1,\infty}(\Omega)}.$$

The remaining arguments of the proof of (c) simplify to $\|\Delta_{\mathrm{pw}}(1-J)v_{\mathrm{M}}\|_{L^2(\Omega)} \leq \sqrt{2}|\!|\!|(1-J)v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}}$, $\pi\|(1-\Pi_0)D\phi\|_{L^2(\Omega)} \leq h_{\max}|\!|\!|\phi|\!|\!|$, and $|\chi|_{W^{1,\infty}(\Omega)} \lesssim \|\chi\|_{H^{2+t}(\Omega)}$ (by embedding $H^{2+t}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $t > 0$). The resulting estimate

$$\int_\Omega \Delta_{\mathrm{pw}}((1-J)v_{\mathrm{M}})((1-\Pi_0)D\phi)\cdot \mathrm{Curl}\chi \,\mathrm{dx} \lesssim h_{\max}|\!|\!|(1-J)v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}}|\!|\!|\phi|\!|\!|\|\chi\|_{H^{2+t}(\Omega)}$$

and Lemma 7.4.e lead to the assertion for one term in the right-hand side of (8.9). The analysis of the other term is analog. Notice that (d) follows even in the form $\Gamma_{\mathrm{pw}}((1-J)v_{\mathrm{M}}, \phi, \chi) \leq C(t)h_{\max}|\!|\!|v - v_{\mathrm{M}}|\!|\!|_{\mathrm{pw}}(|\!|\!|\phi|\!|\!|\|\chi\|_{H^{2+t}(\Omega)} + \|\phi\|_{H^{2+t}(\Omega)}|\!|\!|\chi|\!|\!|)$. □

8.5 Proof of Theorem 8.1

The conditions in Theorem 5.1 are verified to establish the energy norm estimates. The hypotheses (2.3)–(2.6) follow from Lemma 7.7. Hypothesis **(H1)** is verified for Morley/dG/C^0IP in the norm $\| \bullet \|_h$ in [11, Lemma 6.6] and this norm is equivalent to $\|\!| \bullet |\!\|_{\mathrm{pw}}, \| \bullet \|_{\mathrm{dG}}$, and $\| \bullet \|_{\mathrm{IP}}$ by Lemma 7.1.

Recall $a(\bullet, \bullet)$ and $\Gamma(\bullet, \bullet, \bullet)$ from (8.2), $\widehat{\Gamma}(\bullet, \bullet, \bullet) \equiv \Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ from (8.4), and $\widehat{b}(\bullet, \bullet)$ from (3.2) for the regular root $u \in H_0^2(\Omega)$. For $\theta_h \in V_h$ with $\|\theta_h\|_h = 1$, Lemma 8.8.b, and $\|\!| \bullet |\!\|_{\mathrm{pw}} \le \| \bullet \|_h$ provide $\widehat{b}(R\theta_h, \bullet) \in H^{-1-t}(\Omega)$ for $R \in \{\mathrm{id}, I_{\mathrm{M}}, J I_{\mathrm{M}}\}$. There exists a unique $\xi \equiv \xi(\theta_h) \in V \cap H^{3-t}(\Omega)$ such that $a(\xi, \phi) = \widehat{b}(R\theta_h, \phi)$ for all $\phi \in V$ and $\|\xi\|_{H^{3-t}(\Omega)} \lesssim \|\widehat{b}(R\theta_h, \bullet)\|_{H^{-1-t}(\Omega)} \lesssim 1$. The last inequality follows from Lemma 8.8.b and the boundedness of $R \in \{\mathrm{id}, I_{\mathrm{M}}, J I_{\mathrm{M}}\}$ from Lemma 7.7. Since $I_h = \mathrm{id}$ (resp. $I_h = I_{\mathrm{C}}$) for Morley/dG (resp. C^0IP), Lemma 7.1 (resp. Remark 7.9) and Lemma 7.3.d establish **(H2)** with $\delta_2 = \sup\{\|\xi - I_h I_{\mathrm{M}} \xi\|_h : \theta_h \in V_h, \|\theta_h\|_h = 1\} \lesssim h_{\max}^{1-t}$.

Since $\delta_3 = 0$ for $Q = S = J I_{\mathrm{M}}$ it remains $S = \mathrm{id}$ and $S = I_{\mathrm{M}}$ in the sequel to establish **(H3)**. Given θ_h and y_h in $V_h = X_h = Y_h$ of norm one, define $v_2 := S y_h \in P_2(\mathcal{T})$ and observe $Q y_h = J I_{\mathrm{M}} y_h = J I_{\mathrm{M}} v_2$ (by $S = \mathrm{id}, I_{\mathrm{M}}$). Hence with the definition of $\widehat{b}(\bullet, \bullet)$ from (3.2), Lemma 8.9.a shows

$$|\widehat{b}(R\theta_h, (S - Q)y_h)| = |\widehat{b}(R\theta_h, v_2 - J I_{\mathrm{M}} v_2)| \le 2C(t) h_{\max}^{1-t} \|\!| u |\!\| \|R\theta_h\|_h \|v_2\|_h. \tag{8.11}$$

The boundedness of R and I_{M} and the equivalence of norms show $\|R\theta_h\|_h \|v_2\|_h \lesssim 1$ and so $\delta_3 \lesssim h_{\max}^{1-t}$.

Consequently, for the three schemes under question and for a sufficiently small mesh-size $h_{\max}$, (2.9) holds with $\beta_h \ge \beta_0 \gtrsim 1$.

For $u \in H_0^2(\Omega)$ and $\epsilon > 0$, Remark 7.9 establishes **(H4)** with $\delta_4 < \epsilon$ for all the three schemes. The existence and uniqueness of a discrete solution u_h then follows from Theorem 4.1.

For the Morley/dG/C^0IP schemes with $F \in H^{-2}(\Omega)$, Lemma 8.9.a with $v = 0$ for $S = \mathrm{id}$ resp. $S = I_{\mathrm{M}}$, $\| \bullet \|_h \approx \| \bullet \|_{V_h}$ on V_h, and the boundedness of I_{M} show

$$\|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{V_h^*} \lesssim \begin{cases} 0 \text{ for } S = Q = J I_{\mathrm{M}}, \\ h_{\max}^{1-t} \text{ for } S = \mathrm{id} \text{ or } I_{\mathrm{M}}. \end{cases}$$

The energy norm error control then follows from Theorem 5.1.

For $F \in H^{-r}(\Omega)$ with $r < 2$, the energy norm error estimate (8.6) with $t = 0$ can be established by replacing Lemma 8.9.a in the above analysis for $r = 2$ by Lemma 8.9.b. □

8.6 Proof of Theorem 8.5

This subsection establishes the a priori control in weaker Sobolev norms for the Morley/dG/C^0IP schemes of Sect. 8.2. Given $2 - \sigma \le s \le 2$, and $G \in H^{-s}(\Omega)$ with

$\|G\|_{H^{-s}(\Omega)} = 1$, the solution z to the dual problem (6.1) belongs to $V \cap H^{4-s}(\Omega)$ by elliptic regularity. This and Lemma 7.3.d provide

$$|||z - I_{\mathrm{M}} z|||_{\mathrm{pw}} \lesssim h_{\max}^{2-s} \|z\|_{H^{4-s}(\Omega)} \lesssim h_{\max}^{2-s} \|G\|_{H^{-s}(\Omega)} = h_{\max}^{2-s}. \tag{8.12}$$

The assumptions in Theorem 6.2 with $X_s := H^s(\mathcal{T})$ and $z_h := I_h I_{\mathrm{M}} z$ are verified to establish Theorem 8.5.a-e. The control of the linear terms in Theorem 6.2 is identical for the parts (a)-(e) and this is discussed first. The proof starts with a triangle inequality

$$\|u - u_h\|_{H^s(\mathcal{T})} \le \|u - Pu_h\|_{H^s(\mathcal{T})} + \|Pu_h - u_h\|_{H^s(\mathcal{T})} \tag{8.13}$$

in the norm $H^s(\mathcal{T}) = \prod_{T \in \mathcal{T}} H^s(T)$. The Sobolev-Slobodeckii semi-norm over Ω involves double integrals over $\Omega \times \Omega$ and so is larger than or equal to the sum of the contributions over $T \times T$ for all the triangles $T \in \mathcal{T}$, i.e., $\sum_{T \in \mathcal{T}} |\bullet|^2_{H^s(T)} \le |\bullet|^2_{H^s(\Omega)}$ for any $1 < s < 2$. The definition of $\|\bullet\|_{H^s(\mathcal{T})}$ for $1 < s < 2$, Lemma 7.4.f with $t = 1$ and $P = J I_{\mathrm{M}}$ establish

$$\begin{aligned} \|Pu_h - u_h\|_{H^s(\mathcal{T})} &\le \|Pu_h - u_h\|_{H^1(\mathcal{T})} + |\nabla_{\mathrm{pw}}(Pu_h - u_h)|_{H^{s-1}(\mathcal{T})} \\ &\lesssim h_{\max} \|u - u_h\|_h + |\nabla_{\mathrm{pw}}(Pu_h - u_h)|_{H^{s-1}(\mathcal{T})}. \end{aligned} \tag{8.14}$$

The formal equivalence of the Sobolev-Slobodeckii norm and the norm by interpolation of Sobolev spaces provides for $g := \nabla_{\mathrm{pw}}(Pu_h - u_h)$, $\theta := s - 1$ and $K \in \mathcal{T}$ that

$$|g|_{H^\theta(K)} \le C(K, \theta) \|g\|^{1-\theta}_{L^2(K)} |g|^\theta_{H^1(K)}. \tag{8.15}$$

The point is that a scaling argument reveals $C(K, \theta) = C(\theta) \approx 1$ is independent of $K \in \mathcal{T}$ [10]. The Young's inequality $\big(ab \le a^p/p + b^q/q$ for $a, b \ge 0$, $1/p + 1/q = 1\big)$ leads (for $a = h_K^{2\theta(\theta-1)} \|g\|^{2(1-\theta)}_{L^2(K)}$, $b = h_K^{2\theta(1-\theta)} |g|^{2\theta}_{H^1(K)}$, $p = 1/(1-\theta)$, and $q = 1/\theta$) to

$$\begin{aligned} \sum_{K \in \mathcal{T}} \|g\|^{2(1-\theta)}_{L^2(K)} |g|^{2\theta}_{H^1(K)} &= \sum_{K \in \mathcal{T}} h_K^{2\theta(\theta-1)} \|g\|^{2(1-\theta)}_{L^2(K)} h_K^{2\theta(1-\theta)} |g|^{2\theta}_{H^1(K)} \\ &\le \|h_{\mathcal{T}}^{-\theta} g\|^2_{L^2(\Omega)} + |h_{\mathcal{T}}^{1-\theta} g|^2_{H^1(\mathcal{T})}. \end{aligned} \tag{8.16}$$

Since $P = J I_{\mathrm{M}}$ and $g = \nabla_{\mathrm{pw}}(Pu_h - u_h)$, the estimates (7.8)–(7.9) with $t = \theta$ show $\|h_{\mathcal{T}}^{-\theta} g\|^2_{L^2(\Omega)} \lesssim h_{\max}^{1-\theta} \|u - u_h\|_h$. This and Lemma 7.4.f for $t = 2$ provide

$$\|h_{\mathcal{T}}^{-\theta} g\|^2_{L^2(\Omega)} + |h_{\mathcal{T}}^{1-\theta} g|^2_{H^1(\mathcal{T})} \lesssim h_{\max}^{1-\theta} \|u - u_h\|_h. \tag{8.17}$$

The combination of (8.15)–(8.17) reveals $|\nabla_{\mathrm{pw}}(Pu_h - u_h)|_{H^{s-1}(\mathcal{T})} \lesssim h_{\max}^{2-s} \|u - u_h\|_h$ and, with (8.14),

$$\|Pu_h - u_h\|_{H^s(\mathcal{T})} \lesssim h_{\max}^{2-s} \|u - u_h\|_h. \tag{8.18}$$

This leads to the assertion for one term on the right-hand side of (8.13). To estimate the second term, $\|u - Pu_h\|_{H^s(\mathcal{T})} = G(u - Pu_h)$, we verify the assumptions in Theorem 6.1. The hypothesis ($\widehat{\mathbf{H1}}$) for the Morley/dG/C^0IP schemes is derived in [11, Lemma 6.6] for an equivalent norm (by Lemma 7.1) and Lemma 7.7 for $R = JI_{\mathrm{M}}$. The conditions (2.3)–(2.6) also follow from Lemma 7.7 as stated in the proof of Theorem 8.1. Hence, Theorem 6.1 applies and provides

$$\begin{aligned}\|u - Pu_h\|_{H^s(\mathcal{T})} &= G(u - Pu_h) \lesssim \|u - u_h\|_h(\|z - z_h\|_h \\ &\quad + \|u - u_h\|_h) + \Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h) \\ &\quad + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h). \end{aligned} \tag{8.19}$$

Since $\| \bullet \|_{\mathrm{dG}} \approx \| \bullet \|_{\mathrm{pw}}$ in $V + \mathrm{M}(\mathcal{T})$ (by Lemma 7.1), (8.12) establishes

$$\|z - z_h\|_h \lesssim h_{\max}^{2-s} \tag{8.20}$$

for the Morley/dG schemes with $I_h = \mathrm{id}$. Remark 7.9 and (8.12) establish (8.20) for the C^0IP scheme. The combination of (8.19)–(8.20) reads

$$\begin{aligned}\|u - Pu_h\|_{H^s(\mathcal{T})} &\lesssim \|u - u_h\|_h(h_{\max}^{2-s} + \|u - u_h\|_h) + \Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h) \\ &\quad + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h). \end{aligned} \tag{8.21}$$

The combination of (8.13), (8.18), and (8.21) verifies, for each of the Morley/dG/C^0IP schemes, that

$$\begin{aligned}\|u - u_h\|_{H^s(\mathcal{T})} &\lesssim \|u - u_h\|_h(h_{\max}^{2-s} + \|u - u_h\|_h) + \Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h) \\ &\quad + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h). \end{aligned} \tag{8.22}$$

Proof of Theorem 8.5.a. The difference $\Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h)$ vanishes for $P = R = JI_{\mathrm{M}}$ in each of the three schemes. The terms $\Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h)$ in (8.22) are estimated below for $S \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$ and $F \in H^{-2}(\Omega)$. Note that $Qz_h := Jz_h = JI_{\mathrm{M}}z_h$ holds for the Morley scheme. For $S = \mathrm{id}$ and each of the three discretizations, Lemma 8.9.a with $v_2 = z_h$ provides

$$\Gamma_{\mathrm{pw}}(u, u, (1 - JI_{\mathrm{M}})z_h) \lesssim h_{\max}^{1-t} \|\!|u|\!\|^2 \|z - z_h\|_h \lesssim h_{\max}^{3-t-s}$$

with (8.20) in the last step. For $S = I_{\mathrm{M}}$, Lemma 8.9.a with $v_2 = I_{\mathrm{M}}z_h$ and $\| \bullet \|_{\widehat{V}} \approx \| \bullet \|_h$ reveal

$$\Gamma_{\mathrm{pw}}(u, u, (1 - J)I_{\mathrm{M}}z_h) \lesssim h_{\max}^{1-t} \|\!|u|\!\|^2 \|z - I_{\mathrm{M}}z_h\|_h.$$

A triangle inequality and Lemma 7.7 for $R = I_{\mathrm{M}}$ provide $\|z - I_{\mathrm{M}}z_h\|_h \le (1 + \Lambda_{\mathrm{R}})\|z - z_h\|_h \lesssim h_{\max}^{2-s}$ with (8.20) in the last step. Altogether, we obtain $\Gamma_{\mathrm{pw}}(u, u, (1 - J)I_{\mathrm{M}}z_h) \lesssim h_{\max}^{3-t-s}$. The aforementioned estimates and (8.22) conclude the proof. □

Proof of Theorem 8.5.b. All the terms except the last two in (8.22) are already estimated in the proof of (*a*). For $P = Q = JI_{\mathrm{M}}$ and $R = I_{\mathrm{M}}$, elementary algebra reveals

$$\begin{aligned}&\Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h)\\ &\quad = \Gamma_{\mathrm{pw}}((R-P)u_h, Ru_h, Qz_h) + \Gamma_{\mathrm{pw}}(Pu_h, (R-P)u_h, Qz_h)\\ &\quad = \Gamma_{\mathrm{pw}}((1-J)I_{\mathrm{M}}u_h, I_{\mathrm{M}}u_h, JI_{\mathrm{M}}z_h) + \Gamma_{\mathrm{pw}}(JI_{\mathrm{M}}u_h, (1-J)I_{\mathrm{M}}u_h, JI_{\mathrm{M}}z_h).\end{aligned} \tag{8.23}$$

The bound $|||\bullet|||_{\mathrm{pw}} \le \|\bullet\|_h$, a triangle inequality, and Lemma 7.7 for $R = I_{\mathrm{M}}$ result in

$$|||u - I_{\mathrm{M}}u_h|||_{\mathrm{pw}} \le \|u-u_h\|_h + \|u_h - I_{\mathrm{M}}u_h\|_h \le (1+\Lambda_{\mathrm{R}})\|u-u_h\|_h \tag{8.24}$$

as in Remark 2.8. This and Lemma 7.4.e prove

$$|||(1-J)I_{\mathrm{M}}u_h|||_{\mathrm{pw}} \lesssim |||u - I_{\mathrm{M}}u_h|||_{\mathrm{pw}} \lesssim \|u-u_h\|_h. \tag{8.25}$$

A triangle inequality and (8.24)–(8.25) imply

$$|||u - JI_{\mathrm{M}}u_h|||_{\mathrm{pw}} \le |||u - I_{\mathrm{M}}u_h|||_{\mathrm{pw}} + |||(1-J)I_{\mathrm{M}}u_h|||_{\mathrm{pw}} \lesssim \|u-u_h\|_h. \tag{8.26}$$

As in Remark 2.8, analogous arguments plus (8.20) provide

$$|||z - I_{\mathrm{M}}z_h|||_{\mathrm{pw}} \le (1+\Lambda_{\mathrm{R}})\|z-z_h\|_h \text{ and } |||z - JI_{\mathrm{M}}z_h|||_{\mathrm{pw}} \lesssim \|z-z_h\|_h \lesssim h_{\max}^{2-s}. \tag{8.27}$$

Lemma 8.9.c and the equivalence $\|\bullet\|_h \approx |||\bullet|||_{\mathrm{pw}}$ in $V + \mathrm{M}(\mathcal{T})$ (by Lemma 7.1) control the first term on the right-hand side of (8.23), namely

$$\Gamma_{\mathrm{pw}}((1-J)I_{\mathrm{M}}u_h, I_{\mathrm{M}}u_h, JI_{\mathrm{M}}z_h) \lesssim h_{\max}^{1-t}|||u - I_{\mathrm{M}}u_h|||_{\mathrm{pw}}|||I_{\mathrm{M}}u_h|||_{\mathrm{pw}}|||JI_{\mathrm{M}}z_h|||.$$

The first factor is bounded in (8.24). Since the dual solution $z \in V \cap H^{4-s}(\Omega)$ is bounded in $V = H_0^2(\Omega)$ (even in $H^{4-s}(\Omega)$), (8.27) reveals $|||JI_{\mathrm{M}}z_h||| \lesssim 1$. Since $|||I_{\mathrm{M}}u_h|||_{\mathrm{pw}} \lesssim 1$ as well, we infer

$$\Gamma_{\mathrm{pw}}((1-J)I_{\mathrm{M}}u_h, I_{\mathrm{M}}u_h, JI_{\mathrm{M}}z_h) \lesssim h_{\max}^{1-t}\|u-u_h\|_h. \tag{8.28}$$

The anti-symmetry of $\Gamma_{\mathrm{pw}}(\bullet,\bullet,\bullet)$ with respect to the second and third variables allows the application of Lemma 8.9.a to the second term on the right-hand side of (8.23), namely

$$\begin{aligned}\Gamma_{\mathrm{pw}}(JI_{\mathrm{M}}u_h, (1-J)I_{\mathrm{M}}u_h, JI_{\mathrm{M}}z_h) &\lesssim h_{\max}^{1-t}|||JI_{\mathrm{M}}u_h|||\,|||u - I_{\mathrm{M}}u_h|||_{\mathrm{pw}}|||JI_{\mathrm{M}}z_h|||\\ &\lesssim h_{\max}^{1-t}\|u-u_h\|_h.\end{aligned}$$

The last step employed (8.24) and the boundedness $\|\!| JI_\mathrm{M}u_h \|\!| + \|\!| JI_\mathrm{M}z_h \|\!| \lesssim 1$ as well. The combination of the previously displayed estimate with (8.28) and (8.23) leads to

$$\Gamma_\mathrm{pw}(I_\mathrm{M}u_h, I_\mathrm{M}u_h, JI_\mathrm{M}z_h) - \Gamma(JI_\mathrm{M}u_h, JI_\mathrm{M}u_h, JI_\mathrm{M}z_h) \lesssim h_\mathrm{max}^{1-t}\|u - u_h\|_h. \quad (8.29)$$

The estimates of $\Gamma_\mathrm{pw}(u, u, (S-Q)z_h)$ from the above proof of Theorem 8.5.a, (8.29), and (8.22) conclude the proof.

Proof of Theorem 8.5.c. Since $u_h = u_\mathrm{M} = I_\mathrm{M}u_\mathrm{M}$, and $P = Q = J$, for the Morley FEM, the difference $\Gamma_\mathrm{pw}(u_\mathrm{M}, u_\mathrm{M}, JI_\mathrm{M}z_h) - \Gamma(Ju_\mathrm{M}, Ju_\mathrm{M}, JI_\mathrm{M}z_h)$ is controlled by (8.29). This, (8.22), and the estimates from the above proof of Theorem 8.5.a conclude the proof. □

Proof of Theorem 8.5.d. The choice $t{:=}s-1>0$ in the estimates in (a)-(c) concludes the proof. □

Proof of Theorem 8.5.e. For $F \in H^{-r}(\Omega)$ with $r<2$, the lower-order error estimates can be established with $t=0$ by the substitution of the respective assertions of Lemma 8.9.a,c by Lemma 8.9.b,d. □

Remark 8.10 (*weaker Sobolev norm estimates with* $R = \mathrm{id}$) For the dG/C^0IP schemes, (8.23) involves in particular $\Gamma_\mathrm{pw}((1-JI_\mathrm{M})u_h, u_h, JI_\mathrm{M}z_h)$ and improved estimates are unknown.

8.7 WOPSIP scheme

Recall $a_h(\bullet,\bullet) = a_\mathrm{pw}(\bullet,\bullet) + \mathsf{c}_h(\bullet,\bullet)$, $P = Q = JI_\mathrm{M}$ and $\mathsf{c}_h(\bullet,\bullet)$ from Table 3, $a_\mathrm{pw}(\bullet,\bullet)$ from (7.1), and let $u_h \equiv u_\mathrm{P}$ in this subsection. The norm $\|\bullet\|_\mathrm{P}$ from (7.6) for the WOPSIP scheme is *not* equivalent to $\|\bullet\|_h$ from (7.2) and hence **(H1)** and $(\widehat{\mathbf{H1}})$ do *not* follow. This does not prevent rather analog a priori error estimates.

Theorem 8.11 (a priori WOPSIP) *Given a regular root $u \in V$ to (8.3) with $F \in H^{-2}(\Omega)$, $2-\sigma \le s < 2$, and $0<t<1$, there exist $\epsilon, \delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $u_h \in V_h$ to (8.5) with $\|u-u_h\|_\mathrm{P} \le \epsilon$ for the WOPSIP scheme satisfies* (a)–(e).

$$\begin{aligned}(a)\,\|u-u_h\|_\mathrm{P} &\lesssim \|\!|u - I_\mathrm{M}u\|\!|_\mathrm{pw} + \|\!|h_\mathcal{T} I_\mathrm{M}u\|\!|_\mathrm{pw}\\ &+ \begin{cases} 0 & \text{for } S = JI_\mathrm{M},\\ h_\mathrm{max}^{1-t} & \text{for } S = \mathrm{id} \text{ or } I_\mathrm{M}.\end{cases}\end{aligned}$$

Moreover, if $u \in V \cap H^{4-r}(\Omega)$ with $F \in H^{-r}(\Omega)$ for $2-\sigma \le r, s \le 2$, then

$$\begin{aligned}(b)\,\|u-u_h\|_{H^s(\mathcal{T})} &\lesssim \|u-u_h\|_\mathrm{P}(h_\mathrm{max}^{2-s} + \|u-u_h\|_\mathrm{P})\\ &+ \begin{cases} 0 & \text{with } S = JI_\mathrm{M},\\ h_\mathrm{max}^{3-t-s} & \text{for } S = \mathrm{id} \text{ or } I_\mathrm{M}\end{cases} \quad \text{for } R{:=}JI_\mathrm{M}.\\ (c)\,\|u-u_h\|_{H^s(\mathcal{T})} &\lesssim \|u-u_h\|_\mathrm{P}(h_\mathrm{max}^{\min\{2-s,1-t\}} + \|u-u_h\|_\mathrm{P})\end{aligned}$$

$$+\begin{cases} 0 & \textit{for } S = JI_{\mathrm{M}}, \\ h_{\max}^{3-t-s} & \textit{for } S = \mathrm{id} \textit{ or } I_{\mathrm{M}} \end{cases} \quad \textit{for } R{:=}I_{\mathrm{M}}.$$

(d) For $\sigma < 1$, whence $1 < s < 2$, and the WOPSIP scheme with $R \in \{I_{\mathrm{M}}, JI_{\mathrm{M}}\}$,

$$\|u - u_h\|_{H^s(\mathcal{T})} \lesssim \|u - u_h\|_{\mathrm{P}} \left(h_{\max}^{2-s} + \|u - u_h\|_{\mathrm{P}}\right) + \begin{cases} 0 \textit{ for } & S = JI_{\mathrm{M}}, \\ h_{\max}^{4-2s} \textit{ for } & S = \mathrm{id} \textit{ or } I_{\mathrm{M}}. \end{cases}$$

(e) If $F \in H^{-r}(\Omega)$ for some $r < 2$, then (a)-(c) hold with $t = 0$.

The subsequent lemma extends **(H1)** in the analysis of the WOPSIP scheme.

Lemma 8.12 *(*variant of **(H1)***) There exists a constant $\Lambda_{\mathrm{W}} > 0$ such that any $v \in V$ and $v_2 \in P_2(\mathcal{T})$ satisfy*

$$a_h(I_{\mathrm{M}}v, v_2) - a(v, Qv_2) \leq \Lambda_{\mathrm{W}} \left(|||(1 - I_{\mathrm{M}})v|||_{\mathrm{pw}} + |||h_{\mathcal{T}} I_{\mathrm{M}} v|||_{\mathrm{pw}} \right) \|v_2\|_{\mathrm{P}}.$$

Proof Note that $\mathsf{c}_h(I_{\mathrm{M}}v, v_2) = 0$ for $v \in V$ and $v_2 \in P_2(\mathcal{T})$ from Table 3 and the definition of $\mathrm{M}(\mathcal{T})$. Utilize this in $a_h(\bullet, \bullet) = a_{\mathrm{pw}}(\bullet, \bullet) + \mathsf{c}_h(\bullet, \bullet)$ to infer

$$a_h(I_{\mathrm{M}}v, v_2) - a(v, Qv_2) = a_{\mathrm{pw}}((I_{\mathrm{M}} - 1)v, v_2) + a_{\mathrm{pw}}(v, (1 - Q)v_2). \tag{8.30}$$

Lemma 7.3.c implies

$$a_{\mathrm{pw}}((1 - I_{\mathrm{M}})v, v_2) = 0.$$

Since $a_{\mathrm{pw}}((1 - I_{\mathrm{M}})v, (1 - I_{\mathrm{M}})v_2) = 0 = a_{\mathrm{pw}}(I_{\mathrm{M}}v, (1 - J)I_{\mathrm{M}}v_2)$ from Lemma 7.3.c and Remark 7.5,

$$\begin{aligned} a_{\mathrm{pw}}(v, (1 - Q)v_2) &= a_{\mathrm{pw}}(v, (1 - I_{\mathrm{M}})v_2) + a_{\mathrm{pw}}(v, (1 - J)I_{\mathrm{M}}v_2) \\ &= a_{\mathrm{pw}}(I_{\mathrm{M}}v, (1 - I_{\mathrm{M}})v_2) + a_{\mathrm{pw}}((1 - I_{\mathrm{M}})v, (1 - J)I_{\mathrm{M}}v_2) \\ &\leq |||h_{\mathcal{T}} I_{\mathrm{M}}v|||_{\mathrm{pw}} |||h_{\mathcal{T}}^{-1}(1 - I_{\mathrm{M}})v_2|||_{\mathrm{pw}} + |||(1 - I_{\mathrm{M}})v|||_{\mathrm{pw}} |||(1 - J)I_{\mathrm{M}}v_2|||_{\mathrm{pw}}. \end{aligned}$$

Since Lemma 7.4.g provides $|||h_{\mathcal{T}}^{-1}(1 - I_{\mathrm{M}})v_2|||_{\mathrm{pw}} + |||(1 - J)I_{\mathrm{M}}v_2|||_{\mathrm{pw}} \lesssim \|v_2\|_{\mathrm{P}}$, this proves

$$a_{\mathrm{pw}}(v, (1 - Q)v_2) \lesssim (|||h_{\mathcal{T}} I_{\mathrm{M}}v|||_{\mathrm{pw}} + |||(1 - I_{\mathrm{M}})v|||_{\mathrm{pw}}) \|v_2\|_{\mathrm{P}}. \tag{8.31}$$

The combination of (8.30)–(8.31) concludes the proof. □

Proof of* (H2)-(H4) *for the WOPSIP scheme. For a regular root $u \in V$ to (8.3) and any $\theta_h \in P_2(\mathcal{T})$ with $\|\theta_h\|_{\mathrm{P}} = 1$, Lemma 8.8.b, $|||\bullet|||_{\mathrm{pw}} \leq \|\bullet\|_{\mathrm{P}}$, and Lemma 7.1 lead to $\widehat{b}(R\theta_h, \bullet) \in H^{-1-t}(\Omega)$ for $R \in \{\mathrm{id}, I_{\mathrm{M}}, JI_{\mathrm{M}}\}$. Therefore, there exists a unique $\xi \equiv \xi(\theta_h) \in V \cap H^{3-t}(\Omega)$ with $\|\xi\|_{H^{3-t}(\Omega)} \lesssim 1$ such that $a(\xi, \phi) = \widehat{b}(R\theta_h, \phi)$ for all $\phi \in V$. Since $I_h = \mathrm{id}$ and $\|\bullet\|_{\mathrm{P}} = |||\bullet|||_{\mathrm{pw}}$ in $V + \mathrm{M}(\mathcal{T})$ from (7.6), Lemma 7.3.d leads to **(H2)** with $\delta_2 = \sup\{\|\xi - I_h I_{\mathrm{M}}\xi\|_{\mathrm{P}} : \theta_h \in P_2(\mathcal{T}), \|\theta_h\|_{\mathrm{P}} = 1\} \lesssim h_{\max}^{1-t}$.

The proof of **(H3)** starts as in (8.11) and concludes $\delta_3 \lesssim h_{\max}^{1-t}$ from $\| \bullet \|_h \lesssim \| \bullet \|_{\mathrm{P}}$ by Lemma 7.1.

The hypothesis **(H4)** with $\delta_4 = \|u - x_h\|_{\mathrm{P}} < \epsilon$ follows from Remark 7.9. □

Proof of discrete inf-sup condition. The proof of $\beta_0 \gtrsim 1$ in (2.9) follows also for the WOPSIP scheme the above lines until (2.17) with $\xi := A^{-1}(\widehat{b}(Rx_h, \bullet)|_Y) \in X$. Recall that (2.2) leads to $x_h + \xi_h \in P_2(\mathcal{T})$ and then to some $\phi_h \in P_2(\mathcal{T})$ with $\|\phi_h\|_{\mathrm{P}} = 1$ and $\alpha_h \|x_h + \xi_h\|_{\mathrm{P}} = a_h(x_h + \xi_h, \phi_h)$; this time $\epsilon = 0$ can be neglected. An alternative split reads

$$\alpha_h \|x_h + \xi_h\|_{\mathrm{P}} = a_h(x_h, \phi_h) + a_h(\xi_h, \phi_h) - a(\xi, Q\phi_h) + a(\xi, Q\phi_h). \tag{8.32}$$

Lemma 8.12, $\xi_h = I_{\mathrm{M}}\xi$, and $|||(1 - I_{\mathrm{M}})\xi|||_{\mathrm{pw}} \lesssim \delta_2 \lesssim h_{\max}^{1-t}$ from **(H2)** provide

$$a_h(\xi_h, \phi_h) - a(\xi, Q\phi_h) \lesssim \delta_2 + |||h_{\mathcal{T}} I_{\mathrm{M}}\xi|||_{\mathrm{pw}}. \tag{8.33}$$

The arguments in (2.20) lead to $a(\xi, Q\phi_h) \le \widehat{b}(Rx_h, S\phi_h) + \delta_3$. The combination of this with (8.32)–(8.33) provides

$$\|x_h + \xi_h\|_{\mathrm{P}} \lesssim a_h(x_h, \phi_h) + \widehat{b}(Rx_h, S\phi_h) + \delta_2 + \delta_3 + |||h_{\mathcal{T}} I_{\mathrm{M}}\xi|||_{\mathrm{pw}}. \tag{8.34}$$

Replace (2.21) by (8.34) and apply the arguments thereafter to establish the stability condition (2.9) with $\beta_0 := \alpha_h \widehat{\beta} - (\Lambda_{\mathrm{W}} + \alpha_h)\delta_2 - \delta_3 - \Lambda_{\mathrm{W}} |||h_{\mathcal{T}} I_{\mathrm{M}}\xi|||_{\mathrm{pw}}$ for some $\Lambda_{\mathrm{W}} \lesssim 1$. □

Proof of existence and uniqueness of the discrete solution. The analysis follows the proof of Theorem 4.1 verbatim until (4.6). Instead of **(H1)**, Lemma 8.12 and $x_h = I_{\mathrm{M}}u$ in **(H4)** control the first two terms on the right-hand side of (4.6), namely

$$a_h(x_h, y_h) - a(u, Qy_h) \le \Lambda_{\mathrm{W}}(\delta_4 + |||h_{\mathcal{T}} I_{\mathrm{M}}u|||_{\mathrm{pw}}).$$

The remaining steps follow those of the proof of Theorem 4.1 with (4.1) replaced by

$$\begin{aligned}\epsilon_0 := \beta_1^{-1}\big((\Lambda_{\mathrm{W}} + (1 + \Lambda_{\mathrm{R}})(\|R\|\|S\| |||I_{\mathrm{M}}u|||_{\mathrm{pw}} + \|Q\|\|u\|_X)\|\widehat{\Gamma}\|)\delta_4 \\ + \Lambda_{\mathrm{W}} |||h_{\mathcal{T}} I_{\mathrm{M}}u|||_{\mathrm{pw}} + |||I_{\mathrm{M}}u|||_{\mathrm{pw}}\delta_3/2\big).\end{aligned}$$

□

Proof of Theorem 8.11.a. Recall from Lemma 5.2 that $u^* \in X$ and $G(\bullet) = a(u^*, \bullet) \in Y^*$, $u_h^* \in X_h$ and $a_h(u_h^*, \bullet) = G(Q\bullet) \in Y_h^*$. In the proof of Lemma 5.2, set $x_h := I_{\mathrm{M}}u^*$ so that Lemma 8.12 implies

$$\begin{aligned}\alpha_0\|e_h\|_{\mathrm{P}} \le a_h(x_h, y_h) - a(u^*, Qy_h) \le \Lambda_{\mathrm{W}}(|||u^* - I_{\mathrm{M}}u^*|||_{\mathrm{pw}} \\ + |||h_{\mathcal{T}} I_{\mathrm{M}}u^*|||_{\mathrm{pw}}).\end{aligned}$$

Therefore, u^* and u_h^* in Lemma 5.2 satisfy $\|u^* - u_h^*\|_{\mathrm{P}} \le C'_{\mathrm{qo}} \|\!| u^* - I_{\mathrm{M}} u^* \|\!|_{\mathrm{pw}} + \alpha_0^{-1} \Lambda_{\mathrm{W}} \|\!| h_{\mathcal{T}} I_{\mathrm{M}} u^* \|\!|_{\mathrm{pw}}$ for $C'_{\mathrm{qo}} = 1 + \alpha_0^{-1} \Lambda_{\mathrm{W}}$.

The hypotheses (2.3)–(2.6) follow from Lemma 7.7; **(H2)-(H4)** are already verified. The error estimate in Lemma 5.2 applies to Theorem 5.1 with $x_h = I_{\mathrm{M}} u$ and $\| \bullet \|_{\mathrm{P}} = \|\!| \bullet \|\!|_{\mathrm{pw}}$ in $V + \mathrm{M}(\mathcal{T})$ and establishes

$$\|u - u_h\|_{\mathrm{P}} \lesssim \|\!| u - I_{\mathrm{M}} u \|\!|_{\mathrm{pw}} + \|\!| h_{\mathcal{T}} I_{\mathrm{M}} u \|\!|_{\mathrm{pw}} + \|\widehat{\Gamma}(u, u, (S - Q)\bullet)\|_{Y_h^*}$$

For $u \in V$, the last displayed estimate, Lemma 8.9.a with $v = 0$ for $S = \mathrm{id}$ (resp. with $v_2 \in \mathrm{M}(\mathcal{T})$ for $S = I_{\mathrm{M}}$), Lemma 7.1, and the boundedness of I_{M} conclude the proof. □

Proof of Theorem 8.11.b. A triangle inequality leads to

$$\begin{aligned} \|u - u_h\|_{H^s(\mathcal{T})} &\le \|u - Pu_h\|_{H^s(\mathcal{T})} + \|Pu_h - u_h\|_{H^s(\mathcal{T})} \\ &= G(u - Pu_h) + \|Pu_h - u_h\|_{H^s(\mathcal{T})} \end{aligned} \tag{8.35}$$

with $G(u - Pu_h) = \|u - Pu_h\|_{H^s(\mathcal{T})}$ owing to a corollary of the Hahn-Banach theorem as in the proof of Theorem 6.2 in the last step. Since $z \in Y$ solves (6.1), elementary algebra with (3.3)–(3.5) and $z_h := I_{\mathrm{M}} z \in Y_h$ lead to an alternative identity in place of (6.3), namely

$$\begin{aligned} G(u - Pu_h) = (a + b)(u - Pu_h, z) &= a(u, z - Qz_h) + a_{\mathrm{pw}}(u_h - Pu_h, z) \\ &\quad + b(u - Pu_h, z - Qz_h) + b(u - Pu_h, Qz_h) \\ &\quad + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Sz_h) - \Gamma(u, u, Qz_h) \end{aligned} \tag{8.36}$$

with $a_h(u_h, z_h) = a_{\mathrm{pw}}(u_h, z)$ from Lemma 7.3.c in the last step. Since $a_{\mathrm{pw}}(I_{\mathrm{M}} u, z - Qz_h) = 0$ from Lemma 7.3.c and Remark 7.5,

$$a(u, z - Qz_h) = a_{\mathrm{pw}}(u - I_{\mathrm{M}} u, z - Qz_h) \le (1 + \Lambda_{\mathrm{Q}}) \|\!| u - I_{\mathrm{M}} u \|\!|_{\mathrm{pw}} \|\!| z - z_h \|\!|_{\mathrm{pw}}$$

with boundedness of $a_{\mathrm{pw}}(\bullet, \bullet)$ and (2.11) in the last step. A triangle inequality shows that

$$\|\!| u - I_{\mathrm{M}} u \|\!|_{\mathrm{pw}} \le \|\!| u - u_h \|\!|_{\mathrm{pw}} + \|\!| u_h - I_{\mathrm{M}} u_h \|\!|_{\mathrm{pw}} + \|\!| I_{\mathrm{M}}(u - u_h) \|\!|_{\mathrm{pw}} \lesssim \|u - u_h\|_{\mathrm{P}} \tag{8.37}$$

with $\|\!| \bullet \|\!|_{\mathrm{pw}} \le \| \bullet \|_{\mathrm{P}}$, $\|(1 - I_{\mathrm{M}}) u_h\|_{\mathrm{P}} \le \Lambda_{\mathrm{R}} \|u - u_h\|_{\mathrm{P}}$ from Lemma 7.7, and $\|\!| I_{\mathrm{M}}(u - u_h) \|\!|_{\mathrm{pw}} \le \|\!| u - u_h \|\!|_{\mathrm{pw}}$ in the last step. Arguments analogous to (8.31) and Lemma 7.4.g with $v = u$ lead to

$$a_{\mathrm{pw}}(u_h - Pu_h, z) \lesssim (\|\!| h_{\mathcal{T}} I_{\mathrm{M}} z \|\!|_{\mathrm{pw}} + \|\!| (1 - I_{\mathrm{M}}) z \|\!|_{\mathrm{pw}}) \|u - u_h\|_{\mathrm{P}}. \tag{8.38}$$

The combination of (8.36)–(8.38) and the estimates for the remaining terms in the right-hand side of (8.36) from the last part (after (6.4)) of the proof of Theorem 6.1

result in

$$G(u - Pu_h) \lesssim \|u - u_h\|_{\mathrm{P}}(\|\!|z - z_h\|\!|_{\mathrm{pw}} + \|\!|h_{\mathcal{T}} z_h\|\!|_{\mathrm{pw}} + \|u - u_h\|_{\mathrm{P}}) + \Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h) + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h). \tag{8.39}$$

Since $z_h = I_{\mathrm{M}} z$, Lemma 7.3.d provides $\|\!|z - z_h\|\!|_{\mathrm{pw}} \lesssim h_{\max}^{2-s}$ and $\|\!|h_{\mathcal{T}} z_h\|\!|_{\mathrm{pw}} \lesssim h_{\max}$. Lemma 7.4.f and $\| \bullet \|_h \lesssim \| \bullet \|_{\mathrm{P}}$ (by Lemma 7.1) establish $\|Pu_h - u_h\|_{H^s(\mathcal{T})} \lesssim h_{\max}^{2-s}\|u - u_{\mathrm{P}}\|_{\mathrm{P}}$. The combination of those estimates with (8.35) and (8.39) reveals

$$\|u - u_h\|_{H^s(\mathcal{T})} \lesssim \|u - u_h\|_{\mathrm{P}}(h_{\max}^{2-s} + \|u - u_h\|_{\mathrm{P}}) + \Gamma_{\mathrm{pw}}(u, u, (S - Q)z_h) + \Gamma_{\mathrm{pw}}(Ru_h, Ru_h, Qz_h) - \Gamma(Pu_h, Pu_h, Qz_h).$$

The last three terms in the above inequality can be estimated as in the proof of Theorem 8.5.a with $\| \bullet \|_h \lesssim \| \bullet \|_{\mathrm{P}}$ (by Lemma 7.1) and this concludes the proof. □

Proof of Theorem 8.11.c. The arguments in (b) and Theorem 8.5.b establish (c). □

Proof of Theorem 8.11.d. The choice $t{:=}s - 1$ in (b)-(c) concludes the proof. □

Proof of Theorem 8.11.e. For $F \in H^{-r}(\Omega)$ with $r < 2$, the a priori error estimates can be established with $t = 0$ by a substitution of the assertions in Lemma 8.9.a,c by Lemma 8.9.b,d.

9 Application to von Kármán equations

This section verifies **(H1)-(H4)** and ($\widehat{\mathbf{H1}}$), and establishes **(A)-(C)** for the von Kármán equations. Sects. 9.1 and 9.2 present the problem and four discretizations; the a priori error control for the Morley/dG/C^0IP/WOPSIP schemes follows in Sect. 9.3–9.6.

9.1 Von Kármán equations

The von Kármán equations in a polygonal domain $\Omega \subset \mathbb{R}^2$ seek $(u, v) \in H_0^2(\Omega) \times H_0^2(\Omega) = V \times V =: \mathbf{V}$ such that

$$\Delta^2 u = [u, v] + f \text{ and } \Delta^2 v = -\frac{1}{2}[u, u] \text{ in } \Omega. \tag{9.1}$$

The von Kármán bracket $[\bullet, \bullet]$ above is defined by $[\eta, \chi]{:=}\eta_{xx}\chi_{yy}+\eta_{yy}\chi_{xx}-2\eta_{xy}\chi_{xy}$ for all $\eta, \chi \in V$. The weak formulation of (9.1) seeks $u, v \in V$ that satisfy for all $(\varphi_1, \varphi_2) \in \mathbf{V}$

$$a(u, \varphi_1) + \gamma(u, v, \varphi_1) + \gamma(v, u, \varphi_1) = f(\varphi_1) \text{ and } a(v, \varphi_2) - \gamma(u, u, \varphi_2) = 0 \tag{9.2}$$

with $\gamma(\eta, \chi, \varphi) := -\dfrac{1}{2}\int_\Omega [\eta, \chi]\varphi \,\mathrm{dx}$ for all $\eta, \chi, \varphi \in V$ and $a(\bullet, \bullet)$ from (8.2). For all $\Xi = (\xi_1, \xi_2)$, $\Theta = (\theta_1, \theta_2)$, and $\Phi = (\varphi_1, \varphi_2) \in \mathbf{V}$, define the forms

$$A(\Theta, \Phi) := a(\theta_1, \varphi_1) + a(\theta_2, \varphi_2),$$
$$\Gamma(\Xi, \Theta, \Phi) := \gamma(\xi_1, \theta_2, \varphi_1) + \gamma(\xi_2, \theta_1, \varphi_1) - \gamma(\xi_1, \theta_1, \varphi_2), \text{ and } F(\Phi) := f(\varphi_1).$$

Then the vectorised formulation of (9.2) seeks $\Psi = (u, v) \in \mathbf{V}$ such that

$$N(\Psi; \Phi) := A(\Psi, \Phi) + \Gamma(\Psi, \Psi, \Phi) - F(\Phi) = 0 \quad \text{for all } \Phi \in \mathbf{V}. \tag{9.3}$$

The trilinear form $\Gamma(\bullet, \bullet, \bullet)$ inherits symmetry in the first two variables from $\gamma(\bullet, \bullet, \bullet)$. The following boundedness and ellipticity properties hold [5, 16, 22]

$$A(\Theta, \Phi) \le \|\!|\Theta|\!\|\,\|\!|\Phi|\!\|,\ \|\!|\Theta|\!\|^2 \le A(\Theta, \Theta), \text{ and } \Gamma(\Xi, \Theta, \Phi) \lesssim \|\!|\Xi|\!\|\,\|\!|\Theta|\!\|\,\|\!|\Phi|\!\|.$$

9.2 Four quadratic discretizations

This subsection presents the Morley/dG/C^0IP/WOPSIP schemes for (9.3). The spaces and operators employed in the analysis of the von Kármán equations given in Table 5 are vectorised versions (denoted in boldface) of those presented in Table 3, e.g., $\boldsymbol{I}_\mathrm{M} = I_\mathrm{M} \times I_\mathrm{M}$. Recall $a_\mathrm{pw}(\bullet, \bullet)$ from (7.1) and define the bilinear form $a_h : (\mathbf{V}_h + \mathbf{M}(\mathcal{T})) \times (\mathbf{V}_h + \mathbf{M}(\mathcal{T})) \to \mathbb{R}$ by

$$\begin{aligned} a_h(\Theta, \Phi) := {} & a_\mathrm{pw}(\theta_1, \varphi_1) + \mathsf{b}_h(\theta_1, \varphi_1) + \mathsf{c}_h(\theta_1, \varphi_1) \\ & + a_\mathrm{pw}(\theta_2, \varphi_2) + \mathsf{b}_h(\theta_2, \varphi_2) + \mathsf{c}_h(\theta_2, \varphi_2). \end{aligned}$$

The definitions of b_h and c_h for the Morley/dG/C^0IP/WOPSIP schemes from Table 3 are omitted in Table 5 for brevity. For all $\eta, \chi, \varphi \in H^2(\mathcal{T})$, let $\gamma_\mathrm{pw}(\bullet, \bullet, \bullet)$ be the piecewise trilinear form defined by

$$\gamma_\mathrm{pw}(\eta, \chi, \varphi) := -\frac{1}{2}\sum_{K \in \mathcal{T}} \int_K [\eta, \chi]\varphi \,\mathrm{dx}$$

and, for all $\Xi = (\xi_1, \xi_2)$, $\Theta = (\theta_1, \theta_2)$, $\Phi = (\varphi_1, \varphi_2) \in \mathbf{H}^2(\mathcal{T})$, let

$$\widehat{\Gamma}(\Xi, \Theta, \Phi) := \Gamma_\mathrm{pw}(\Xi, \Theta, \Phi) := \gamma_\mathrm{pw}(\xi_1, \theta_2, \varphi_1) + \gamma_\mathrm{pw}(\xi_2, \theta_1, \varphi_1) - \gamma_\mathrm{pw}(\xi_1, \theta_1, \varphi_2). \tag{9.4}$$

For all the schemes and a regular root $\Psi \in \mathbf{V}$ to (9.3), let $\widehat{b}(\bullet, \bullet) := 2\Gamma_\mathrm{pw}(\Psi, \bullet, \bullet)$ in (3.2). For $R, S \in \{\mathbf{id}, \boldsymbol{I}_\mathrm{M}, \boldsymbol{J}\boldsymbol{I}_\mathrm{M}\}$, the discrete scheme seeks a root $\Psi_h := (u_h, v_h) \in \mathbf{V}_h$ to

$$N_h(\Psi_h; \Phi_h) := a_h(\Psi_h, \Phi_h) + \Gamma_\mathrm{pw}(R\Psi_h, R\Psi_h, S\Phi_h) - F(\boldsymbol{J}\boldsymbol{I}_\mathrm{M}\Phi_h) = 0 \quad \text{for all} \Phi_h \in \mathbf{V}_h. \tag{9.5}$$

Table 5 Spaces, operators, and norms in Sect. 9

Scheme	Morley	dG	C^0IP	WOPSIP
$X_h = Y_h = \mathbf{V}_h$	$\mathbf{M}(\mathcal{T})$	$\boldsymbol{P}_2(\mathcal{T})$	$S_0^2(\mathcal{T})$	$\boldsymbol{P}_2(\mathcal{T})$
$\widehat{X} = \widehat{Y} = \widehat{\mathbf{V}} = \mathbf{V} + \mathbf{V}_h$	$\mathbf{V} + \mathbf{M}(\mathcal{T})$	$\mathbf{V} + \boldsymbol{P}_2(\mathcal{T})$	$\mathbf{V} + S_0^2(\mathcal{T})$	$\mathbf{V} + \boldsymbol{P}_2(\mathcal{T})$
$\|\bullet\|_{\widehat{X}}$	$\|\!\|\bullet\|\!\|_{\mathrm{pw}}$	$\|\bullet\|_{\mathrm{dG}}$	$\|\bullet\|_{\mathrm{IP}}$	$\|\bullet\|_{\mathrm{P}}$
$P = Q$	$\boldsymbol{J}$	$\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$
I_h	$\mathbf{id}$	$\mathbf{id}$	$\boldsymbol{I}_{\mathrm{C}}$	$\mathbf{id}$
$I_{X_h} = I_{\mathbf{V}_h} = I_h \boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{I}_{\mathrm{C}}\boldsymbol{I}_{\mathrm{M}}$	$\boldsymbol{I}_{\mathrm{M}}$

9.3 Main results

The main results on a priori error control in energy and weaker Sobolev norms for the Morley/dG/C^0IP/ WOPSIP schemes of Sect. 9.2 are stated in this and verified in the subsequent subsections. Unless stated otherwise, $R \in \{\mathbf{id}, \boldsymbol{I}_{\mathrm{M}}, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\}$ is arbitrary.

Theorem 9.1 *(*A priori energy norm error control*) Given a regular root $\Psi \in \mathbf{V}$ to (9.3) with $F \in \mathbf{H}^{-2}(\Omega)$, there exist $\epsilon, \delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_h \in \mathbf{V}_h$ to (9.5) with $\|\Psi - \Psi_h\|_h \le \epsilon$ for the Morley/dG/C^0IP schemes satisfies*

$$\|\Psi - \Psi_h\|_h \lesssim \min_{\Psi_h \in \mathbf{V}_h} \|\Psi - \Psi_h\|_h + \begin{cases} 0 \text{ for } S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \\ h_{\max} \text{ for } S = \mathbf{id} \text{ or } \boldsymbol{I}_{\mathrm{M}}. \end{cases}$$

The a priori estimates in Table 1 hold for von Kármán equations component-wise for $F \in \mathbf{H}^{-r}(\Omega)$, $2 - \sigma \le r \le 2$ and $\Psi \in \mathbf{V} \cap \mathbf{H}^{4-r}(\Omega)$.

Remark 9.2 (*Comparison*) Suppose $\Psi \in \mathbf{V}$ is a regular root to (9.3) with $F \in \mathbf{H}^{-2}(\Omega)$ and $S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$. If $h_{\max}$ is sufficiently small, then the respective local discrete solutions $\Psi_{\mathrm{M}}, \Psi_{\mathrm{dG}}, \Psi_{\mathrm{IP}} \in \mathbf{V}_h$ to (9.5) for the Morley/dG/C^0IP schemes satisfy

$$\|\Psi - \Psi_{\mathrm{M}}\|_h \approx \|\Psi - \Psi_{\mathrm{dG}}\|_h \approx \|\Psi - \Psi_{\mathrm{IP}}\|_h \approx \|(1 - \Pi_0) D^2 \Psi\|_{\boldsymbol{L}^2(\Omega)}. \qquad \square$$

Theorem 9.3 *(*a priori error control in weaker norms*) Given a regular root $\Psi \in \mathbf{V} \cap \mathbf{H}^{4-r}(\Omega)$ to (9.3) with $F \in \mathbf{H}^{-r}(\Omega)$ for $2 - \sigma \le r, s \le 2$, there exist $\epsilon, \delta > 0$ such that, for any $\mathcal{T} \in \mathbb{T}(\delta)$, the unique discrete solution $\Psi_h \in \mathbf{V}_h$ to (9.5) with $\|\Psi - \Psi_h\|_h \le \epsilon$ satisfies*

$$\|\Psi - \Psi_h\|_{\mathbf{H}^s(\mathcal{T})} \lesssim \|\Psi - \Psi_h\|_h \big(h_{\max}^{2-s} + \|\Psi - \Psi_h\|_h\big) + \begin{cases} 0 \text{ for } S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \\ h_{\max}^{3-s} \text{ for } S = \mathbf{id} \text{ or } \boldsymbol{I}_{\mathrm{M}} \end{cases}$$

(a) for the Morley/dG/C^0IP schemes and $R = \{\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \boldsymbol{I}_{\mathrm{M}}\}$ and (b) for the Morley scheme and $R = \mathbf{id}$.

Theorem 9.4 (a priori WOPSIP) *Given a regular root* $\Psi \in \mathbf{V}$ *to* (9.3) *with* $F \in \mathbf{H}^{-2}(\Omega)$, *there exist* $\epsilon, \delta > 0$ *such that, for any* $\mathcal{T} \in \mathbb{T}(\delta)$, *the unique discrete solution* $\Psi_h \in \mathbf{V}_h$ *to* (9.5) *with* $\|\Psi - \Psi_h\|_{\mathrm{P}} \le \epsilon$ *for the WOPSIP scheme satisfies*

$$(a) \|\Psi - \Psi_h\|_{\mathrm{P}} \lesssim \|\Psi - \boldsymbol{I}_{\mathrm{M}}\Psi\|_{\mathrm{pw}} + \|h_{\mathcal{T}}\boldsymbol{I}_{\mathrm{M}}\Psi\|_{\mathrm{pw}} + \begin{cases} 0 \text{ for } S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \\ h_{\max} \text{ for } S = \mathbf{id} \text{ or } \boldsymbol{I}_{\mathrm{M}}. \end{cases}$$

Moreover, if $F \in \mathbf{H}^{-r}(\Omega)$ *for* $2 - \sigma \le r, s \le 2$ *and* $R \in \{\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \boldsymbol{I}_{\mathrm{M}}\}$, *then*

$$(b) \|\Psi - \Psi_h\|_{\mathbf{H}^s(\mathcal{T})} \lesssim \|\Psi - \Psi_h\|_{\mathrm{P}}\big(h_{\max}^{2-s} + \|\Psi - \Psi_h\|_{\mathrm{P}}\big) + \begin{cases} 0 \text{ for } S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \\ h_{\max}^{3-s} \text{ for } S = \mathbf{id} \text{ or } \boldsymbol{I}_{\mathrm{M}}. \end{cases}$$

9.4 Preliminaries

Two lemmas on the trilinear form $\Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ from (9.4) are crucial for the a priori error control.

Lemma 9.5 (boundedness) *For any* $0 < t < 1$ *there exists a constant* $C(t) > 0$ *such that any* $\widehat{\Phi}, \widehat{\boldsymbol{\chi}} \in \mathbf{V} + \boldsymbol{P}_2(\mathcal{T})$, $\widehat{\Xi} \in \mathbf{V} + \mathbf{M}(\mathcal{T})$, *and* $\Xi \in \mathbf{V}$ *satisfy*

$$(a) \Gamma_{\mathrm{pw}}(\widehat{\Phi}, \widehat{\boldsymbol{\chi}}, \widehat{\Xi}) \lesssim \|\widehat{\Phi}\|_{\mathrm{pw}} \|\widehat{\boldsymbol{\chi}}\|_{\mathrm{pw}} \|\widehat{\Xi}\|_{\mathrm{pw}} \text{ and}$$
$$(b) \Gamma_{\mathrm{pw}}(\widehat{\Phi}, \widehat{\boldsymbol{\chi}}, \Xi) \le C(t) \|\widehat{\Phi}\|_{\mathrm{pw}} \|\widehat{\boldsymbol{\chi}}\|_{\mathrm{pw}} \|\Xi\|_{\mathbf{H}^{1+t}(\Omega)}.$$

Proof of (a). The definition of $\gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$, Hölder inequalities, and $\|\bullet\|_{L^\infty(\Omega)} \lesssim \|\bullet\|_{\mathrm{pw}}$ in $V + \mathrm{M}(\mathcal{T})$ from [8, Lemma 4.7] establish, for $\widehat{\phi}, \widehat{\chi} \in V + P_2(\mathcal{T})$, $\widehat{\xi} \in V + \mathrm{M}(\mathcal{T})$, that

$$\gamma_{\mathrm{pw}}(\widehat{\phi}, \widehat{\chi}, \widehat{\xi}) \le \|\widehat{\phi}\|_{\mathrm{pw}} \|\widehat{\chi}\|_{\mathrm{pw}} \|\widehat{\xi}\|_{L^\infty(\Omega)} \lesssim \|\widehat{\phi}\|_{\mathrm{pw}} \|\widehat{\chi}\|_{\mathrm{pw}} \|\widehat{\xi}\|_{\mathrm{pw}}.$$

Proof of (b). For $\widehat{\phi}, \widehat{\chi} \in V + P_2(\mathcal{T})$ and $\xi \in V$, the definition of $\gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$, Hölder inequalities, and the continuous Sobolev embedding $H^{1+t}(\Omega) \hookrightarrow L^\infty(\Omega)$ [4, Corollary 9.15] for $t > 0$ show

$$\gamma_{\mathrm{pw}}(\widehat{\phi}, \widehat{\chi}, \xi) \le \|\widehat{\phi}\|_{\mathrm{pw}} \|\widehat{\chi}\|_{\mathrm{pw}} \|\xi\|_{L^\infty(\Omega)} \lesssim \|\widehat{\phi}\|_{\mathrm{pw}} \|\widehat{\chi}\|_{\mathrm{pw}} \|\xi\|_{H^{1+t}(\Omega)}.$$

This and (9.4) conclude the proof. □

Lemma 9.6 (approximation) *Any* $\widehat{\boldsymbol{\chi}} \in \mathbf{V} + \boldsymbol{P}_2(\mathcal{T})$, $\Phi, \mathbf{v} \in \mathbf{V}$, *and* $(\mathbf{v}_2, \mathbf{v}_{\mathrm{M}}) \in \boldsymbol{P}_2(\mathcal{T}) \times \mathbf{M}(\mathcal{T})$ *satisfy*

(a) $\Gamma_{\mathrm{pw}}(\Phi, \widehat{\boldsymbol{\chi}}, (1 - \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}})\mathbf{v}_2) \lesssim h_{\max} \|\Phi\| \|\widehat{\boldsymbol{\chi}}\|_{\mathrm{pw}} \|\mathbf{v} - \mathbf{v}_2\|_h$,
(b) $\Gamma_{\mathrm{pw}}((1 - \boldsymbol{J})\mathbf{v}_{\mathrm{M}}, \mathbf{v}_2, \Phi) \lesssim h_{\max} \|\mathbf{v} - \mathbf{v}_{\mathrm{M}}\|_{\mathrm{pw}} \|\mathbf{v}_2\|_{\mathrm{pw}} \|\Phi\|$.

Proof of (a). For $\phi \in V$, $\widehat{\chi} \in V + P_2(\mathcal{T})$ and $v_2 \in P_2(\mathcal{T})$, the definition of $\gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$, Hölder inequalities, and an inverse estimate $h_T \|(1 - JI_{\mathrm{M}})v_2\|_{L^\infty(T)} \lesssim \|(1 - JI_{\mathrm{M}})v_2\|_{L^2(T)}$ lead to

$$\begin{aligned}\gamma_{\mathrm{pw}}(\phi, \widehat{\chi}, (1 - JI_{\mathrm{M}})v_2) &\le \|\!|\phi\|\!| \, \|\!|\widehat{\chi}\|\!|_{\mathrm{pw}} \|(1 - JI_{\mathrm{M}})v_2\|_{L^\infty(\Omega)} \\ &\lesssim \|\!|\phi\|\!| \, \|\!|\widehat{\chi}\|\!|_{\mathrm{pw}} \|h_{\mathcal{T}}^{-1}(1 - JI_{\mathrm{M}})v_2\|.\end{aligned}$$

This, Lemma 7.4.f, and the definition of $\Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ conclude the proof of (a).

Proof of (b). For $\phi \in V$, $v_2 \in P_2(\mathcal{T})$, and $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$, an introduction of $\Pi_0\phi$ and $\gamma_{\mathrm{pw}}((1 - J)v_{\mathrm{M}}, v_2, \Pi_0\phi) = 0$ from Lemma 7.3.c and Remark 7.5 provide

$$\gamma_{\mathrm{pw}}((1 - J)v_{\mathrm{M}}, v_2, \phi) = \gamma_{\mathrm{pw}}((1 - J)v_{\mathrm{M}}, v_2, \phi - \Pi_0\phi). \tag{9.6}$$

Hölder inequalities and the estimate $\|\phi - \Pi_0\phi\|_{L^\infty(\Omega)} \lesssim h_{\max}\|\!|\phi\|\!|$ [15, Theorem 3.1.5] provide

$$\begin{aligned}\gamma_{\mathrm{pw}}((1 - J)v_{\mathrm{M}}, v_2, \phi - \Pi_0\phi) &\lesssim h_{\max}\|\!|(1 - J)v_{\mathrm{M}}\|\!|_{\mathrm{pw}}\|\!|v_2\|\!|_{\mathrm{pw}}\|\!|\phi\|\!| \\ &\lesssim h_{\max}\|\!|v - v_{\mathrm{M}}\|\!|_{\mathrm{pw}}\|\!|v_2\|\!|_{\mathrm{pw}}\|\!|\phi\|\!|\end{aligned}$$

with $\|\!|(1 - J)v_{\mathrm{M}}\|\!|_{\mathrm{pw}} \lesssim \|\!|v - v_{\mathrm{M}}\|\!|_{\mathrm{pw}}$ from Lemma 7.4.e in the last step. Recall (9.4) and (9.6) to conclude the proof of (b). □

9.5 Proof of Theorem 9.1

The conditions in Theorem 5.1 are verified to establish the energy norm estimates. The hypotheses (2.3)–(2.6) follow from Lemma 7.7 (component-wise). The paper [11] has verified hypothesis **(H1)** for Morley/dG/C^0IP in the norm $\| \bullet \|_h$ that is equivalent to $\|\!| \bullet \|\!|_{\mathrm{pw}}$, $\| \bullet \|_{\mathrm{dG}}$, and $\| \bullet \|_{\mathrm{IP}}$ by Lemma 7.1.

For any $\boldsymbol{\theta}_h \in \mathbf{V}_h$ with $\|\boldsymbol{\theta}_h\|_{\mathbf{V}_h} = 1$, Lemma 9.5.b with $\|\!| \bullet \|\!|_{\mathrm{pw}} \le \| \bullet \|_h$ implies $\widehat{b}(R\boldsymbol{\theta}_h, \bullet) \in \mathbf{H}^{-1-t}(\Omega)$ for $R \in \{\mathbf{id}, \boldsymbol{I}_{\mathrm{M}}, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\}$. Therefore, there exists a unique $\boldsymbol{\chi} \in \mathbf{V} \cap \mathbf{H}^{3-t}(\Omega)$ with $\|\boldsymbol{\chi}\|_{\mathbf{H}^{3-t}(\Omega)} \lesssim 1$ such that $A(\boldsymbol{\chi}, \Phi) = \widehat{b}(R\boldsymbol{\theta}_h, \Phi)$ for all $\Phi \in \mathbf{V}$. Hence, for Morley/dG schemes (resp. C^0IP scheme), the boundedness of R (from Lemma 7.7), Lemma 7.1 (resp. Remark 7.9), and Lemma 7.3.d provide **(H2)** with $\delta_2 \lesssim h_{\max}^{1-t}$.

The proof of **(H3)** starts as in Sect. 8.5 and adopts Lemma 9.6.a (in place of Lemma 8.9.a) to establish (8.11) with $t = 0$ and the slightly sharper version $\delta_3 \lesssim h_{\max}$.

Since $\delta_3 = 0$ for $S = Q = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$, it remains $S = \mathbf{id}$ and $= \boldsymbol{I}_{\mathrm{M}}$ in the sequel to establish **(H3)**. Given $\boldsymbol{y}_h$ and $\boldsymbol{\theta}_h \in \mathbf{V}_h$ of norm one, define $\mathbf{v}_2 := S\boldsymbol{y}_h \in \boldsymbol{P}_2(\mathcal{T})$ and observe $Q\boldsymbol{y}_h = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\boldsymbol{y}_h = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\mathbf{v}_2$ (by $S = \mathbf{id}, \boldsymbol{I}_{\mathrm{M}}$). Hence with the definition of $\widehat{b}(\bullet, \bullet)$, Lemma 9.6.a shows

$$|\widehat{b}(R\boldsymbol{\theta}_h, (S - Q)\boldsymbol{y}_h)| = |\widehat{b}(R\boldsymbol{\theta}_h, \mathbf{v}_2 - \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\mathbf{v}_2)| \lesssim h_{\max}\|\!|u\|\!| \, \|\!|R\boldsymbol{\theta}_h\|\!|_{\mathrm{pw}}\|\mathbf{v}_2\|_h.$$

The boundedness of R and $\boldsymbol{I}_{\mathrm{M}}$ and the equivalence of norms show $\|\!|R\boldsymbol{\theta}_h\|\!|_{\mathrm{pw}}\|\mathbf{v}_2\|_h \lesssim 1$ and hence $\delta_3 \lesssim h_{\max}$.

As in the application for Navier-Stokes equations, Remark 7.9 leads to hypothesis **(H4)** with $\delta_4 < \epsilon$. The existence and uniqueness of a discrete solution Ψ_h then follows from Theorem 4.1.

Note that for $\mathbf{v}_h \in \mathbf{M}(\mathcal{T})$, $Q\mathbf{v}_h = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\mathbf{v}_h$. For Morley/dG/$C^0$IP, Lemma 9.6.a with $\mathbf{v} = 0$ for $S = \mathbf{id}$; and Lemma 9.6.a with $\mathbf{v}_2 \in \mathbf{M}(\mathcal{T})$ and $\mathbf{v} = 0$ for $S = \boldsymbol{I}_{\mathrm{M}}$ show

$$\|\widehat{\Gamma}(\Psi, \Psi, (S - Q)\bullet)\|_{\mathbf{V}_h^*} \lesssim \begin{cases} 0 \text{ for } S = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}, \\ h_{\max} \text{ for } S = \mathbf{id} \text{ or } \boldsymbol{I}_{\mathrm{M}}. \end{cases}$$

The energy norm error control then follows from Theorem 5.1. □

9.6 Proof of Theorem 9.3

Given $2 - \sigma \le s \le 2$ and $G \in \mathbf{H}^{-s}(\Omega)$ with $\|G\|_{\mathbf{H}^{-s}(\Omega)} = 1$, the solution $z \in \mathbf{V}$ to the dual problem (6.1) belongs to $\mathbf{V} \cap \mathbf{H}^{4-s}(\Omega)$ by elliptic regularity. This and Lemma 7.3.d verify

$$|||z - \boldsymbol{I}_{\mathrm{M}}z|||_{\mathrm{pw}} \lesssim h_{\max}^{2-s}\|z\|_{\mathbf{H}^{4-s}(\Omega)} \lesssim h_{\max}^{2-s}. \tag{9.7}$$

Proof of Theorem 9.3.a. for $R = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$. The assumptions in Theorem 6.2 with $X_s := \mathbf{H}^s(\mathcal{T})$ are verified to establish the lower-order estimates. Hypothesis ($\widehat{\mathbf{H1}}$) for Morley/dG/C^0IP schemes is verified in [11, Lemma 6.6] for an equivalent norm (with Lemma 7.1) and Lemma 7.7 for $R = JI_{\mathrm{M}}$ (applied component-wise to vector functions). The conditions (2.3)–(2.6) follow from Lemma 7.7. In Theorem 6.2, set $z_h = \boldsymbol{I}_h\boldsymbol{I}_{\mathrm{M}}z$ with $\boldsymbol{I}_h = \mathbf{id}$ for Morley/dG resp. $\boldsymbol{I}_h = \boldsymbol{I}_{\mathrm{C}}$ for C^0IP. Notice that (9.7) implies

$$\|z - z_h\|_h \lesssim h_{\max}^{2-s} \tag{9.8}$$

for Morley/dG with $\|\bullet\|_{\mathrm{dG}} \approx |||\bullet|||_{\mathrm{pw}}$ in $\mathbf{V} + \mathbf{M}(\mathcal{T})$. Remark 7.9 and (9.7) provide (9.8) for C^0IP. For Morley/dG/C^0IP, Lemma 7.4.f implies $\|\Psi_h - P\Psi_h\|_{\mathbf{H}^s(\mathcal{T})} \lesssim h_{\max}^{2-s}\|\Psi - \Psi_h\|_h$.

The difference $\Gamma_{\mathrm{pw}}(R\Psi_h, R\Psi_h, Qz_h) - \Gamma(P\Psi_h, P\Psi_h, Qz_h)$ vanishes for $R = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}} = P$ (for all schemes). It remains to control the term $\widehat{\Gamma}(\Psi, \Psi, (S - Q)z_h)$ for $S \in \{\mathbf{id}, \boldsymbol{I}_{\mathrm{M}}, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\}$.

For $S = Q = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$, $\Gamma_{\mathrm{pw}}(\Psi, \Psi, (S - Q)z_h) = 0$. For $S = \mathbf{id}$, Lemma 9.6.a and (9.8) establish

$$\Gamma_{\mathrm{pw}}(\Psi, \Psi, (1 - \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}})z_h) \lesssim h_{\max}|||\Psi|||^2\|z - z_h\|_h \lesssim h_{\max}^{3-s}.$$

For $S = \boldsymbol{I}_{\mathrm{M}}$, Lemma 9.6.a applies to $\mathbf{v}_h = \boldsymbol{I}_{\mathrm{M}}z_h$. A triangle inequality and Lemma 7.7 reveal $\|z - \boldsymbol{I}_{\mathrm{M}}z_h\|_h \lesssim \|z - z_h\|_h \lesssim h_{\max}^{2-s}$ with (9.8) in the last step. Hence,

$$\Gamma_{\mathrm{pw}}(\Psi, \Psi, (\boldsymbol{I}_{\mathrm{M}} - \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}})z_h) \lesssim h_{\max}|||\Psi|||^2\|z - z_h\|_h \lesssim h_{\max}^{3-s}.$$

□

Proof of Theorem 9.3.a. for $R = \boldsymbol{I}_{\mathrm{M}}$. Elementary algebra and the symmetry of $\Gamma_{\mathrm{pw}}(\bullet, \bullet, \bullet)$ with respect to the first and second argument recast the last two terms on the right-hand side of Theorem 6.2 as

$$\begin{aligned}\Gamma_{\mathrm{pw}}&(\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h) - \Gamma_{\mathrm{pw}}(\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h)\\ &=2\Gamma_{\mathrm{pw}}((1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h)\\ &\quad - \Gamma_{\mathrm{pw}}((1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, (1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h). \end{aligned} \tag{9.9}$$

The arguments in (8.24)–(8.26) for (Ψ, Ψ_h) replacing (u, u_h) and (9.8) reveal

$$\|\!|\Psi - \boldsymbol{I}_{\mathrm{M}}\Psi_h\|\!|_{\mathrm{pw}} \lesssim \|\Psi - \Psi_h\|_h \text{ and } \|\!|z - \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h\|\!|_{\mathrm{pw}} \lesssim h_{\max}^{2-s}.$$

This and Lemma 9.6.b for the first term in (9.9) (resp. Lemma 9.5.a and 7.4 .e for the second) show

$$\begin{aligned}\Gamma_{\mathrm{pw}}((1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h) &\lesssim h_{\max}\|\Psi - \Psi_h\|_h\\ \Gamma_{\mathrm{pw}}((1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, (1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h) &\lesssim \|\!|(1-\boldsymbol{J})\boldsymbol{I}_{\mathrm{M}}\Psi_h\|\!|_{\mathrm{pw}}^2 \lesssim \|\Psi - \Psi_h\|_h^2.\end{aligned}$$

This leads in (9.9) to

$$\begin{aligned}\Gamma_{\mathrm{pw}}(\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h) &- \Gamma_{\mathrm{pw}}(\boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}\Psi_h, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h)\\ &\lesssim \|\Psi - \Psi_h\|_h(h_{\max} + \|\Psi - \Psi_h\|_h). \end{aligned} \tag{9.10}$$

The remaining terms are controlled as in the above case $R = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$. This concludes the proof. □

Proof of Theorem 9.3.b. Since $\Psi_h = \boldsymbol{I}_{\mathrm{M}}\Psi_{\mathrm{M}}$, and $P = Q = \boldsymbol{J}$ for the Morley FEM, the last two terms of Theorem 6.2 read $\Gamma_{\mathrm{pw}}(\Psi_{\mathrm{M}}, \Psi_{\mathrm{M}}, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h) - \Gamma(\boldsymbol{J}\Psi_{\mathrm{M}}, \boldsymbol{J}\Psi_{\mathrm{M}}, \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}z_h)$ and are controlled in (9.10). This, Theorem 6.2, and the above estimates from the proof for $R = \boldsymbol{J}\boldsymbol{I}_{\mathrm{M}}$ in (a) conclude the proof. □

Proof of Theorem 9.4. The proofs at the abstract level in Sects. 2–6 follow as further explained for the Navier Stokes equations. A straightforward adoption of the arguments provided in the proofs of Theorem 9.1 and 9.3.a lead to **(H2)-(H4)** and the a priori error control. □

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